Hitting times of points for symmetric Lévy processes with completely monotone jumps

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Notation

Let X be a one-dimensional Lévy process with Lévy–Khintchine exponent Ψ:

$$\mathbf{E}e^{iyX_t} = e^{-t\Psi(y)}.$$

We assume that X is symmetric, so that:

$$\Psi(y) = ay^2 + \int_{\mathbf{R}\setminus\{0\}} (1 - \cos(yz))\nu(dz),$$

where $a \geq 0$ is the Gaussian component and ν is the (symmetric) Lévy measure.

Notation

The first hitting time of a point $x \in \mathbf{R}$ is defined by the formula:

$$\tau_x = \inf\{t \geq 0 : X_t = x\}.$$

Under a number of assumptions, we find estimates for the

- $\mathbf{P}(\tau_x > t)$
- the density function of τ_x, denoted by $p_x(t)$
- all time derivatives of $p_x(t)$

for small x and large t.

Basic example: Wiener process

Lévy–Khintchine exponent: $\Psi(y) = |y|^2$.

- The answer is known explicitly: by the reflection principle:

$$\mathbf{P}(\tau_x \leq t) = 2\mathbf{P}(X_t \geq x),$$

and so

$$\mathbf{P}(\tau_x > t) = \mathbf{P}(|X_t| < x).$$

- It is easy to see that:

$$\mathbf{P}(\tau_x > t) \approx 1 \wedge \frac{x}{\sqrt{t}} \qquad \text{for all } t, x$$

$$p_x(t) \approx \frac{x}{t^{3/2}} \qquad \text{when } t > Cx^2.$$

Assumptions

(A) X is a subordinated Brownian motion and the Lévy measure μ of subordinator is absolutely continuous, with a completely monotone density on $(0, \infty)$.

Recall that f is completely monotone if $(-1)^n f^{(n)}(x) \geq 0$ for $n = 0, 1, 2, \ldots$ (equivalently: f is the Laplace transform of a nonnegative measure).

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Equivalently:

(A") Lévy measure ν of X is absolutely continuous with a completely monotone density on $(0, \infty)$ (in this case we say that X has completely monotone jumps)

Assumptions

(B) X satisfies a scaling-type condition of order $\alpha \in (1, 2]$:

$$\frac{x\Psi''(x)}{\Psi'(x)} \geq \alpha - 1 \qquad (x > 0).$$

This condition is equivalent to:

$$\frac{\Psi'(x_2)}{\Psi'(x_1)} \geq \left(\frac{x_2}{x_1}\right)^{\alpha-1} \qquad (0 < x_1 < x_2)$$

and it implies that:

$$\frac{\Psi(x_2)}{\Psi(x_1)} \geq \left(\frac{x_2}{x_1}\right)^{\alpha} \qquad (0 < x_1 < x_2),$$

which motivates the name scaling-type condition.

Assumptions

Recall that Ψ is regularly varying at 0 with index γ if:

$$\lim_{x \to 0^+} \frac{\Psi(kx)}{\Psi(x)} = k^\gamma.$$

Similarly we say that Ψ is regularly varying at ∞ with index δ if:

$$\lim_{x \to \infty} \frac{\Psi(kx)}{\Psi(x)} = k^\delta.$$

Our results are expressed in terms of Ψ and its partial inverse Ψ^{-1}. Conditions (A), (B) give estimates for $\mathbf{P}(\tau_x > t)$ and its time derivatives. Additional assumption of regular variation give asymptotic behaviour for small x and large t.

Examples

Assumptions (A) and (B) are satisfied by the following processes, provided that $1 < \alpha < \beta \leq 2$:

- Symmetric stable processes: $\Psi(y) = y^\alpha$
- Sums of independent symmetric stable processes: $\Psi(y) = y^\alpha + y^\beta$
- Relativistic Lévy processes: $\Psi(y) = (1 + |y|^\beta)^{\alpha/\beta} - 1$
- Process with Lévy–Khintchine exponent: $\Psi(y) = \frac{y^2}{\log(1+y^2)} - 1$

Spectrally negative processes

For Lévy processes with negative jumps only:

- For $x > 0$, τ_x is equal to the first passage time: $\tau_x = \inf\{t \geq 0 : X_t \geq x\}$.
- Fluctuation theory of Lévy processes provides a lot of information about τ_x.
- Standard references for the fluctuation theory are [Bertoin, 1996], [Kyprianou, 2006], [Sato, 1999].

