Dimension Theory of self-affine and almost self-affine sets and measures

Károly Simon

Department of Stochastics
Institute of Mathematics
Budapest University of Technology and Economics
www.math.bme.hu/~simonk
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History I

The research related to transversality condition is a continuation of the following results:

- Marstrand Projection Theorem: Given a set $A \subset \mathbb{R}^2$ Borel set. Let $\Pi^{\alpha}(A)$ its pojection to the line of angle α. Then for lebesgue almost all α:
  (a) $\dim_{\mathrm{H}}(\Pi^{\alpha}(A)) = \min\{1, \dim_{\mathrm{H}}(A)\}$.
  (b) $\mathcal{L}\mathrm{eb}(\Pi^{\alpha}(A)) > 0$ if $\dim_{\mathrm{H}}(A) > 1$.
  Matilla generalized it to higher dimension
- Falconer papers on the dimension of "typical"-self similar and self-affine sets.

M. Keane's " $\{0,1,3\}$ " problem:

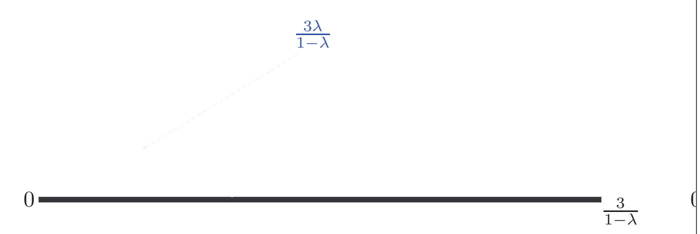
For every $\lambda \in (\frac{1}{4}, \frac{2}{5})$ consider the following self-similar set:

$$\Lambda_\lambda := \left\{ \sum_{i=0}^{\infty} a_i \lambda^i : a_i \in \{0,1,3\} \right\}.$$

Then Λ_λ is the attractor of the one-parameter (λ) family IFS:

$$\left\{ S_i^\lambda(x) := \lambda \cdot x + i \right\}_{i=0,1,3}$$

$\{0,1,3\}$ problem II.

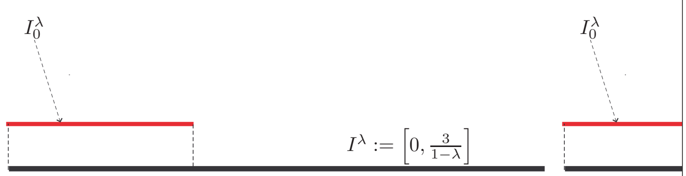


$\Pi_\lambda : \{0,1,3\}^{\mathbb{N}} \mapsto \Lambda_\lambda$

Let $k \in \mathbb{N}$ and $\mathbf{i} = (i_0, i_1, \dots) \in \underbrace{\{0,1,3\}^{\mathbb{N}}}_{\Sigma}$.

$I^\lambda_{i_0,\dots,i_k} := S^\lambda_{i_0} \circ \cdots \circ S^\lambda_{i_k}(I^\lambda)$ and $\Pi_\lambda(\mathbf{i}) := \bigcap_{k=1}^{\infty} I^\lambda_{i_0,\dots,i_k}$.

Example: $\Pi_\lambda(0,3,1,0,\dots)$



The dimension of the attractor

Mike Keane asked: is the function $\lambda \to \dim_{\mathrm{H}} \Lambda_\lambda$ continuous on $\lambda \in (1/4, 1/3)$?

Theorem 1.1 (Pollicott, S. (1994))

- *For Lebesgue almost all $\lambda \in (1/4, 1/3)$ we have $\dim_{\mathrm{H}} \Lambda_\lambda = \frac{\log 3}{\log(1/\lambda)}$ (which is the similarity dimension).*
- *There is an exceptional set E which is dense in $[1/4, 1/3]$ such that for $\lambda \in E$ we have $\dim_{\mathrm{H}} \Lambda_\lambda < \frac{\log 3}{\log(1/\lambda)}$.*

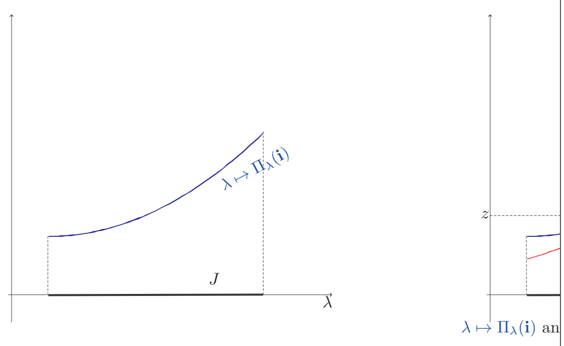
Transversality condition (Pollicott, S. 1995)[9]

