Hitting times of points for symmetric Lévy processes with completely monotone jumps

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joint work with Mateusz Kwaśnicki
Let $X$ be a one-dimensional Lévy process with Lévy–Khintchine exponent $\Psi$:

$$Ee^{iyX_t} = e^{-t\Psi(y)}.$$ 

We assume that $X$ is symmetric, so that:

$$\Psi(y) = ay^2 + \int_{\mathbb{R}\setminus\{0\}} (1 - \cos(yz))\nu(dz),$$

where $a \geq 0$ is the Gaussian component and $\nu$ is the (symmetric) Lévy measure.
The first hitting time of a point $x \in \mathbb{R}$ is defined by the formula:

$$
\tau_x = \inf\{t \geq 0 : X_t = x\}.
$$

Under a number of assumptions, we find estimates for the

- $\mathbb{P}(\tau_x > t)$
- the density function of $\tau_x$, denoted by $p_x(t)$
- all time derivatives of $p_x(t)$

for small $x$ and large $t$.  

Basic example: Wiener process

Lévy–Khintchine exponent: \( \Psi(y) = |y|^2 \).

The answer is known explicitly: by the reflection principle:

\[
P(\tau_x \leq t) = 2P(X_t \geq x),
\]

and so

\[
P(\tau_x > t) = P(|X_t| < x).
\]

It is easy to see that:

\[
P(\tau_x > t) \approx 1 \land \frac{x}{\sqrt{t}} \quad \text{for all } t, x
\]

\[
p_x(t) \approx \frac{x}{t^{3/2}} \quad \text{when } t > Cx^2.
\]
Assumptions

(A) $X$ is a subordinated Brownian motion and the Lévy measure $\mu$ of subordinator is absolutely continuous, with a completely monotone density on $(0, \infty)$.

Recall that $f$ is completely monotone if $(-1)^n f^{(n)}(x) \geq 0$ for $n = 0, 1, 2, \ldots$ (equivalently: $f$ is the Laplace transform of a nonnegative measure).
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(A'') Lévy measure $\nu$ of $X$ is absolutely continuous with a completely monotone density on $(0, \infty)$ (in this case we say that $X$ has completely monotone jumps)
(B) $X$ satisfies a **scaling-type condition** of order $\alpha \in (1, 2]$:

$$\frac{x\Psi''(x)}{\Psi'(x)} \geq \alpha - 1 \quad (x > 0).$$

This condition is equivalent to:

$$\frac{\Psi'(x_2)}{\Psi'(x_1)} \geq \left( \frac{x_2}{x_1} \right)^{\alpha - 1} \quad (0 < x_1 < x_2)$$

and it implies that:

$$\frac{\Psi(x_2)}{\Psi(x_1)} \geq \left( \frac{x_2}{x_1} \right)^\alpha \quad (0 < x_1 < x_2),$$

which motivates the name scaling-type condition.
Recall that $\Psi$ is regularly varying at 0 with index $\gamma$ if:

$$\lim_{x \to 0^+} \frac{\Psi(kx)}{\Psi(x)} = k^\gamma.$$  

Similarly we say that $\Psi$ is regularly varying at $\infty$ with index $\delta$ if:

$$\lim_{x \to \infty} \frac{\Psi(kx)}{\Psi(x)} = k^\delta.$$  

Our results are expressed in terms of $\Psi$ and its partial inverse $\Psi^{-1}$. Conditions (A), (B) give estimates for $P(\tau_x > t)$ and its time derivatives. Additional assumption of regular variation give asymptotic behaviour for small $x$ and large $t$. 
Assumptions (A) and (B) are satisfied by the following processes, provided that $1 < \alpha < \beta \leq 2$:

- **Symmetric stable processes**: $\Psi(y) = y^\alpha$
- **Sums of independent symmetric stable processes**: $\Psi(y) = y^\alpha + y^\beta$
- **Relativistic Lévy processes**: $\Psi(y) = \left(1 + |y|^\beta\right)^{\alpha/\beta} - 1$
- **Process with Lévy–Khintchine exponent**: $\Psi(y) = \frac{y^2}{\log(1+y^2)} - 1$
Spectrally negative processes