α-stable Lévy processes

For (also non-symmetric) α-stable Lévy processes:

- $\mathbf{P}(\tau_x > t) = \mathbf{P}(\tau_1 > x^{-\alpha} t)$ for $x > 0$,
  $\mathbf{P}(\tau_x > t) = \mathbf{P}(\tau_{-1} > (-x)^{-\alpha} t)$ for $x < 0$.
- Full series expansion in [Cordero, 2010] in the symmetric case and in [Kuznetsov–Kyprianou–Pardo–Watson, 2014] in general.
- For *most* irrational α, with $\varrho = \mathbf{P}(X_t > 0)$:

$$p_1(t) = \frac{\sin\frac{\pi}{\alpha}}{\pi\sin(\pi\varrho)} \sum_{k=1}^{\infty} \frac{\Gamma(\frac{k}{\alpha}+1)\sin((k+1)\pi\varrho)\sin\frac{k\pi}{\alpha}}{k!\sin\frac{(k+1)\pi}{\alpha}} \frac{(-1)^{k-1}}{t^{1+k/\alpha}}$$
$$- \frac{(\sin\frac{\pi}{\alpha})^2}{\pi\sin(\pi\varrho)} \sum_{k=1}^{\infty} \frac{\Gamma(k-\frac{1}{\alpha})\sin(\pi\alpha\varrho k)}{\Gamma(\alpha k-1)\sin(\pi\alpha k)} \frac{1}{t^{k+1-1/\alpha}}.$$

Simmilar results

In 2015 T. Grzywny and M. Ryznar proved that:

$$\mathbf{P}(\tau_x > t) \asymp 1 \wedge \frac{1}{t\Psi^{-1}(\frac{1}{t})|x|\Psi(\frac{1}{|x|})}$$

under weaker assumptions. They assume that:

(a) X_t is a symmetric Lévy process whose Lévy measure has a decreasing density,

(b) $\Psi(x)$ satisfies weak lower scaling condition (WLSC) for constants $\alpha > 1$ and $C \in (0, 1]$, that is:

$$\frac{\Psi(x_2)}{\Psi(x_1)} \geq C \left(\frac{x_2}{x_1}\right)^{\alpha} \qquad (0 < x_1 < x_2)$$

Theorem 1

Suppose that (A) and (B) hold.

(a) For $t > 0$ and $x \in \mathbf{R} \setminus \{0\}$:

$$\mathbf{P}(\tau_x > t) \asymp 1 \wedge \frac{1}{t\Psi^{-1}(\frac{1}{t})|x|\Psi(\frac{1}{|x|})}.$$

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(b) For $t > 0$ and $x \in \mathbf{R} \setminus \{0\}$ such that $t\Psi(\frac{1}{x}) \geq C(\alpha)$:

$$p_x(t) \asymp \frac{1}{t^2\Psi^{-1}(\frac{1}{t})|x|\Psi(\frac{1}{|x|})}.$$

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(c) For $n \geq 0$, $t > 0$ and $x \in \mathbf{R} \setminus \{0\}$, if $t\Psi(\frac{1}{x}) \geq C(\alpha, n)$:

$$(-\tfrac{d}{dt})^n \mathbf{P}(\tau_x > t) \asymp \frac{1}{t^{n+1}\Psi^{-1}(\frac{1}{t})|x|\Psi(\frac{1}{|x|})}.$$

Theorem 2

Suppose that (A) and (B) hold.

(a) If Ψ is regularly varying at infinity with index $\gamma \in (1, 2]$, then as $x \to 0$:

$$(-\tfrac{d}{dt})^n \mathbf{P}(\tau_x > t) \sim \frac{C}{|x|\Psi(\frac{1}{|x|})}$$

for all $n \geq 0$ and $t > 0$.

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for all $n \geq 0$ and $t > 0$.

(b) If Ψ is regularly varying at zero with index $\delta \in (1,2]$, then as $t \to \infty$:

$$(-\tfrac{d}{dt})^n \mathbf{P}(\tau_x > t) \sim \frac{C}{t^{n+1}\Psi^{-1}(\frac{1}{t})}$$

for all $n \geq 0$ and $x \in \mathbf{R} \setminus \{0\}$.

The constants are given explicitly, by rather complicated expressions.

Example: relativistic process

Lévy–Khintchine exponent: $\Psi(y) = (1 + |y|^\beta)^{\alpha/\beta} - 1$, $1 < \alpha < \beta \leq 2$.