We say that the transversality condition holds if, for every distinct $\mathbf{i}, \mathbf{j} \in \Sigma := \{1, \dots, m\}^{\mathbb{N}}$ the graph of the functions

$$\lambda \mapsto \Pi_\lambda(\mathbf{i}) \text{ and } \lambda \mapsto \Pi_\lambda(\mathbf{j})$$

have transversal intersection. That is if these two graphs intersect then their tangent lines are different. This is a generalization of Marstrand theorem.

$\Pi_\lambda(\mathbf{i}) := \cap_{k=0}^{\infty} I^\lambda_{i_0,\dots,i_k}$, $\Pi_\lambda(\mathbf{j}) := \cap_{k=0}^{\infty} I^\lambda_{j_0,\dots,j_k}$



Transversality condition can hold for:

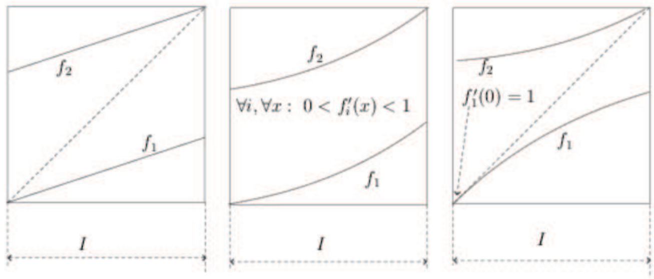
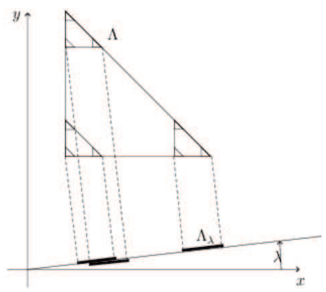


Figure: Linear, hyperbolic and parabolic Cantor sets

Examples for transversality condition I



Let $\Lambda \subset \mathbb{R}^2$ be the attractor of a self-similar sets with disjoint cylinders of similarities of the form $S_i(x) = \lambda_i x + t_i$ .Let $J := [0, \pi]$. Let Λ_λ be the projection of Λ to a line L_λ having angle $\lambda \in J$ with the positive part of the x axis on the plane. The transversality condition holds.

Examples for transversality condition II

(1) $$K_u^r := \left\{ \sum_{n=0}^{\infty} a_n r^n : a_n \in \{0, 1, u\} \right\}.$$

We get a one-parameter family if we fix one of the two parameters r, u. The cylinders intersect and the transversality condition holds in both of the following one-parameter families:

- Fix $u \in [2, 4]$, and the parameter in K_u^r is $r \in \left(\frac{1}{1+u}, \frac{1}{3}\right)$
- Fix $r \in \left(\frac{1}{5}, \frac{1}{3}\right)$ be fixed. The parameter in K_u^r is $u \in \left[\frac{1-r}{r}, \frac{2(1-r)}{1-3r}\right]$.

Examples for transversality condition III

Example 1.2

Let $f_1(x), \dots, f_m(x) : \mathbb{R} \to \mathbb{R}$ such that for every $i = 1, \dots, m$ we assume that $f_i'(x)$ exists for all $x \in J$ and $|f_i'(x)| < \frac{1}{2}$ for every $x \in J$. Fix a $j \in \{1, \dots, m\}$ then the one parameter family of contracting IFS

$$\{f_1(x), \dots, f_i(x) + \lambda, \dots, f_m(x)\}$$

satisfies transversality holds.

Examples for transversality condition IV

Example 1.3 (R. Lyons' continued fraction example [2])

Let $f_1^\alpha(x) := \frac{x+\alpha}{1+x+\alpha}$ and $f_2^\alpha := \frac{x}{1+x}$ for $\lambda \in J = (0.215, 0.5)$. Then the transversality condition holds. The invariant measure ν_λ for this IFS above is the same as the distribution of the random continued fractions $y = [1, Y_1, 1, Y_2, 1, Y_3, \cdots]$, where $Y_i = 0, \alpha$ independently with $\frac{1}{2}, \frac{1}{2}$ probability.

