Sudakov Minoration

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Random vectors.

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- Usually \( X \) is in the isotropic position, i.e. \( \mathbb{E}X_i = 0 \) and \( \mathbb{E}X_i X_j = 0 \) if \( i \neq j \) and \( \mathbb{E}X_i^2 = 1 \).
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The main question: for a given norm $\| \cdot \|$ on $\mathbb{R}^d$, how to estimate $\mathbf{E}\|X\|$?

More generally: for each $T \subset \mathbb{R}^d$ obtain bounds for

$$\mathbf{E}\sup_{t \in T} \langle t, X \rangle = \mathbf{E}\sup_{t \in T} X_t,$$

where $X_t = \langle t, X \rangle$. 

Random vectors.
Upper bound.

- Suppose that $T \subset \mathbb{R}^d$, $T$-finite and $|T| \leq e^p$, where $p \geq 1$. 

Is it possible to reverse this estimate?
Suppose that $T \subset \mathbb{R}^d$, $T$-finite and $|T| \leq e^p$, where $p \geq 1$.

Let $\|X_t\|_p = (\mathbb{E}|X_t|^p)^{\frac{1}{p}} = (\mathbb{E}|\langle t, X \rangle|^p)^{\frac{1}{p}} \leq A$, for all $t \in T$. 

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- Then

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E \sup_{t \in T} X_t \leq E \sup_{t \in T} |X_t| = E (\sup_{t \in T} |X_t|^p)^{\frac{1}{p}} \leq (E \sum_{t \in T} |X_t|^p)^{\frac{1}{p}} \leq (e^p A^p)^{\frac{1}{p}} = eA.
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Log concave distribution.

Vector $X$ has a log concave distribution $\mu_X$ if for any non-empty compact sets $A, B \subset \mathbb{R}^d$ and $\alpha + \beta = 1$, $\alpha, \beta \geq 0$

$$\mu_X(\alpha A + \beta B) \geq \mu_X(A)^\alpha \mu_X(B)^\beta.$$
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- If the support of $\mu_X$ is $\mathbb{R}^d$ then there exists density $f_X$ of $\mu_X$ such that $f_X = \exp(-U_X)$, where $U_X : \mathbb{R}^d \to \mathbb{R}$ is convex.
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For log concave $X$, all $t \in \mathbb{R}^d$ and $p \geq 1$ we have

$$\|X_t\|_p = \|\langle t, X \rangle\|_p < \infty.$$
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It is known that $\|X_t\|_p \leq \frac{p}{q} \|X_t\|_q$ for all $1 \leq q \leq p$. 

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How to compute the norms?

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\|X_t\|_p = \sup \left\{ \sum_{i \in l(t)} a_i t_i : \mathbb{P} \left( \bigcap_{i \in l(t)} \{ X_i \leq a_i \} \right) \geq e^{-p} \right\}.
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- In particular for $X_i$ iid $\mathcal{N}(0, 1)$ then $\|X_t\|_p \sim \sqrt{p}\|t\|_2$.
- If $X_i$ iid symmetric $\mathbb{P}(|X_i| > t) = C_\alpha \exp(-|t|^\alpha)$, $1 \leq \alpha \leq 2$,
  $\|X_t\|_p \sim \sqrt{p}\|t\|_2 + p^{\frac{1}{\alpha}}\|t\|_\beta$, where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. 
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- If $X_i$ iid $\mathcal{U}(\sqrt{3}, -\sqrt{3})$ then

$$
\|X_t\|_p \sim \sum_{i=1}^{p} |t_i^*| + \sqrt{p}(\sum_{i>p} |t_i^*|^2)^{\frac{1}{2}}, \text{ where } |t_i^*| \geq |t_{i+1}^*|.
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- Suppose that for each $s, t \in T$, $s \neq t$

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\|X_t - X_s\|_p = \left( \mathbb{E}|X_t - X_s|^p \right)^{\frac{1}{p}} = \left( \mathbb{E}|\langle t - s, X \rangle|^p \right)^{\frac{1}{p}} \geq A.
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- Does it imply that

$$\mathbb{E}\sup_{t \in T} X_t = \mathbb{E}\sup_{t \in T} \langle t, X \rangle \geq K^{-1}A,$$

where $K$ is an absolute constant?
Motivation.

