Gaussian kernels have also Gaussian minimizers

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joint work with Franck Barthe (University of Toulouse)

Lieb's principle: Gaussian kernels have Gaussian maximizers

For $p, q \geq 1$ consider $T \colon L^p(\mathbb{R}^{n_1}) \to L^q(\mathbb{R}^{n_2})$

$$Tf(x) = \int_{\mathbb{R}^{n_1}} K(x,y) f(y) \, dy$$

Question: $\|T\| =?$

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Gaussian function: $f(x) = \exp(-Q(x) + \ell(x) + c)$

- Q is a psd quadratic form; if Q is pd then f is a non-degenerate Gaussian function
- ℓ is a linear form; if $\ell \equiv 0$ then f is centered
- $\mathcal{CG}$ — non-degenerate and centered Gaussian functions

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Theorem (Lieb '90)

If $K(x,y) = \exp\big(-Q_1(x) - Q_2(y) - B(x,y)\big)$ then

$$\|T\| = \sup\left\{ \frac{\|Tf\|_{L^q}}{\|f\|_{L^p}} : f \in \mathcal{CG} \right\}$$

Lieb's principle: more general setting

- $H, H_1, \ldots, H_m$ Euclidean spaces; $c_1, \ldots, c_m > 0$ exponents
- $f_i \colon H_i \to [0, \infty)$ measurable with $0 < \int_{H_i} f_i < \infty$
- $B_i \colon H \to H_i$ surjective linear maps
- $Q \colon H \to H$ self-adjoint, psd ($Q \geq 0$)

Example: $H = \mathbb{R}^n, \qquad H_i = \mathbb{R}, \qquad B_i = \langle \cdot, u_i \rangle, \quad u_i \in H$

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The functional J

$$J(f_1, \dots, f_m) = \frac{\int_H e^{-\langle Qx, x\rangle} \prod_{i=1}^m f_i^{c_i}(B_i x)\,dx}{\prod_{i=1}^m \left(\int_{H_i} f_i \right)^{c_i}}$$

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Theorem (Brascamp-Lieb '76 ($Q = 0, dim H_i = 1$), Lieb '90)

$$\sup_{f_1, \ldots, f_m} J(f_1, \ldots, f_m) = \sup_{g_1, \ldots, g_m \in \mathcal{CG}} J(g_1, \ldots, g_m)$$

Lieb's principle: remarks

- $h_i \in L^{p_i}(H_i), \quad p_i = 1/c_i, \quad f_i = |h_i|^{p_i}$

$$\frac{\int_H e^{-\langle Qx,x\rangle} \prod h_i(B_i x)\, dx}{\prod \|h_i\|_{L^{p_i}}} \leq J(|h_1|^{p_1}, \ldots, |h_m|^{p_m})$$

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- $J(g_1, \dots, g_m)$ for $g_i \in \mathcal{CG}$:
  let A be positive definite and $g_A(x) = e^{-\pi\langle Ax,x\rangle}$

$$\int g_A(x)\, dx = (\det A)^{-1/2}$$

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$J(g_{A_1}, \dots, g_{A_m})$

$$= \frac{\int e^{-\langle Qx,x\rangle} \prod e^{-c_i\langle A_i(B_i x),(B_i x)\rangle}\,dx}{\prod (\det A_i)^{-c_i/2}} = \left(\frac{\det(Q + \sum c_i B_i^* A_i B_i)}{\prod (\det A_i)^{c_i}}\right)^{-\frac{1}{2}}$$

Lieb's principle: examples

Nelson's hypercontractivity ('73): $(P_t)_{t\geq 0}$ acting on $f\colon \mathbb{R}\to\mathbb{R}$

$$(P_t f)(x) = \int f(e^{-t}x + \sqrt{1-e^{-2t}}y)\,\gamma(dy)$$

[$P_t\colon L^p(\gamma)\to L^p(\gamma)$ has norm 1 for all $t\geq 0$ and $1\leq p\leq\infty$]

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Theorem (Nelson '73)