Also we can define the same distribution as the stationary measure of the sequence of random matrix products:

$$\begin{pmatrix} 1 & Y_n \\ 1 & 1+Y_n \end{pmatrix} \cdots \begin{pmatrix} 1 & Y_1 \\ 1 & 1+Y_1 \end{pmatrix}.$$

Examples for transversality condition V

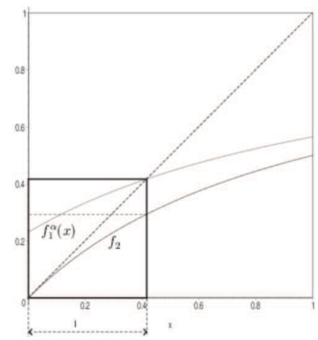


Figure: $f_2(x) = \frac{x}{1+x}$ and $f_1^\alpha(x) = f_2(x + \alpha)$

The parabolic IFS $\{f_1^\alpha, f_2\}$ satisfies transversality condition on the parameter interval $\alpha \in [0.215, 0.5]$ .Using that we can compute the dimension of the attractor and the dimension of invariant measures. See [10], [11].

Some consequences of the transversality condition for the dimension I

Theorem 1.4

Let $S_i^\lambda : \mathbb{R} \to \mathbb{R}$,

$$S_i^\lambda := r_i(\lambda) \cdot x + t_i(\lambda),$$

$i = 1, \dots, m$ and $\lambda \in J$. We assume that $r_i(\lambda), t_i(\lambda) \in \mathcal{C}^\infty(J)$ and there exist β, γ such that for all $i = 1, \dots, m$ and for all $\lambda \in J$ we have $0 < \beta < r_i(\lambda) < \gamma < 1$. Let Λ_λ be the attractor of S_i^λ.

Some consequences of the transversality condition for the dimension II

Theorem 1.4 (Cont.)

Let us call $\mathcal{IP}$ the set of those parameters λ for which the cylinders of Λ_λ intersect. That is

$$\mathcal{IP} := \{\lambda \in J : \exists \mathbf{i} \neq \mathbf{j} \text{ such that } \Pi_\lambda(\mathbf{i}) = \Pi_\lambda(\mathbf{j})\}.$$

Further, we assume that the **transversality condition holds** .

Some consequences of the transversality condition for the dimension III

Theorem 1.4 (Cont.)

Then

(i) $\dim_{\mathrm{H}} \Lambda_\lambda = s(\lambda)$, where $s(\lambda)$ is the similarity dimension,

(ii) for Lebesgue almost all $\lambda \in \mathcal{IP}$ we have

(2) $$\mathcal{H}^{s(\lambda)}(\Lambda_\lambda) = 0,$$

Some consequences of the transversality condition for the dimension IV

Theorem 1.4 (Cont.)

Then

(iii) assuming that $\forall \lambda \in J$, $\#\Lambda_\lambda > 1$ we get that the set $\{\lambda \in \mathcal{IP} : \mathcal{H}^{s(\lambda)}(\Lambda_\lambda) = 0\}$ is a G_δ dense set in $\mathcal{IP}$,

Some consequences of the transversality condition for the dimension V

Theorem 1.4 (Cont.)

Then

(iv) if we assume that there exists a function $\varphi(\lambda)$ and constants $r_1, \dots, r_m$ such that for all $\lambda \in J$, $r_i(\lambda) = r_i^{\varphi(\lambda)}$ then for almost all $\lambda \in J$

(3) $$0 < \mathcal{P}^{s(\lambda)}(\Lambda_\lambda) < \infty,$$

where $\mathcal{P}^s$ is the s-dimensional Packing Measure

Radon measure definition

μ is a Radon measure if

(a) Borel measure,

(b) $\forall K \subset X$ compact: $\mu(K) < \infty$,

(c) $\forall V \subset X$ open: $\mu(V) = \sup\{\mu(K) : K \subset V \text{ is compact }\}$

(d) $\forall A \subset X$: $\mu(A) = \inf\{\mu(V) : A \subset \text{ and } V \text{ is open }\}$.

Theorem 2.1

A measure μ on $\mathbb{R}^d$ is a Radon measure if and only if it is locally finite and Borel regular

Proof: See Mattila's book [4, p. 11-12].

Mass Distribution Principle

We say that a Borel measure μ on the set X is a mass distribution if $0 < \mu(X) < \infty$.

Lemma 2.2 (Mass Distribution Principle)

If $A \subset X$ supports a mass distribution μ such that for a constant C and for every Borel set D we have

$$\mu(D) \le \mathrm{const} \cdot |D|^t$$

Then $\dim_{\mathrm{H}}(A) \geq t$.

