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SDE's for particle systems and applications in harmonic analysis

Consider the following system of SDEs

$$d\lambda_i = \sigma_i(\lambda_i)dB_i + \left(b_i(\lambda_i) + \sum_{j \neq i} \frac{H_{ij}(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j} \right) dt, \quad i = 1, \dots, p, \tag{1}$$

describing ordered particles $\lambda_1(t) \leq \ldots \leq \lambda_p(t)$, $t \geq 0$ on $\mathbf{R}$. Here B_i denotes a collection of one-dimensional independent Brownian motions.

Let $Sym(p \times p)$ be the vector space of symmetric real $p \times p$ matrices. The SDEs systems (1) contain the systems describing, for the starting point having no collisions, the eigenvalues of the $Sym(p \times p)$-valued process X_t satisfying the following matrix valued stochastic differential equation

$$dX_t = g(X_t)dW_t h(X_t) + h(X_t)dW_t^T g(X_t) + b(X_t)dt,$$

where the functions g, h, b act spectrally on $Sym(p \times p)$, and W_t is a Brownian matrix of dimension $p \times p$. Thus the systems (1) contain Dyson Brownian Motions, Squared Bessel particle systems, Jacobi particle systems, their β-versions and other particle systems crucial in mathematical physics and physical statistics ([5],[6]).

Note that the functions $\frac{H_{ij}(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j}$ describe the repulsive forces with which the particle λ_i acts on the particle λ_j. On the other hand the singularities $\frac{1}{\lambda_i - \lambda_j}$ make the SDEs system (1) difficult to solve, especially when the starting point $\Lambda(0)$ has a collision $\lambda_i(0) = \lambda_j(0)$. The most degenerate case $\lambda_1(0) = \ldots = \lambda_p(0)$ is of great importance in applications.

In some particular cases (Dyson Brownian Motions, some Squared Bessel particle systems), the existence of strong solutions of (1) has been established by Cépa and Lépingle, using the technique of Multivalued SDEs ([1], [7]).

We prove the existence of strong and pathwise unique non-colliding solutions of (1), with a degenerate colliding initial point $\Lambda(0)$ in the whole generality, under natural assumptions on the coefficients of the equations in (1). Our approach is based on the classical Itô calculus, applied to elementary symmetric polynomials in p variables $X = (x_1, \ldots, x_p)$

$$e_n(X) = \sum_{i_1 < \ldots < i_n} x_{i_1} x_{i_2} \ldots x_{i_n},$$

as well as to symmetric polynomials of squares of differences between particles

$$V_n = e_n(A), \quad \text{where } A = \{a_{ij} = (\lambda_i - \lambda_j)^2 : 1 \leq i < j \leq p\}.$$

In the case of Squared Bessel particle systems

$$d\lambda_i = 2\sqrt{\lambda_i}dB_i + \left(\alpha + \sum_{j\neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j}\right) dt,$$

describing the eigenvalues of the matrix Squared Bessel process

$$dX_t = \sqrt{X_t}dW_t + dW_t^T \sqrt{X_t} + \alpha I dt,$$

we use our stochastic approach in order to determine the so-called Wallach set of permissible parameters α, known before only by harmonic analysis methods. We also determine the admissible starting points of such processes for $\alpha = 1, \ldots, p-2$.

References

[1] E. Cépa, D. Lépingle, *Diffusing particles with electrostatic repulsion*, Probab. Theory Related Fields 107 (1997), no. 4, 429-449.

[2] P. Graczyk, J. Małecki, *Multidimensional Yamada-Watanabe theorem and its applications to particle systems*, J. Math. Phys. 54, 021503 (2013)

[3] P. Graczyk, J. Malecki *Strong solutions of non-colliding particle systems*, Electron. J. Probab., 19(2014), no. 119, 1-21.

[4] P. Graczyk, J. Malecki *Generalized Squared Bessel particle systems and Wallach set*, preprint (2015).

[5] M. Katori and H. Tanemura, *Noncolliding processes, matrix-valued processes and determinantal processes*, Sugaku Expositions 24 (2011), no. 2, 263-289.

[6] M. Katori and H. Tanemura, *Noncolliding Squared Bessel processes*, J. Stat. Phys.(2011) **142**, 592-615

[7] D. Lépingle, *Boundary behavior of a constrained Brownian motion between reflecting-repellent walls*, Probab. Math. Statist. 30 (2010), no. 2, 273-287.