Poisson boundary : from discrete to continuous groups

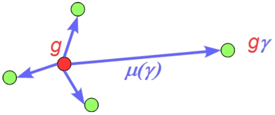
Sara Brofferio - University Paris Sud

Bedlewo, May 2015

1. Harmonic functions from discrete to continuous groups
   - Bounded μ-harmonic functions on a group
   - Countable Γ in continuous G
2. G-Poisson Boundary
   - Example ; Affine group
3. From Γ-boundary to G-boundary
   - Poisson Boundary of Baumslag-Solitar Group
   - Open Questions

μ-harmonic functions on a group

- G be locally compact group, e.g. group of matrices such as :
  $SL_2(\mathbb{R})$ or $Aff(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} | a \in \mathbb{R}^*_+, b \in \mathbb{R} \right\}$
- μ probability on G



Bounded μ-harmonic function on G

$$f(g) = \int_G f(g\gamma) d\mu(\gamma) \qquad \text{for } \lambda\text{-almost all } g \in G.$$

λ-Haar measure on G

Spaces of bounded G-harmonic function $H^\infty(G) \subset L^\infty(G, \lambda)$

State of art : a Quick tour

Questions

- There exist **bounded harmonic functions on** (G, λ) ?
- If yes, describe the space $H^\infty(G)$ via an integral representation.

Quite well understood if :

- μ generic for special situation :
  - for Abelian Groups $\Rightarrow$ no bounded harmonic function (Choquet and Deny,...)
  - similar result for nilpotent groups (Guivarc'h ('73), Breuillard ('02),..)
  - *NA* group for measure μ with specific drift (Raugi('77),..)
  - ...
- μ absolutely continuous w.r. to λ (Frustemberg ('63)...)
- **in particular if** G **is countable** thus λ =counting measure (Derriennic, Kaimanovich and Vershik('83),...)

Purely atomic μ

μ is **purely atomic**
supported by a countable subgroup Γ dense a continuous group G

Exemples :

- μ supported by $\begin{bmatrix} 2 & \pm 1 \\ 0 & 1 \end{bmatrix}^{\pm 1}$ and $\begin{bmatrix} 3 & \pm 1 \\ 0 & 1 \end{bmatrix}^{\pm 1}$.
  Then μ is supported by a dense subgroup of $Aff(\mathbb{R})$
- More generally μ supported by matrices with rational coefficient as a sub group of real matrices. E.g. μ supprted by $SL_2(\mathbb{Q}) \subset SL_2(\mathbb{R})$.

Harmonic functions on G or on Γ

$$f(g) = \int_\Gamma f(g\gamma)\, d\mu(\gamma)$$

↗ for λ-almost all $g \in G \Rightarrow f$ is G-harmonic

↘ for all $g \in \Gamma \Rightarrow f$ is Γ-harmonic

G-harmonic functions vs Γ-harmonic functions

$$L^\infty(G,\lambda) \supset H^\infty(G) \neq H^\infty(\Gamma) \subset L^\infty(\Gamma, \text{counting})$$

Remarks

- if $f \in H^\infty(G)$ is **continuous** then
  - $f(\gamma)$ is well defined for $\gamma \in \Gamma$ and Γ-harmonic.
  - If Γ is dense in G, then f is uniquely determined by $f\big|_\Gamma$

  $f \mapsto f\big|_\Gamma$ is an isometric embedding $H^\infty(G) \cap C^0(G) \hookrightarrow H^\infty(\Gamma)$
- every function $f \in H^\infty(G)$ is λ-a.s. limit **continuous** functions :

$$f_n(g) = \int \alpha_n(h) f(hg)\lambda(dh) \to f(g) \text{ if } \alpha_n(h)\lambda(dh) \to \delta_e$$

Idea

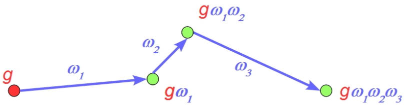
- If $H^\infty(\Gamma)$ can be described
- if $\Gamma \hookrightarrow G$ is well understood