Proof For all $\{A_j\}_{j=1}^\infty$

$$A \subset \bigcup_{j=1}^{\infty} A_j \Rightarrow \sum_i |A_j|^t \geq C^{-1} \sum_i \mu(A_j) \geq \frac{\mu(A)}{C}.$$

Frostman's Energy method

Let μ be a mass distribution on $\mathbb{R}^d$. The t-energy of μ is defined by

$$\mathcal{E}_t(\mu) := \iint |x-y|^{-t} d\mu(x) d\mu(y).$$

Lemma 2.3 (Frostman (1935))

For a Borel set $\Lambda \subset \mathbb{R}^d$ and for a mass distribution μ supported by Λ we have

$$\mathcal{E}_t(\mu) < \infty \Longrightarrow \dim_{\mathrm{H}}(\Lambda) \geq t.$$

In this case $\mathcal{H}^t(\Lambda) = \infty$.

Proof of Frostman Lemma I

This proof if due to Y. Peres. Let

$$\Phi_t(\mu, x) := \int \frac{d\mu(y)}{|x-y|^t}.$$

Then $\mathcal{E}_t(\mu) = \int \Phi_t(\mu, x) d\mu(x)$. Let

$$\Lambda_M := \{x \in \Lambda : \Phi_t(\mu, x) \leq M\}.$$

Since $\int \Phi_t(\mu, x) d\mu(x) = \mathcal{E}_t(\mu) < \infty$ we have M such that $\mu(\Lambda_M) > 0$. Fix such an M.

Proof of Frostman Lemma II

Let

$$\nu := \mu|_{\Lambda_M}$$

Then ν is a mass distribution supported by Λ. (That is ν satisfies one of the assumptions of the Mass Distribution Principle above.) Now we show that for every bounded set D:

$$\nu(D) < \mathrm{const} \cdot |D|^t. \quad (4)$$

If $D \cap \Lambda_M = \emptyset$ then (4) holds obviously. From now we assume that D is a bounded set such that $D \cup \Lambda_m \neq \emptyset$.

Proof of Frostman Lemma III

Pick an arbitrary $x \in D \cap \Lambda_M$. We define

$$m := \max\left\{k \in \mathbb{Z} : B(x, 2^{-k}) \supset D\right\}.$$

Then

$$|D| \geq 2^{-(m+1)} \text{ and } |D| < 2 \cdot 2^{-m}. \quad (5)$$

Proof of Frostman Lemma IV

Observe that from the right hand side of (5): $y \in D$ we have $|x-y|^{-t} \geq |D|^{-t} \geq 2^{-t} \cdot 2^{mt}$. So,

$$M \geq \int \frac{d\nu(y)}{|x-y|^t} \geq \int_D \frac{d\nu(y)}{|x-y|^t} \geq \nu(D) \cdot 2^{-t} \cdot 2^{m \cdot t}.$$

Using this and the left hand side of (5) we obtain

$$\nu(D) \leq M \cdot 2^t \cdot 2^t \cdot 2^{-(m+1)t} \leq M \cdot 2^{2t} \cdot |D|^t.$$

So, the mass distribution ν satisfies the assumptions of the Mass Distribution Principle which completes the proof of the Lemma.

Radon measures IV

Definition 2.4

Let μ, η be Radon measures on $\mathbb{R}^d$. We define the upper and lower derivatives of μ with respect to η:

$$\overline{\underline{D}}(\mu, \eta, x) := \overline{\underline{\lim}}_{r \to 0} \frac{\mu(B(x,r))}{\eta(B(x,r))}.$$

If the limit exists then we write $D(\mu, \eta, x)$ for this common value and we call it the derivative of the measure μ with respect to the measure η.

Radon measures V

Theorem 2.5

Let μ, η be Radon measures on $\mathbb{R}^d$.

(i) *The derivative $D(\mu, \eta, x)$ exists and is finite for η almost all $x \in \mathbb{R}^d$. [3, Theorem 2.12]*

(ii) *For all Borel sets $B \subset \mathbb{R}^d$ we have*

$$\int_B D(\mu, \eta, x) d\eta(x) \leq \mu(B) \quad (6)$$

with equality if $\mu \ll \eta$. [3, Theorem 2.12]

Radon measures VI

Theorem 2.5 (Cont.)

(iii) $\mu \ll \eta$ if and only if $\underline{D}(\mu, \eta, x) < \infty$ for μ almost all $x \in \mathbb{R}^d$. [3, Theorem 2.12]

(iv) If $\mu \ll \eta$ then

$$\int D(\mu, \eta, x)^2 d\eta(x) = \int D(\mu, \eta, x) d\mu(x).$$

This is [3, Exercise 6 on p. 43]

Radon measures VII

Theorem 2.5 (Cont.)

