Sudakov Minoration

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- In this setting $\mathbf{E}\langle t, X\rangle^2 = \|t\|_2^2$ for all $t \in \mathbb{R}^d$.
- The main question: for a given norm $\|\cdot\|$ on $\mathbb{R}^d$, how to estimate $\mathbf{E}\|X\|$?
- More generally: for each $T \subset \mathbb{R}^d$ obtain bounds for

$$\mathbf{E}\sup_{t\in T}\langle t, X\rangle = \mathbf{E}\sup_{t\in T} X_t,$$

where $X_t = \langle t, X\rangle$.

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- Is it possible to reverse this estimate?

Log concave distribution.

- Vector X has a log concave distribution μ_X if for any non-empty compact sets $A, B \subset \mathbb{R}^d$ and $\alpha + \beta = 1$, $\alpha, \beta \geqslant 0$

$$\mu_X(\alpha A + \beta B) \geqslant \mu_X(A)^\alpha \mu_X(B)^\beta.$$

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- Vector X is unconditional if X and $(\varepsilon_1 X_1, \ldots, \varepsilon_d X_d)$, where ε_i are independent random signs $\mathbf{P}(\varepsilon_i = \pm 1) = \frac{1}{2}$.

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- Vector X is unconditional if X and $(\varepsilon_1 X_1, \ldots, \varepsilon_d X_d)$, where ε_i are independent random signs $\mathbf{P}(\varepsilon_i = \pm 1) = \frac{1}{2}$.
- It is known that $\|X_t\|_p \leqslant \frac{p}{q}\|X_t\|_q$ for all $1 \leqslant q \leqslant p$.

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- If X_i iid $\mathcal{U}(-\sqrt{3}, \sqrt{3})$ then $\|X_t\|_p \sim \sum_{i=1}^p |t_i^*| + \sqrt{p}(\sum_{i>p} |t_i^*|^2)^{\frac{1}{2}}$, where $|t_i^*| \geqslant |t_{i+1}^*|$.

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- Does it imply that

$$\mathbf{E}\sup_{t\in T} X_t = \mathbf{E}\sup_{t\in T}\langle t, X\rangle \geqslant K^{-1}A,$$

where K is an absolute constant?

Motivation.

- Dimension free estimate for $\mathbf{E}\sup_{t\in T} X_t$ in a particular case where $|T| \sim e^p$, $0 \in T$ and $\|X_t - X_s\|_p \sim A$ for all $s, t \in T$, $s \neq t$.

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- First step in order to establish dimension free estimates for $\mathbf{E}\|X\|$ by the generic chaining approach.
- Concentration inequalities of the type

$$\mathbf{P}(\|X\| \geqslant K(\mathbf{E}\|X\| + \sup_{\|x^*\|\leqslant 1} \|\langle x^*, X\rangle\|_p)) \leqslant e^{-p}.$$

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- Paouris type estimates

$$(\mathbf{E}\|X\|^p)^{\frac{1}{p}} \leqslant K(\mathbf{E}\|X\| + \sup_{\|x^*\|\leqslant 1} \|\langle x^*, X\rangle\|_p).$$

Gaussian case

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- Recall that $\|X_t\|_p \sim \sqrt{p}\|t\|_2$ and hence
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$$\|X_t - X_s\|_p \sim \sqrt{p}\|t-s\|_2 \geqslant A, \quad \text{then} \quad a = \frac{A}{\sqrt{p}}.$$
- Therefore for $|T| = e^p$
$$\mathbf{E} \sup_{t \in T} X_t \geqslant K^{-1} \frac{A}{\sqrt{p}} \sqrt{p} = K^{-1} A.$$

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- Let $N(T, D(a))$ denotes the smallest number of shifts of the set $D(a)$ that covers T.
- Then $b(T) \geqslant K^{-1} a \sqrt{\ln N(T, D(a))}$.
- If $\|X_t - X_s\|_p \geqslant A$, then $t - s \notin AB_1 + \frac{A}{\sqrt{p}} B_2$ and hence either $b(T) \geqslant K^{-1}A$ or T is covered by at least e^p shifts of $D(\frac{A}{\sqrt{p}})$ which means

$$b(T) \geqslant K^{-1} \frac{A}{\sqrt{p}} \sqrt{p} = K^{-1} A.$$

Symmetric exponentials

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- The question can be reduced to the following one: suppose $t_i \in \{0, k_i\}$, $k_i \geqslant 1$ for all $i \in \{1, 2, \ldots, d\}$ then

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- This fact was established by Talagrand and generalized by Latala and then by Latala and Tkocz.

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- (Sufficient separation) For each $s, t \in T$, $s \neq t$

$$\|X_t - X_s\|_p = \| \sum_{i \in I(t) \setminus I(s)} k_i X_i - \sum_{i \in I(s) \setminus I(t)} k_i X_i \| \geqslant p = A.$$

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- Does it imply that $\mathbf{E} \sup_{t \in T} X_t \geqslant K^{-1} p = K^{-1} A$?

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- The condition is verified when supports are disjoint or intersects in few coordinates.

Exponential inequality

- For log concave measures (unconditional) the following inequality holds

$$\mathbf{P}(X \in A + \alpha(\sqrt{u}B_2 + uB_1)) \geqslant 1 - e^{-u}, \quad \text{for } u > 0,$$

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$$\mathbf{P}(\sup_{t \in T} \sup_{s \in J(t)} |\sum_{i \in I(t) \setminus I(s)} k_i X_i| \leqslant K^{-1} p) \geqslant \frac{1}{2}.$$

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- Therefore $\mathbf{P}(X \in A) \geqslant \frac{1}{2}$ for $A = \{x \in \mathbb{R}^d : \sup_{t\in T} \sup_{s\in J(t)} |\sum_{i\in I(t)\setminus I(s)} k_i x_i| \leqslant K^{-1}p\}$.

Set of non-negligible measure

- Recall that $|T| \geqslant e^{Cp}$, C-large. Clearly

$$e^{(C-1)p} \leqslant e^{-p}|T| \leqslant \sum_{t \in T} \mathbf{P}\Big(\bigcap_{i \in I(t)} \{X_i \geqslant a_i(t)\}\Big) = \mathbf{E}N,$$

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- Consequently $\mathbf{P}(X \in B) \geqslant \frac{1}{2}e^{-p}$ for $B = \{y \in \mathbb{R}^d : \ \exists S \subset T,\ |S| \geqslant \frac{1}{2}e^{(C-1)p},\ y_i \geqslant a_i(t)\ \forall i \in I(t), t \in S\}$.

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- Consequently if $\mathbf{P}(X \in A) \geqslant \frac{1}{2}$

$$\frac{1}{2} e^{-p} \leqslant \mathbf{P}(X \in B) \leqslant \mathbf{P}(X \notin A + C(\sqrt{p} B_2 + p B_1)) \leqslant e^{-2p}.$$

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- The contradiction implies that $\mathbf{P}(X \in A) \leqslant \frac{1}{2}$ and hence the minoration holds.

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$$\mathbf{E} \sup_{t \in T} X_t \geqslant K^{-1} p = K^{-1} A.$$

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- There is no chance to remove the common witness assumption from the argument described above.
- Still there is a possibility to strengthen the induction argument which is the core of the main Latala's approach to the Sudakov minoration for canonical processes.

Thank you for your attention