On should be able to understand $H^\infty(G)$

Random walks

- $(\Omega, \mathbb{P}) = (G, \mu)^{\mathbb{N}}$, be the space of random steps
- the right random walk with starting point g and steps $\{\omega_i\} \in \Omega$:

$$gr_n(\omega) = g\omega_1 \cdots \omega_n$$



Harmonic function $f \in L^\infty(G, \lambda)$

- the process $f(gr_n(\omega))$ is a well defined martingale on $\rho(dg) \times \mathbb{P}(d\omega)$ if ρ is a probability absolutely continuous with respect to λ
- $\lim_{n\to\infty} f(gr_n(\omega)) =: Z_f(g, \omega)$ exists $\rho(dg)\mathbb{P}(d\omega)$-almost surely.
- $f(g) = \int Z_f(g, \omega) d\mathbb{P}(\omega)$

Poisson transform

(G, μ)-spaces

a measurable space (X, ν)

- G acts on X
- ν is an μ invariant probability $\int_X \phi(x) d\nu(x) = \int_X \phi(g \cdot x) d\nu(x) d\mu(g)$

Poisson transform

$$\mathcal{P}_\nu : \phi \mapsto f_\phi(g) := \int_X \phi(g \cdot x) d\nu(x)$$

- f_ϕ is harmonic (since ν is μ-invariant)

$$\int_G f(g\gamma) d\mu(\gamma) = \int_G \int_X \phi(g\gamma \cdot x) d\nu(x) d\mu(\gamma) = \int_X \phi(g \cdot x) d\nu(x) = f(g)$$

Poisson Boundary

$$\mathcal{P}_\nu : \phi \mapsto f_\phi(g) := \int_X \phi(g\cdot x)d\nu(x)$$

Problem : $g * \nu$ can be singular w.r. to ν thus $\|\phi\|_\infty^\nu \neq \|\phi(g\cdot\ )\|_\infty^\nu$

- ρ is a probability absolutely continuous with respect to λ
- $\|\phi\|_\infty^{\rho*\nu} = \|\phi(g\cdot\ )\|_\infty^{\rho*\nu}$
- $\mathcal{P}_\nu$ is well defined on $L^\infty(X, \rho * \nu)$

$$\mathcal{P}_\nu : L^\infty(X, \rho * \nu) \to H^\infty(G)$$

G-Poisson Boundary

(X, ν) is a G-poisson boundary if

$$\mathcal{P}_\nu : \quad L^\infty(X, \rho * \nu) \quad \longleftrightarrow \quad H^\infty(G)$$

is a bijection.

It can be shown that the Poisson boundary is unique as a G-measurable

Exemple : Affine group

- $Aff(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} | a \in \mathbb{R}_+^*, b \in \mathbb{R} \right\}$
- μ on $Aff(\mathbb{R})$ with suitable moments.

$Aff(\mathbb{R})$ acts on $\mathbb{R}$ by $x \mapsto ax + b$.

Results for $Aff(\mathbb{R})$**-Poisson Boundary** (Raugi....)

- If $\int \log a \; \mu(da) = 0$ the $Aff(\mathbb{R})$-Poisson Boundary is trivial
- If $\int \log a \; \mu(da) < 0$
  - Then if $\begin{bmatrix} a_n & b_n \\ 0 & 1 \end{bmatrix}$ i.i.d.

    $$Z = \sum_{n=1}^{\infty} a_1 \cdots a_{n-1} b_n$$

    converges in $\mathbb{R}$.
  - $(\mathbb{R}, \nu \sim Z)$ is the $Aff(\mathbb{R})$-Poisson Boundary
- if $\int \log a \; \mu(da) > 0$ no general results. Trivial if μ a.c.