(v) Assume that $\mu \ll \eta$. Then $D(\mu, \eta, x)$ is a version of the Radon-Nikodym derivative $\frac{d\mu(x)}{d\eta(x)}$. So, we know that $\int_{\mathbb{R}} D(\mu, \eta, x) d\eta(x) < \infty$. Further, by (iv) above, we have:

(7)
$$\int_{\mathbb{R}} D(\mu, \eta, x) d\mu(x) < \infty \Longrightarrow \frac{d\mu(x)}{d\eta(x)} \in L^2(\mathbb{R}).$$

This argument appears in [7, p.233].

Infinite Bernoulli convolution I

For a $\lambda \in (0, 1)$ we define the random variable

$$Y_\lambda := \sum_{n=0}^{\infty} \pm \lambda^n.$$

ν_λ be the distribution of Y_λ. On the other hand ν_λ is the self similar measure of the IFS. That is for $\lambda \in (0, 1)$, $x \in [0, 1/(1-\lambda)]$

$$S_1^\lambda(x) := \lambda x + 1, S_{-1}^\lambda(x) := \lambda x - 1,$$

with weights $1/2, 1/2$
$(\nu_\lambda(A) = \frac{1}{2}\nu_\lambda((S_1^\lambda)^{-1}(A)) + \frac{1}{2}\nu_\lambda((S_{-1}^\lambda)^{-1}(A)))$.

Infinite Bernoulli convolution II

$$\nu_\lambda = (\Pi_\lambda)_*(\{1/2, 1/2\}^{\mathbb{N}}),$$
$$\Pi_\lambda(i_0, i_1, i_2, \dots) = i_0 + i_1\lambda + i_2\lambda^2 + \cdots$$

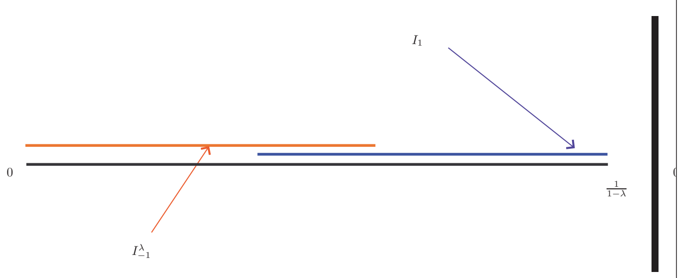
Let $I_\lambda := \left[0, \frac{1}{1-\lambda}\right]$. Yet again we write

$$I^\lambda_{i_0 \dots i_k} := S_{i_0 \dots i_k}(I^\lambda).$$

Then
$$\Pi_\lambda(i_0, i_1, \dots) = \bigcap_{k=0}^{\infty} I^\lambda_{i_0 \dots i_k}.$$

Infinite Bernoulli convolution III

Cylinders for $\lambda \in (0.5, 1)$



Law of pure type

Theorem 3.1 (Jensen, Wintner 1935)

Either $\nu_\lambda \ll \mathcal{L}\text{eb}$ *or* $\nu_\lambda \perp \mathcal{L}\text{eb}$

It was proved by Parry and York that for every λ we have

(8) Either $\nu_\lambda \sim \mathcal{L}\text{eb}$ or $\nu_\lambda \perp \mathcal{L}\text{eb}$.

Solomyak's Theorem (1995)

After 60 years after that in 1930's P. Erdős started to investigate the infinite Bernoulli convolutions Boris Solomyak made the following major achievement:

Theorem 3.2 (Solomyak (1995))

1. $\nu_\lambda \ll \mathcal{L}eb$ *with a density in* $L^2(\mathbb{R})$ *for a.e.* $\lambda \in (1/2, 1)$.
2. $\nu_\lambda \ll \mathcal{L}eb$ *with a density in* $C(\mathbb{R})$ *for a.e.* $\lambda \in (2^{-1/2}, 1)$.

$$\hat{\nu}_\lambda(x) := \int_{\mathbb{R}} e^{itx} d\nu_\lambda(t) = \Pi_{n=0}^{\infty} \cos(\lambda^n x).$$

Hence for every $k \geq 2$ we have

(9)
$$\hat{\nu}_\lambda(x) = \prod_{i=0}^{k-1} \hat{\nu}_{\lambda^k}(\lambda^i x).$$

Using this if we have absolute continuity on $\lambda \in \left[\frac{1}{2}, \frac{1}{\sqrt{2}}\right]$ then we have absolute continuity for the whole $\lambda \in \left[\frac{1}{2}, 1\right]$. This and Solomyak theorem implies that

$$k \geq 2, \text{ then for a.a. } \lambda \in \left(2^{-1/k}, 1\right), \text{ then } \hat{\nu}_\lambda \in L^{2/k}.$$

In particular, for $\lambda \in \left(2^{-1/2}, 1\right)$, ν_λ has bounded density.