For Lévy processes with negative jumps only:

- For $x > 0$, $\tau_x$ is equal to the first passage time: $\tau_x = \inf \{ t \geq 0 : X_t \geq x \}$.
- Fluctuation theory of Lévy processes provides a lot of information about $\tau_x$.
- Standard references for the fluctuation theory are [Bertoin, 1996], [Kyprianou, 2006], [Sato, 1999].
\(\alpha\)-stable Lévy processes

For (also non-symmetric) \(\alpha\)-stable Lévy processes:

- \(\mathbb{P}(\tau_x > t) = \mathbb{P}(\tau_{1 > x^{-\alpha}t})\) for \(x > 0\),
  \(\mathbb{P}(\tau_x > t) = \mathbb{P}(\tau_{-1 > (-x)^{-\alpha}t})\) for \(x < 0\).


- For most irrational \(\alpha\), with \(\varrho = \mathbb{P}(X_t > 0)\):

\[
p_1(t) = \sum_{k=1}^{\infty} \frac{\sin \left( \frac{\pi}{\alpha} \right)}{\pi \sin(\pi \varrho)} \frac{\Gamma \left( \frac{k}{\alpha} + 1 \right) \sin((k + 1)\pi \varrho) \sin \left( \frac{k\pi}{\alpha} \right)}{k! \sin \left( \frac{(k+1)\pi}{\alpha} \right)} (-1)^{k-1} t^{1+k/\alpha} - \left( \sin \left( \frac{\pi}{\alpha} \right) \right)^2 \sum_{k=1}^{\infty} \frac{\Gamma \left( k - \frac{1}{\alpha} \right) \sin(\pi \varrho \alpha k)}{\Gamma(\alpha k - 1) \sin(\pi \alpha k)} \frac{1}{t^{k+1-1/\alpha}}.
\]
In 2015 T. Grzywny and M. Ryznar proved that:

\[ P(\tau_x > t) \asymp 1 \wedge \frac{1}{t\psi^{-1}(\frac{1}{t})|x|\psi(\frac{1}{|x|})} \]

under weaker assumptions. They assume that:

(a) \( X_t \) is a symmetric Lévy process whose Lévy measure has a decreasing density,

(b) \( \Psi(x) \) satisfies weak lower scaling condition (WLSC) for constants \( \alpha > 1 \) and \( C \in (0, 1] \), that is:

\[ \frac{\Psi(x_2)}{\Psi(x_1)} \geq C \left( \frac{x_2}{x_1} \right)^\alpha \quad (0 < x_1 < x_2) \]
Theorem 1

Suppose that (A) and (B) hold.

(a) For $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$:

$$P(\tau_x > t) \asymp 1 \wedge \frac{1}{t\psi^{-1}(\frac{1}{t})|x|\psi(\frac{1}{|x|})}.$$
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\]

(b) For $t > 0$ and $x \in \mathbb{R} \setminus \{0\}$ such that $t \psi(\frac{1}{x}) \geq C(\alpha)$:
\[
p_x(t) \asymp \frac{1}{t^2 \psi^{-1}(\frac{1}{t}) \frac{1}{x} \psi(\frac{1}{|x|})}.
\]
Theorem 1

Suppose that (A) and (B) hold.

(a) For \( t > 0 \) and \( x \in \mathbb{R} \setminus \{0\} \):

\[
P(\tau_x > t) \cong 1 \wedge \frac{1}{t \psi^{-1}(\frac{1}{t})|x| \psi(\frac{1}{|x|})}.
\]

(b) For \( t > 0 \) and \( x \in \mathbb{R} \setminus \{0\} \) such that \( t \psi(\frac{1}{x}) \geq C(\alpha) \):

\[
p_x(t) \cong \frac{1}{t^2 \psi^{-1}(\frac{1}{t})|x| \psi(\frac{1}{|x|})}.
\]

(c) For \( n \geq 0 \), \( t > 0 \) and \( x \in \mathbb{R} \setminus \{0\} \), if \( t \psi(\frac{1}{x}) \geq C(\alpha, n) \):

\[
(-\frac{d}{dt})^n P(\tau_x > t) \cong \frac{1}{t^{n+1} \psi^{-1}(\frac{1}{t})|x| \psi(\frac{1}{|x|})}.
\]
Theorem 2

Suppose that (A) and (B) hold.