If $1<p<q<\infty$ and $t>0$ satisfy $e^{-2t}\leq \frac{p-1}{q-1}$ then

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$\int_{\mathbb{R}}(P_t f)(x)g(x)\,\gamma(dx) \leq \Big(\int |f|^p\,d\gamma\Big)^{1/p}\Big(\int |g|^{q'}\,d\gamma\Big)^{1/q'}$, or

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$$\int_{\mathbb{R}^2} e^{-Q_t(x,y)} f_1^{1/p}(x) f_2^{1/q'}(y)\,dx\,dy \leq \Big(\int f_1\,dx\Big)^{1/p}\Big(\int f_2\,dx\Big)^{1/q'}$$

Young's convolution inequality with sharp constant (Beckner '75, Brascamp-Lieb '76)

Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$
For all $f \in L^p$, $g \in L^q$,

$$\|f * g\|_{L^r} \leq C_{p,q} \|f\|_{L^p} \|g\|_{L^q}$$

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Sharp for $f(x) = e^{-p'x^2}, g(x) = e^{-q'x^2}, h(x) = e^{-r'x^2}$

Converse inequalities: Borell's reversed hypercontractivity

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$p < 1$, any measure, $f \geq 0$

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Theorem (Borell '82)

$-\infty < q < p < 1$. If $e^{-2t} \leq \frac{1-p}{1-q}$ then

$$\|P_t f\|_{L^q(\gamma)} \geq \|f\|_{L^p(\gamma)}$$

Converse Young's convolution ineq. (Brascamp-Lieb '76)

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For $\lambda \in (0, 1)$, taking $p = \frac{t}{\lambda}, q = \frac{t}{1-\lambda}$ and letting $t \to 0^+$

$$\int \operatorname*{ess\,sup}_y f\left(\frac{x-y}{\lambda}\right)^{\lambda} g\left(\frac{y}{1-\lambda}\right)^{1-\lambda} dx \geq \left(\int f\right)^{\lambda} \left(\int g\right)^{1-\lambda}$$

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Converse Brascamp-Lieb ineq. (Chen-Dafnis-Paouris '13)

Under certain assumptions on c_i and B_i (some of $c_i \geq 1$, the remaining $c_i < 0$):

$$\int_H \prod_{i=1}^m f_i^{c_i}(B_i x)\, dx \geq \prod_{i=1}^m \left(\int_{H_i} f_i\right)^{c_i}$$

Main result

$c_i \in \mathbb{R} \setminus \{0\}, \qquad c_1, \dots, c_{m^+} > 0, \quad c_{m^+ + 1}, \dots, c_m < 0$

$Q \colon H \to H$ self-adjoint

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Theorem (Barthe-W. '14)

If

(i) $x \mapsto (B_1 x, \ldots, B_{m^+} x)$ from H to $H_1 \times \cdots H_{m^+}$ is onto

(ii) $s^+(Q) + \dim H_1 + \ldots + \dim H_{m^+} \leq \dim H$

then

$$\inf_{f_1,\ldots,f_m} J(f_1, \ldots, f_m) = \inf_{g_1,\ldots,g_m \in \mathcal{CG}} J(g_1, \ldots, g_m)$$

Moreover, if (i) or (ii) fails then either $J \equiv \infty$ or $\inf J = 0$.

Proof: the setting

Setting: $Q = 0$, $H = \mathbb{R}^n$, $H_i = \mathbb{R}$, $B_i = \langle \cdot, u_i \rangle$, assume $\int f_i = 1$

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Example: converse Young's convolution inequality

$$H = \mathbb{R}^2, m = 3; c_1 = \frac{1}{p}, c_2 = \frac{1}{q} \geq 1, c_3 = \frac{1}{r'} < 0$$

$$\langle x, u_1 \rangle = x_1, \langle x, u_2 \rangle = x_1 - x_2, \langle x, u_3 \rangle = x_2$$

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$$\prod g_{a_i}^{c_i}(\langle x, u_i \rangle) = \prod a_i^{c_i/2} \exp\Big(-\pi \Big\langle \big(\sum c_i a_i u_i \otimes u_i \big) x, x \Big\rangle \Big)$$