Erdős Results form the 1930's

Theorem 3.3 (Pál Erdős 1940)

There exists a $t < 1$ (rather close to 1) such that for a.e. $\lambda \in (t, 1)$ we have $\nu_\lambda \ll \mathcal{L}eb$. More precisely,

$$\exists a_k \uparrow 1 \text{ s.t. } \quad \frac{d\nu_\lambda}{dx} \in C^k(\mathbb{R}) \text{ for } \lambda \in (a_k, 1).$$

Problem 3.4 (Erdős)

Is it true that $\nu_\lambda \ll \mathcal{L}eb$ holds for a.e. $\lambda \in (1/2, 1)$?

The only known counter examples are the reciprocals of the so-called PV number or Pisot or Pisot Vayangard numbers (they are the same but nobody can pronounce Vayangard properly so people avoid using his name). The most beautiful account olf this field was given by Solomyak [13].

Definition of PV numbers

Definition 4.1

We say that the algebraic integer $\theta > 1$ is a PV number if all of the other roots of its minimal polynomials are less than one in modulus.

We study the distribution of $Y_\lambda := \sum_{i=0}^{\infty} \pm \lambda^i$ for a $\lambda \in (0, 1)$. For such a λ:

- We denote $\#_\lambda(n)$ the number of distinct points in $\sum_{k=0}^{n-1} \pm \lambda^k$.
- We denote by $\omega_\lambda(n)$ the minimal distance between two distinct points in $\sum_{k=0}^{n-1} \pm \lambda^k$.

Properties of PV numbers

1. If θ is a PV number then there exists an $\eta \in (0, 1)$ such that
$$\|\theta^n\|_{\mathbb{Z}} < \eta^n.$$
2. If $\lambda \in (0.5, 1)$ and $\lambda = \theta^{-1}$ for a PV number θ then
$$\omega_\lambda(n) \geq C_1 \cdot \lambda^n \text{ and } C_2 \cdot \lambda^{-n} \leq \#_\lambda(n) \leq C_3 \lambda^{-n}$$

for some constants $C_1, C_2, C_3 > 0$. The golden ratio $\frac{1+\sqrt{5}}{2}$ is the only quadratic PV number in $(1, 2)$ and the smallest limit point of the closed set of PV numbers. The smallest Pisot number is $\theta = 1.32478$ which is the root of $x^3 - x - 1 = 0$.

Theorem 4.2 (Erdős 1939)

If $\lambda \neq \frac{1}{2}$ and $\frac{1}{\lambda}$ is a Pisot number then

(a) $\nu_\lambda \perp \mathcal{L}\text{eb}$.

(b) $\lim_{\xi \to \infty} \hat{\nu}(\xi) \not\to 0$.

Clearly, if ν_λ was absolute continuous then $\lim_{\xi \to \infty} \hat{\nu}(\xi) \to 0$. So, the second part is stronger.

Theorem 4.3 (Salem 1944)

If $\lambda \in (0, 1)$ and λ^{-1} is NOT a Pisot number then

$$\lim_{\xi \to \infty} \hat{\nu}_\lambda(\xi) = 0.$$

The Proof of the previous Erdős Theorem

This sketch of the proof is from Slomyak's survey paper [13]. Using a theorem of Pisot, Erdős proved that (10)

$$\exists \gamma > 0, \quad \hat{\nu}_\lambda(\xi) = \mathcal{O}\left(|\xi|^{-\gamma}\right) \quad \text{for a.a. } \lambda \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right).$$

Now we combine formulas (9) and (10) to obtain that

$$|\hat{\nu}_\lambda(\xi)| = \mathcal{O}\left(|\xi|^{-k\gamma}\right), \quad \text{for a.e. } \lambda \in \left(\frac{1}{2^{1/k}}, \frac{1}{2^{1/(2k)}}\right).$$

The Proof of the previous Erdős Theorem (Cont.)