Countable subgroups of $Aff(\mathbb{R})$

Baumsalg-Solitar group :

$$BS(1,2) = \left\langle \begin{bmatrix} 2 & \pm 1 \\ 0 & 1 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 2^{k_1} & k_2 2^{k_3} \\ 0 & 1 \end{bmatrix} | k_i \in \mathbb{Z} \right\} = \mathbb{Z}(\frac{1}{2}) \rtimes \mathbb{Z}.$$

The action $x \mapsto ax + b$ is defined on $\mathbb{R}$ and in $\mathbb{Q}_2$

- $\int \log a \; \mu(da) < 0$
  - $Z = \sum_{n=1}^{\infty} a_1 \cdots a_{n-1} b_n$ converges in $\mathbb{R}$.
  - $(\mathbb{R}, \nu \sim Z)$ is the $BS(1,2)$-Poisson Boundary
- $\int \log a \; \mu(da) > 0 \Longrightarrow \int \log |a|_2 \; \mu(da) = -\int \log a \; \mu(da) < 0$
  - $Z = \sum_{n=1}^{\infty} a_1 \cdots a_{n-1} b_n$ converges in $\mathbb{Q}_2$.
  - $(\mathbb{Q}_2, \nu \sim Z)$ is the $BS(1,2)$-Poisson Boundary

Question :

If $BS(1,2)$-Poisson Boundary is $\mathbb{Q}_2$, there exist **real** harmonic functions ?

Same question for μ supported by $\begin{bmatrix} 2 & \pm 1 \\ 0 & 1 \end{bmatrix}^{\pm 1}$ and $\begin{bmatrix} 3 & \pm 1 \\ 0 & 1 \end{bmatrix}^{\pm 1}$

G-action on Γ-space

- μ supported on countable Γ dense in G.
- (X, ν) Γ-Poisson boundary

Goal

Build the G-boundary on the Γ-space X

If f **continuous** G-harmonic function

- $f\big|_\Gamma \in H^\infty(\Gamma) \Rightarrow f(\gamma) = \int_X \phi(\gamma \cdot x) d\nu(x) \qquad \forall \gamma \in \Gamma.$
  Do not hold for $\gamma = g \in G \setminus \Gamma$, since X is not a priori a G-space.
- For fixed g $f(g\cdot)$ is Γ-harmonic thus exist $\phi(g, \cdot) \in L^\infty(X, \nu)$:

$$f(g\gamma) = \int_X \phi(g, \gamma \cdot x) d\nu(x) \qquad \forall \gamma \in \Gamma.$$

- Since $f(g) = f(g\gamma^{-1}\gamma) \Rightarrow$

$$f(g) = \int_X \phi(g, x) d\nu(x) = \int_X \phi(g\gamma^{-1}, \gamma \cdot x) d\nu(x) = f(g\gamma^{-1}\gamma)$$

From Γ-boundary to G-boundary

Theorem

- μ supported by a countable subgroup Γ of G
- X be the Γ-Poisson Boundary

For all $f \in H^\infty(G)$ there exits $\phi \in L^\infty(G \times X, \rho \times \nu)$ such that

$$\phi(g, x) = \phi(g\gamma^{-1}, \gamma \cdot x) \text{ for all } \gamma \in \Gamma \text{ and } \rho(dg)\nu(dx) \text{ a.s.}$$

and

$$f(g) = \int_X \phi(g, x)\nu(dx) \quad \rho(dg) \text{ a.s..}$$

The G-Poisson boundary is the **measurable** quotient space

$$(G \times X, \rho \times \nu) /_\Gamma$$

for the Γ-action $\gamma * (g, x) = (g\gamma^{-1}, \gamma \cdot x)$

$(G \times X)/_{\Gamma}$

Remarks :