- Theorem 1(a) states that for all t, x:

$$\mathbf{P}(\tau_x > t) \approx 1 \wedge \frac{|x|^{\alpha-1}(1+|x|)^{\beta-\alpha}}{t^{1-1/\alpha}(1+t)^{1/\alpha-1/\beta}} .$$

- By Theorem 1(b), if $t > C|x|^\alpha(1+|x|)^{\beta-\alpha}$, then

$$p_x(t) \approx \frac{|x|^{\alpha-1}(1+|x|)^{\beta-\alpha}}{t^{2-1/\alpha}(1+t)^{1/\alpha-1/\beta}} .$$

- Theorem 2 asserts that:

$$\mathbf{P}(\tau_x > t) \overset{t\to\infty}{\sim} Ct^{1/\beta-1}, \qquad \mathbf{P}(\tau_x > t) \overset{x\to 0}{\sim} C|x|^{\alpha-1} ,$$

$$p_x(t) \overset{t\to\infty}{\sim} Ct^{1/\beta-2}, \qquad p_x(t) \overset{x\to 0}{\sim} C|x|^{\alpha-1} .$$

Generalised eigenfunction expansion [Kwaśnicki, 2012]

If (A) and (B) hold, then, for $t > 0$ and $x \in \mathbf{R}$:

$$\mathbf{P}(t < \tau_x) = \frac{1}{\pi} \int_0^\infty \cos \vartheta_\lambda e^{-t\Psi(\lambda)} \frac{\Psi'(\lambda)}{\Psi(\lambda)} F_\lambda(x) d\lambda.$$

Here F_λ is an eigenfunction of the generator of X in $\mathbf{R} \setminus \{0\}$ with Dirichlet boundary condition $F_\lambda(0) = 0$:

$$F_\lambda(x) = \sin(\lambda|x| + \vartheta_\lambda) - G_\lambda(x),$$

where

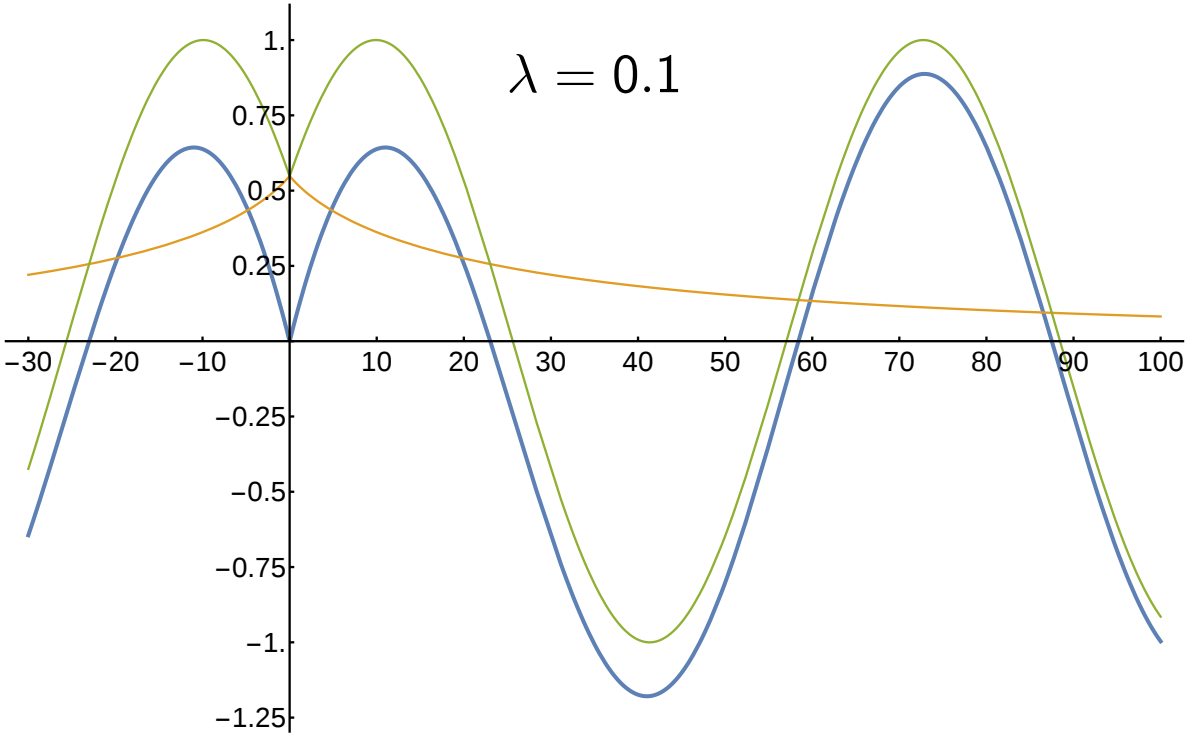
$$\vartheta_\lambda = \arctan\left(\frac{1}{\pi} \int_0^\infty \left(\frac{\Psi'(\lambda)}{\Psi(\xi) - \Psi(\lambda)} - \frac{2\lambda}{\xi^2 - \lambda^2}\right) d\xi\right)$$

and

$$G_\lambda(x) = \frac{\cos \vartheta_\lambda}{\pi} \int_0^\infty \left(\frac{\Psi'(\lambda)}{\Psi(\xi) - \Psi(\lambda)} - \frac{2\lambda}{\xi^2 - \lambda^2}\right) \cos(\xi x) d\xi.$$

Example: sum of two independent stable processes

Lévy–Khintchine exponent: $\Psi(x) = |x|^2 + |x|^{1.1}$.