$$(11) \quad \exists \alpha > 1, \ |\hat{\nu}_\lambda(\xi)| = \mathcal{O}\left(|\xi|^{-\alpha}\right) \Longrightarrow \hat{\nu}_\lambda \in L^1(\mathbb{R})$$
$$\Longrightarrow \nu_\lambda \ll \mathcal{L}\text{eb with } \frac{d\nu_\lambda}{dx} \in C(\mathbb{R}).$$

If $\alpha > k + 1$ then in distributional sense

$$(12) \quad \frac{d}{dx^k} \widehat{\left(\frac{d\nu_\lambda}{dx}\right)} = \xi^k \hat{\nu}_\lambda(\xi) \in L^1(\mathbb{R}).$$

The Proof of the previous Erdős Theorem (Cont.)

Formula (12) implies that

$$\frac{d\nu_\lambda}{dx} \in \mathcal{C}^k(\mathbb{R})$$

The definition of Garcia numbers

The largest collection of numbers λ for which $\nu_\lambda \ll \mathcal{L}eb$ is the reciprocals of the so called Garcia numbers.

Definition 4.4

Garsia numbers are those algebraic integers in $(1,2)$ for which the minimal polynomial has another root out of the unit circle and the constant coefficient is ± 2.

Examples for Garsia numbers

Example 4.5

Examples for the reciprocal of Garsia numbers

- $2^{-1/k}$ for $k \geq 1$ (with polynomial $x^k - 2$).
- $\approx .5651977175\ldots$ (with polynomial $x^3 - 2x - 2$).
- The reciprocal of the largest root of $x^{n+p} - x^n - 2$ such that $p, n \geq 1$ and $\max\{p, n\} \geq 2$ (e.g. $0.6572981061\ldots$ with the polynomial $x^3 - x - 2$).

Solomyak's Theorem (1995)

After 60 years after that in 1930's P. Erdős started to investigate the infinite Bernoulli convolutions Boris Solomyak made the following major achievement:

Theorem 5.1 (Solomyak (1995))

1. *$\nu_\lambda \ll \mathcal{L}eb$ with a density in $L^2(\mathbb{R})$ for a.e. $\lambda \in (1/2, 1)$.*
2. *$\nu_\lambda \ll \mathcal{L}eb$ with a density in $\mathcal{C}(\mathbb{R})$ for a.e. $\lambda \in (2^{-1/2}, 1)$.*

A generalization of Solomyak's Theorem

Let $\mathbf{p} = (p_1, \ldots, p_m)$ be a probability vector and $D = \{d_1, \ldots, d_m\} \subset \mathbb{R}$ be the set of digits. Let ν_λ be the distribution of the random series $\sum_{n=0}^{\infty} a_n \lambda^n$, where a_i is chosen from D independently in every steps with probabilities p_i. Then ν_λ is the self-similar measure for the IFS $\{S_i(x) = \lambda x + d_i\}_{i=1}^m$ with probabilities given by $\mathbf{p}$. That is

$$\nu_\lambda = \sum_{i=1}^{m} p_i \cdot \left(\nu_\lambda \circ S_i^{-1}\right). \tag{13}$$

A generalization II.

Theorem 5.2 (Peres, Solomyak)

Let $J \subset [0,1]$ be a closed interval on which the transversality condition holds. Then

1. *$\nu_\lambda \ll \mathcal{L}eb$ for a.e. $\lambda \in J \cap [\prod_{i=1}^m p_i^{p_i}, 1]$ and ν_λ is singular for all $\lambda < \prod_{i=1}^m p_i^{p_i}$.*
2. *$\nu_\lambda \ll \mathcal{L}eb$ with a density in $L^2(\mathbb{R})$ for a.e.*

$$\lambda \in J \cap \left(\sum_{i=1}^{m} p_i^2, 1\right).$$

The transversality interval in case of the Bernoulli convolution $J = [0.5, 0.668]$.

Comments on the theorem

Let $\mu := (p_1, \ldots, p_n)^{\mathbb{N}}$ the Bernoulli measure on $\Sigma = \{d_1, \ldots, d_m\}^{\mathbb{N}}$. Then it follows from (13) that $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$, where $\Pi_\lambda(i_0, i_1, i_2, \ldots) = i_0 + i_1\lambda + i_2\lambda^2 + \cdots$. Clearly the entropy of μ is

$$h_\mu = -\log \prod_{i=1}^{m} p_i^{p_i}.$$

Thus for $\lambda_0 = \prod_{i=1}^m p_i^{p_i}$ we have

$$\dim_{\mathrm{H}}(\nu_{\lambda_0}) \leq \frac{h_\mu}{\log(1/\lambda_0)} = 1.$$

Further comments to Theorem 5.2

Consider the special case in Theorem 5.2 when the IFS is

$$\{S_{-1}(x) = \lambda x - 1, S_1(x) = \lambda x + 1\}$$

and the probabilities $(p, 1-p)$. The invariant measure is ν_λ^p. We know that ν_λ^p is the distribution of

$$\sum_{i=0}^{\infty} \pm\lambda^n,$$

where the $-$ and $+$ signs are chosen with probability p and $1-p$ respectively.