- Dimension free estimate for $\mathbb{E} \sup_{t \in T} X_t$ in a particular case where $|T| \sim e^p$, $0 \in T$ and $\|X_t - X_s\|_p \sim A$ for all $s, t \in T$, $s \neq t$. 
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- First step in order to establish dimension free estimates for $\mathbb{E}\|X\|$ by the generic chaining approach.
- Concentration inequalities of the type

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P(\|X\| \geq K(\mathbb{E}\|X\| + \sup_{\|x^*\| \leq 1} \|\langle x^*, X \rangle\|_p)) \leq e^{-p}.$$
Motivation.

- Dimension free estimate for $E \sup_{t \in T} X_t$ in a particular case where $|T| \sim e^p$, $0 \in T$ and $\|X_t - X_s\|_p \sim A$ for all $s, t \in T, s \neq t$.
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- Paouris type estimates

$$(E\|X\|^p)^{\frac{1}{p}} \leq K(E\|X\| + \sup_{\|x^*\| \leq 1} \|\langle x^*, X \rangle\|_p).$$
Gaussian case

Let $X_i$ be iid $\mathcal{N}(0, 1)$, $|T| \geq e^p$, $\|X_t - X_s\|_p \geq A$. 
Gaussian case

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- Sudakov minoration: if $\|t - s\|_2 \geq a$ for all $s, t \in T, s \neq t$ then

$$\mathbf{E} \sup_{t \in T} X_t \geq K^{-1} a \sqrt{\ln |T|}.$$
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Recall that $\|X_t\|_p \sim \sqrt{p} \|t\|_2$ and hence

$$\|X_t - X_s\|_p \sim \sqrt{p} \|t - s\|_2 \geq A, \text{ then } a = \frac{A}{\sqrt{p}}.$$
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- Recall that $\|X_t\|_p \sim \sqrt{p} \|t\|_2$ and hence
  $$\|X_t - X_s\|_p \sim \sqrt{p} \|t - s\|_2 \geq A,$$ then $a = \frac{A}{\sqrt{p}}$.
- Therefore for $|T| = e^p$
  $$\mathbb{E} \sup_{t \in T} X_t \geq K^{-1} \frac{A}{\sqrt{p}} \sqrt{p} = K^{-1} A.$$
Bernoulli case

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Talagrand's minoration: let $b(T) = \mathbb{E} \sup_{t \in T} X_t$ and $D(a) = b(T)B_1 + aB_2$, $B_p = \{x \in \mathbb{R}^d : \sum_{i=1}^d |x_i|^p \leq 1\}$. 
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- Let \( N(T, D(a)) \) denotes the smallest number of shifts of the set \( D(a) \) that covers \( T \).
- Then \( b(T) \geq K^{-1} a \sqrt{\ln N(T, D(a))} \).
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- Let $N(T, D(a))$ denotes the smallest number of shifts of the set $D(a)$ that covers $T$.
- Then $b(T) \geq K^{-1} a \sqrt{\ln N(T, D(a))}$.
- If $\|X_t - X_s\|_p \geq A$, then $t - s \notin AB_1 + \frac{A}{\sqrt{p}} B_2$ and hence either $b(T) \geq K^{-1} A$ or $T$ is covered by at least $e^p$ shifts of $D(\frac{A}{\sqrt{p}})$ which means

$$b(T) \geq K^{-1} \frac{A}{\sqrt{p}} \sqrt{p} = K^{-1} A.$$
Symmetric exponentials

Let $X_i$ be iid, symmetric $\mathbf{P}(|X_i| \geq x) = e^{-x}$, $|T| \geq e^p$, $\|X_t - X_s\|_p \geq A$ for all $s \neq t$. 

Recall that $\|X_t - X_s\|_p \sim \sqrt{p} \|t - s\|_2 + p \|s - t\|_\infty \geq A$. 

Sudakov minoration: $E\sup_{t \in T} X_t \geq K^{-1}A$?

This fact was established by Talagrand and generalized by Latala and then by Latala and Tkocz.
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- Sudakov minoration: $E \sup_{t \in T} X_t \geq K^{-1}A$?
- The question can be reduced to the following one: suppose $t_i \in \{0, k_i\}$, $k_i \geq 1$ for all $i \in \{1, 2, \ldots, d\}$ then
  \[
  E \sup_{t \in T} X_t = E \sup_{t \in T} \sum_{i \in I(t)} k_i X_i \geq K^{-1} \ln |T|.
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► Let $X_i$ be iid, symmetric $\mathbf{P}(|X_i| \geq x) = e^{-x}$, $|T| \geq e^p$, $\|X_t - X_s\|_p \geq A$ for all $s \neq t$.