(a) If $\Psi$ is regularly varying at infinity with index $\gamma \in (1, 2]$, then as $x \to 0$:

$$(-\frac{d}{dt})^nP(\tau_x > t) \sim \frac{C}{|x|\psi\left(\frac{1}{|x|}\right)}$$

for all $n \geq 0$ and $t > 0$. 
Theorem 2

Suppose that (A) and (B) hold.

(a) If $\Psi$ is regularly varying at infinity with index $\gamma \in (1, 2]$, then as $x \to 0$:

$$(-\frac{d}{dt})^n P(\tau_x > t) \sim \frac{C}{|x| \Psi\left(\frac{1}{|x|}\right)}$$

for all $n \geq 0$ and $t > 0$.

(b) If $\Psi$ is regularly varying at zero with index $\delta \in (1, 2]$, then as $t \to \infty$:

$$(-\frac{d}{dt})^n P(\tau_x > t) \sim \frac{C}{t^{n+1} \Psi^{-1}\left(\frac{1}{t}\right)}$$

for all $n \geq 0$ and $x \in \mathbb{R} \setminus \{0\}$.

The constants are given explicitly, by rather complicated expressions.
Lévy–Khintchine exponent: $\Psi(y) = (1 + |y|^\beta)^{\alpha/\beta} - 1$, $1 < \alpha < \beta \leq 2$.

- Theorem 1(a) states that for all $t, x$:

$$
P(\tau_x > t) \approx 1 \wedge \frac{|x|^\alpha (1 + |x|)^{\beta - \alpha}}{t^{1-1/\alpha}(1 + t)^{1/\alpha - 1/\beta}}.
$$

- By Theorem 1(b), if $t > C|x|^\alpha (1 + |x|)^{\beta - \alpha}$, then

$$
p_x(t) \approx \frac{|x|^\alpha (1 + |x|)^{\beta - \alpha}}{t^{2-1/\alpha}(1 + t)^{1/\alpha - 1/\beta}}.
$$

- Theorem 2 asserts that:

$$
P(\tau_x > t) \xrightarrow{t \to \infty} Ct^{1/\beta - 1}, \quad P(\tau_x > t) \xrightarrow{x \to 0} C|x|^\alpha - 1,
$$

$$
p_x(t) \xrightarrow{t \to \infty} Ct^{1/\beta - 2}, \quad p_x(t) \xrightarrow{x \to 0} C|x|^\alpha - 1.
$$
Generalised eigenfunction expansion [Kwaśnicki, 2012]

If (A) and (B) hold, then, for \( t > 0 \) and \( x \in \mathbb{R} \):

\[
P(t < \tau_x) = \frac{1}{\pi} \int_{0}^{\infty} \cos \vartheta_{\lambda} e^{-t\psi(\lambda)} \frac{\psi'(\lambda)}{\psi(\lambda)} F_{\lambda}(x) d\lambda.
\]

Here \( F_{\lambda} \) is an eigenfunction of the generator of \( X \) in \( \mathbb{R} \setminus \{0\} \) with Dirichlet boundary condition \( F_{\lambda}(0) = 0 \):

\[
F_{\lambda}(x) = \sin(\lambda|x| + \vartheta_{\lambda}) - G_{\lambda}(x),
\]

where

\[
\vartheta_{\lambda} = \arctan \left( \frac{1}{\pi} \int_{0}^{\infty} \left( \frac{\psi'(\lambda)}{\psi(\xi) - \psi(\lambda)} - \frac{2\lambda}{\xi^2 - \lambda^2} \right) d\xi \right)
\]

and

\[
G_{\lambda}(x) = \frac{\cos \vartheta_{\lambda}}{\pi} \int_{0}^{\infty} \left( \frac{\psi'(\lambda)}{\psi(\xi) - \psi(\lambda)} - \frac{2\lambda}{\xi^2 - \lambda^2} \right) \cos(\xi x) d\xi.
\]
Example: sum of two independent stable processes

Lévy–Khintchine exponent: $\Psi(x) = |x|^2 + |x|^{1.1}$.