- $\gamma * (g, x) = (g\gamma^{-1}, \gamma \cdot x)$ is (after conjugation) equal to is the action of Γ on $G \times X$:

$$\gamma \overset{\mathrm{d}}{\cdot} (g, x) = (\gamma g, \gamma \cdot x).$$

- $\eta(g, x) \in (G \times X)/_{\Gamma}$ is a measurable equivalence class containing (g, x) :

$$\eta(g, x) = \eta(g\gamma^{-1}, \gamma \cdot x) \text{ for all } \gamma \in \Gamma$$

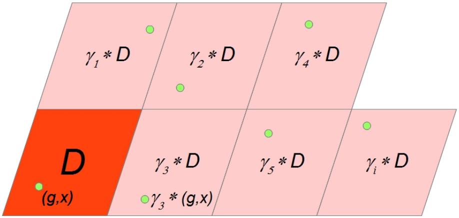
- $(G \times X)/_{\Gamma}$ is a measurable G-space

$$g_0\eta(g, x) = \eta(g_0 g, x)$$

- The G-action coincide with the Γ-action on X :

$$\gamma\eta(e, x) = \eta(\gamma, x) = \eta(e, \gamma \cdot x).$$

Actions with fundamental domain



Corollary

Suppose there exists measurable fundamental domain $D \subset G \times X$ for the action $*$ of Γ that is

- $\Gamma * D = G \times X \qquad \rho \times \nu$-a.s.
- $\rho \times \nu(D \cap \gamma * D) = 0 \quad \forall \gamma \in \Gamma - \{e\}$

then $D = {(G \times X)}/_{\Gamma}$ is the G-Poisson boundary

Poisson Boundary of Baumslag-Solitar Group

Corollary

$$BS(1,2) = \left\langle \begin{bmatrix} 2 & \pm 1 \\ 0 & 1 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 2^{k_1} & k_2 2^{k_3} \\ 0 & 1 \end{bmatrix} | k_i \in \mathbb{Z} \right\} = \mathbb{Z}(\frac{1}{2}) \rtimes \mathbb{Z}.$$

- μ with first logarithmic moment on $\mathbb{R}$ and $\mathbb{Q}_2$.
- $\int \log a \; \mu(da) > 0$,(thus $BS(1,2)$-Poisson boundary is $X = \mathbb{Q}_2$).

then the **real** Poisson boundary is is the p-solenoid :

$$D = [0,1) \times \mathbb{Z}_2 = {}^{\mathbb{R} \times \mathbb{Q}_2} \Big/ {}_{\mathbb{Z}(\frac{1}{2})}.$$

That is any bounded function on D rise in a harmonic function on $\left\{ \begin{bmatrix} 2^m & b \\ 0 & 1 \end{bmatrix} | m \in \mathbb{Z}, b \in \mathbb{R} \right\} = \mathbb{R} \rtimes \mathbb{Z}$

Proof : For all $(x, \xi) \in \mathbb{R} \times \mathbb{Q}_2$ there exist a unique $k \in \mathbb{Z}(\frac{1}{2})$ such that $0 < x - k \leq 1$ and $|\xi + k| \leq 1$.

Open questions

- μ supported by $\Gamma = \left\langle \begin{bmatrix} 2 & \pm 1 \\ 0 & 1 \end{bmatrix}^{\pm 1}, \begin{bmatrix} 3 & \pm 1 \\ 0 & 1 \end{bmatrix}^{\pm 1} \right\rangle$.

  If $\int \log a \; \mu(da) > 0$, the Γ-Poisson boundary is $\mathbb{Q}_2$ or $\mathbb{Q}_2 \times \mathbb{Q}_3$
  The $Aff(\mathbb{R})$-Poisson Boundary is it trivial ?

- μ supported by

$$\Gamma = SL_2(\mathbb{Z}(1/2)) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | ad - cd = 1, a, b, c \text{ et } d \in \mathbb{Z}/2^m \right\}$$

  The Γ-Poisson boundary is $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{Q}_2)$.
  The $SL_2(\mathbb{R})$-Poisson boundary is $\mathbb{P}^1(\mathbb{R})$?