Plots of $F_\lambda(x)$ (blue), $G_\lambda(x)$ (yellow) and $\sin(\lambda|x| + \vartheta_\lambda)$ (green)

Main estimate

With no loss of generality assume that $x > 0$. Write:

$$\mathbf{P}(\tau_x > t) = \frac{1}{\pi}\int_0^\infty \cos\vartheta_\lambda e^{-t\Psi(\lambda)}\frac{\Psi'(\lambda)}{\Psi(\lambda)}F_\lambda(x)d\lambda = I + J,$$

where:

$$I = \frac{1}{\pi}\int_{1/x}^\infty (\cdots)d\lambda, \qquad J = \frac{1}{\pi}\int_0^{1/x}(\cdots)d\lambda.$$

To estimate I, we use $|F_\lambda(x)| \le 2$ and $|\cos\vartheta_\lambda| \le 1$:

$$\begin{aligned}|I| &\le \frac{2}{\pi}\int_{1/x}^\infty e^{-t\Psi(\lambda)}\frac{\Psi'(\lambda)}{\Psi(\lambda)}d\lambda \\ &= \frac{2}{\pi}\int_{t\Psi(1/x)}^\infty \frac{e^{-s}}{s}ds.\end{aligned}$$

Main estimate

Estimates of J require estimates of F_λ: we show that if $\lambda x \leq \frac{\pi}{2} - \vartheta_\lambda$, then:

$$F_\lambda(x) \asymp \int_{\frac{1}{x}}^{\infty} \left(\frac{\Psi'(\lambda)}{\Psi(\xi) - \Psi(\lambda)} \right) d\xi \asymp \frac{\Psi'(\lambda)}{x\Psi(\frac{1}{x})}.$$

If we know that $\frac{\pi}{2} - \vartheta_\lambda > C$, then $\cos\vartheta_\lambda \asymp 1$ and so:

$$J \asymp \frac{1}{\pi} \int_0^{1/x} e^{-t\Psi(\lambda)} \frac{\Psi'(\lambda)}{\Psi(\lambda)} \frac{\Psi'(\lambda)}{x\Psi(\frac{1}{x})} d\lambda.$$

Since $\lambda\Psi'(\lambda) \asymp \Psi(\lambda)$ (by (B)):

$$\begin{aligned} J &\asymp \frac{1}{\pi} \int_0^{1/x} e^{-t\Psi(\lambda)} \frac{1}{\lambda} \frac{\Psi'(\lambda)}{x\Psi(\frac{1}{x})} d\lambda \\ &= \frac{1}{\pi} \frac{1}{x\Psi(\frac{1}{x})} \int_0^{t\Psi(1/x)} \frac{e^{-s}}{t\Psi^{-1}(s/t)} ds. \end{aligned}$$

Main estimate

It follows that if $t\Psi(\frac{1}{x})$ is large enough, then:

$$J \asymp \frac{1}{x\Psi(\frac{1}{x})} \frac{1}{t\Psi^{-1}(\frac{1}{t})}$$

and $|I| \leq \frac{1}{2}J$, which proves the main theorem.

Estimates of ϑ_λ

Recall that:

$$\vartheta_\lambda = \arctan\left(\frac{1}{\pi}\int_0^\infty \left(\frac{\Psi'(\lambda)}{\Psi(\xi)-\Psi(\lambda)} - \frac{2\lambda}{\xi^2-\lambda^2}\right) d\xi\right)$$

(B) implies that:

$$\frac{1}{\pi}\int_0^\infty \left(\frac{\Psi'(\lambda)}{\Psi(\xi)-\Psi(\lambda)} - \frac{2\lambda}{\xi^2-\lambda^2}\right) d\xi \geq$$
$$\geq \frac{2}{\pi}\int_0^\infty \left(\frac{\frac{\alpha}{2}\lambda^{\alpha-1}}{\xi^\alpha-\lambda^\alpha} - \frac{\lambda}{\xi^2-\lambda^2}\right) d\xi,$$

so that:

$$\vartheta_\lambda \leq \arctan\left(K_\lambda\right) = \frac{\pi}{\alpha} - \frac{\pi}{2}$$