$\lambda = 0.1$

Plots of $F_\lambda(x)$ (blue), $G_\lambda(x)$ (yellow) and $\sin(\lambda|x| + \vartheta_\lambda)$ (green)
Main estimate

With no loss of generality assume that \( x > 0 \). Write:

\[
\mathbb{P}(\tau_x > t) = \frac{1}{\pi} \int_0^\infty \cos \vartheta_\lambda e^{-t\psi(\lambda)} \frac{\psi'(\lambda)}{\psi(\lambda)} F_\lambda(x) d\lambda = I + J,
\]

where:

\[
I = \frac{1}{\pi} \int_{1/x}^\infty (\cdots) d\lambda, \quad J = \frac{1}{\pi} \int_0^{1/x} (\cdots) d\lambda.
\]

To estimate \( I \), we use \( |F_\lambda(x)| \leq 2 \) and \( |\cos \vartheta_\lambda| \leq 1 \):

\[
|I| \leq \frac{2}{\pi} \int_{1/x}^\infty e^{-t\psi(\lambda)} \frac{\psi'(\lambda)}{\psi(\lambda)} d\lambda
\]

\[
= \frac{2}{\pi} \int_{t\psi(1/x)}^\infty \frac{e^{-s}}{s} ds.
\]
Main estimate

Estimates of $J$ require estimates of $F_\lambda$: we show that if $\lambda x \leq \frac{\pi}{2} - \vartheta_\lambda$, then:

$$F_\lambda(x) \lesssim \int_{\frac{1}{x}}^{\infty} \left( \frac{\psi'(\lambda)}{\psi(\xi) - \psi(\lambda)} \right) d\xi \lesssim \frac{\psi'(\lambda)}{x \psi(\frac{1}{x})}.$$  

If we know that $\frac{\pi}{2} - \vartheta_\lambda > C$, then $\cos \vartheta_\lambda \approx 1$ and so:

$$J \approx \frac{1}{\pi} \int_0^{1/x} e^{-t \psi(\lambda)} \frac{\psi'(\lambda)}{\psi(\lambda)} \frac{\psi'(\lambda)}{x \psi(\frac{1}{x})} d\lambda.$$  

Since $\lambda \psi'(\lambda) \approx \psi(\lambda)$ (by (B)):

$$J \approx \frac{1}{\pi} \int_0^{1/x} e^{-t \psi(\lambda)} \frac{1}{\lambda} \frac{\psi'(\lambda)}{x \psi(\frac{1}{x})} d\lambda$$  

$$= \frac{1}{\pi} \frac{1}{x \psi\left(\frac{1}{x}\right)} \int_0^{t \psi(1/x)} \frac{e^{-s}}{t \psi^{-1}\left(s/t\right)} ds.$$
Main estimate

It follows that if $t\Psi\left(\frac{1}{x}\right)$ is large enough, then:

$$J \lesssim \frac{1}{x\Psi\left(\frac{1}{x}\right)} \frac{1}{t\Psi^{-1}\left(\frac{1}{t}\right)}$$

and $|I| \leq \frac{1}{2} J$, which proves the main theorem.
Estimates of $\vartheta_\lambda$

Recall that:

$$\vartheta_\lambda = \arctan \left( \frac{1}{\pi} \int_0^\infty \left( \frac{\psi'(\lambda)}{\psi(\xi) - \psi(\lambda)} - \frac{2\lambda}{\xi^2 - \lambda^2} \right) d\xi \right)$$

(B) implies that:

$$\frac{1}{\pi} \int_0^\infty \left( \frac{\psi'(\lambda)}{\psi(\xi) - \psi(\lambda)} - \frac{2\lambda}{\xi^2 - \lambda^2} \right) d\xi \geq \frac{2}{\pi} \int_0^\infty \left( \frac{\alpha \lambda^{\alpha-1}}{\xi^\alpha - \lambda^\alpha} - \frac{\lambda}{\xi^2 - \lambda^2} \right) d\xi,$$

so that:

$$\vartheta_\lambda \leq \arctan (K_\lambda) = \frac{\pi}{\alpha} - \frac{\pi}{2}.$$