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The basic simplification

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- (Short supports) For each $t$ the support satisfies $|I(t)| \leq \delta p$, where $\delta$ is sufficiently small. In fact $\sum_{i \in I(t)} k_i \leq \delta p$.
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- (Short supports) For each $t$ the support satisfies $|I(t)| \leq \delta p$, where $\delta$ is sufficiently small. In fact $\sum_{i \in I(t)} k_i \leq \delta p$.
- (Sufficient separation) For each $s, t \in T$, $s \neq t$

$$\|X_t - X_s\|_p = \left\| \sum_{i \in I(t) \setminus I(s)} k_i X_i - \sum_{i \in I(s) \setminus I(t)} k_i X_i \right\|_p \geq p = A.$$
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- (Cube-like) Let $T$ consists of $t$ that satisfies $t_i \in \{0, k_i\}$, where $k_i \geq 1$.
- (Short supports) For each $t$ the support satisfies $|l(t)| \leq \delta p$, where $\delta$ is sufficiently small. In fact $\sum_{i \in l(t)} k_i \leq \delta p$
- (Sufficient separation) For each $s, t \in T$, $s \neq t$

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- Does it imply that $\mathbb{E} \sup_{t \in T} X_t \geq K^{-1} p = K^{-1} A$?
The idea of common witness

- Let $J(t)$ consists of points $s \in T$ such that

$$\| \sum_{i \in l(t) \setminus l(s)} k_i X_i \|_p \geq \frac{p}{2}.$$
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  \[ \| \sum_{i \in I(t) \setminus I(s)} k_i X_i \|_p \geq \frac{p}{2}. \]

- Suppose that for each $t \in T$ one can select $a_i(t) \geq 1$, $i \in I(t)$ such that
  \[ \sum_{i \in I(t) \setminus I(s)} k_i a_i(t) \geq C^{-1} p \text{ for all } s \in J(t). \]
The idea of common witness

- Let $J(t)$ consists of points $s \in T$ such that
  \[ \| \sum_{i \in I(t) \setminus l(s)} k_i X_i \|_p \geq \frac{p}{2}. \]

- Suppose that for each $t \in T$ one can select $a_i(t) \geq 1$, $i \in I(t)$ such that
  \[ \sum_{i \in I(t) \setminus l(s)} k_i a_i(t) \geq C^{-1} p \text{ for all } s \in J(t) \]

- and
  \[ P \left( \bigcap_{i \in I(t)} \{ X_i \geq a_i(t) \} \right) \geq e^{-p}. \]
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- The condition is verified when supports are disjoint or intersects in few coordinates.
Exponential inequality

For log concave measures (unconditional) the following inequality holds

\[ P(X \in A + \alpha(\sqrt{u}B_2 + uB_1)) \geq 1 - e^{-u}, \quad \text{for } u > 0, \]

where \( P(X \in A) \geq \frac{1}{2}, \) \( \alpha \)-constant (best result \( \alpha \sim \log d \)).
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- Assume that

\[ P(\sup_{t \in T} \sup_{s \in J(t)} | \sum_{i \in I(t) \setminus I(s)} k_i X_i | \leq K^{-1} p) \geq \frac{1}{2}. \]

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- Therefore \( P(X \in A) \geq \frac{1}{2} \) for

\( A = \{ x \in \mathbb{R}^d : \sup_{t \in T} \sup_{s \in J(t)} \left| \sum_{i \in I(t) \setminus I(s)} k_i x_i \right| \leq K^{-1} p \}. \)
Recall that $|T| \geq e^{Cp}$, $C$-large. Clearly

$$e^{(C-1)p} \leq e^{-p} |T| \leq \sum_{t \in T} P\left( \bigcap_{i \in I(t)} \{X_i \geq a_i(t)\} \right) = EN,$$

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Consequently $P(X \in B) \geq \frac{1}{2} e^{-p}$ for $B = \{ y \in \mathbb{R}^d : \exists S \subset T, |S| \geq \frac{1}{2} e^{(C-1)p}, y_i \geq a_i(t) \ \forall i \in I(t), t \in S \}$. 
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- Consider points $x \in A$ and $y \in B$. 
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- The contradiction implies that $P(X \in A) \leq \frac{1}{2}$ and hence the minoration holds.
Further thoughts

- This argument shows

\[ \mathbb{E} \sup_{t \in T} X_t \geq K - 1_p = K - 1_A. \]

There is no chance to remove the common witness assumption from the argument described above.

Still there is a possibility to strengthen the induction argument which is the core of the main Latala's approach to the Sudakov minoration for canonical processes.
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**Theorem**

If in the simplified setting the common witness exists for each \( t \in T \) and the exponential inequality holds with dimension free \( \alpha \) then

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Thank you for your attention