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If $\sum c_i a_i u_i \otimes u_i > 0$ (is p.d.) then

$$J(g_{a_1}, \ldots, g_{a_m}) = \left(\frac{\prod a_i^{c_i}}{\det\left(\sum c_i a_i u_i \otimes u_i \right)} \right)^{1/2}$$

Proof: Gaussian computation

$$J(g_{a_1}, \dots, g_{a_m}) = \begin{cases} \left(\dfrac{\prod a_i^{c_i}}{\det\left(\sum c_i a_i u_i \otimes u_i\right)} \right)^{1/2} & \text{if } \sum c_i a_i u_i \otimes u_i > 0 \\ +\infty & \text{otherwise} \end{cases}$$

Define

$$D^{1/2} := \inf_{a_1, \dots, a_m > 0} J(g_{a_1}, \dots, g_{a_m})$$

i.e. $D \geq 0$ is the best (greatest) constant s.t. for all $a_i > 0$

$$\sum c_i a_i u_i \otimes u_i \geq 0 \implies \prod a_i^{c_i} \geq D \cdot \det\left(\sum c_i a_i u_i \otimes u_i\right) \quad (1)$$

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Goal: $J(f_1, \dots, f_m) \geq D^{1/2}$

Proof: monotone mass transport

- Fix $f_i \colon \mathbb{R} \to [0, \infty)$ regular
  $I_i := \{f_i > 0\}$ bd open int. for $i \leq m^+$ and $I_i = \mathbb{R}$ for $i > m^+$

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Monotone transport $f_i(x)dx$ onto $g_{a_i}(y)dy$ by $\theta_i \colon I_i \to \mathbb{R}$

$$\int_{-\infty}^{x} f_i(t)\, dt = \int_{-\infty}^{\theta_i(x)} g_{a_i}(y)\, dy, \text{ i.e. } f_i(x) = g_{a_i}(\theta_i(x))\theta_i'(x)$$

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$$\int_{-\infty}^{x} f_i(t)\,dt = \int_{-\infty}^{\theta_i(x)} g_{a_i}(y)\,dy, \text{ i.e. } f_i(x) = g_{a_i}(\theta_i(x))\theta_i'(x)$$

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Proof: monotone mass transport

- Fix $f_i\colon \mathbb{R} \to [0,\infty)$ regular
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On $\Omega_+ := \{x \in \Omega \colon D\theta(x) = \sum c_i \theta_i'(\langle x, u_i\rangle) u_i \otimes u_i \ge 0\}$ use (1)

Proof: duality of quadratic forms

Recall: $\theta(x) = \sum c_i u_i \theta_i(\langle x, u_i \rangle)$

$$J(f_1, \ldots, f_m) \overset{(1)}{\geq} D \cdot \int_{\Omega_+} \prod g_{a_i}^{c_i}\big(\theta_i(\langle x, u_i \rangle)\big) \cdot \det D\theta(x)\, dx$$

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Proof: optimizing over Gaussian functions

For any $a_i > 0$ s.t. $\sum c_i a_i^{-1} u_i \otimes u_i > 0$ we have proved

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Take *sup* over $a_i > 0$ to get $J(f_1, \dots, f_m) \geq D \cdot D^{-1/2} = D^{1/2}$.

Proof: Surjectivity of $\theta(x) = \sum c_i u_i \theta_i(\langle x, u_i \rangle) \colon \Omega_+ \to \mathbb{R}^n$ ...?

Let $\phi_i \colon I_i \to \mathbb{R}$ s.t. $\phi_i' = \theta_i$; then ϕ_i convex.

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Note that $\theta(x) = \nabla \phi(x)$, where

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$$x_0 = \arg\min \phi_+(x) - \phi_-(x) - \langle x, y_0 \rangle$$

Note that $x_0 \in \Omega$. Therefore

- $\nabla(\phi_+ - \phi_- - \langle \cdot, y_0 \rangle)(x_0) = 0$, i.e. $\nabla\phi(x_0) = y_0$
- $D^2(\phi_+ - \phi_- - \langle \cdot, y_0 \rangle)(x_0) = D\theta(x_0) \ge 0$, i.e. $x_0 \in \Omega_+$