Further comments to Theorem 5.2 (Cont.)

Theorem 5.2 gives L^2 density only for λ from

$$J_p := \left(p^2 + (1-p)^2, 1\right)$$

in the following way: Let

$$J_k := \left(\left(p^2 + (1-p)^2\right)^{(k-1)/2}, \left(p^2 + (1-p)^2\right)^{k/2}\right)$$

Assume that for a $k \geq 1$ we have

(14) $$\hat{\nu}_\lambda^p \in L^2, \quad \forall \lambda \in J_k.$$

We prove that this holds for J_1 by transversality condition then we proceed by induction:

Further comments to Theorem 5.2 (Cont.)

Observe that

$$\sum \pm \left(\sqrt{\lambda}\right)^n = \sum \pm(\lambda)^n + \sqrt{\lambda}\sum \pm(\lambda)^n$$

Since the random signs are independent we obtain:

(15) $$\hat{\nu}_{\sqrt{\lambda}}^p(u) = \hat{\nu}_\lambda^p(u) \cdot \hat{\nu}_\lambda^p(\sqrt{\lambda} \cdot u).$$

So, if ν_λ^p has L^2 density then by Plancherel Theorem, $(\hat{\nu})_\lambda^p \in L^2(\mathbb{R})$. Then by (15)

Further comments to Theorem 5.2 (Cont.)

(16) $\hat{\nu}_{\sqrt{\lambda}}^p \in L^1(\mathbb{R}) \Longrightarrow \nu_{\sqrt{\lambda}}^p$ has continouous density.

So, $\nu_{\sqrt{\lambda}}^p$ has L^2 density and we can continue the induction to show that for all k, the measure ν_λ^p has L^2 density for $\lambda \in J_k$.

Let μ be an ergodic measure on the symbolic space $\Sigma := \{1, \dots, m\}^{\mathbb{N}}$.

Definition 5.3 (L^q-dimension of μ)

Let $q > 1$. We define the L^q-dimension of m by

$$D_q(\mu) := \frac{1}{q-1} \liminf_{n\to\infty} \frac{-\log \sum\limits_{\mathbf{i}\in\{1,\dots,m\}^n} \mu([\mathbf{i}])^q}{n \log m}$$

If $\mu = \{p_1, \dots, p_m\}^{\mathbb{N}}$ then

$$m^{-D_q(\mu)} = \left[p_1^q + \cdots + p_m^q\right]^{1/(q-1)}.$$

The following Peres-Solomyak theorem is from:[8, Theorem 1.3]

Theorem 5.4 (Peres and Solomyak)

Let

$$S_i(x) = \lambda x + d_i(\lambda), i = 1, \dots, m.$$

and $\Pi_\lambda(\mathbf{i}) := \sum\limits_{k=0}^{\infty} d_{i_k}\lambda^k$. *Given a probability vector* $\mathbf{p} = (p_1, \dots, p_m)$. *Let*

$$\mu := \{p_1, \dots, p_m\}^{\mathbb{N}}$$

and

$$\nu_\lambda := \Pi_*(\mu).$$

Theorem (Cont)

Suppose that $J \subset (0,1)$ is an interval such that the transversality condition holds. Then

(a) ν_λ is absolute continuous if $\lambda > \prod_{i=1}^m p_i^{p_i}$ and singular if $\lambda < \prod_{i=1}^m p_i^{p_i}$.

(b) Let $q \in (1,2]$. then for a.e. $\lambda > [p_1^q + \cdots + p_m^q]^{1/(q-1)}$ such that $\lambda \in J$ the measure $\nu_\lambda \ll \mathcal{L}eb$ with L^q density

(c) For any $q > 1$ and all $\lambda \in (0,1)$, if $\nu_\lambda \ll \mathcal{L}eb$ with L^q density then $\lambda > [p_1^q + \cdots + p_m^q]^{1/(q-1)}$.

Example

Example 5.5

Let the digit set be $D := \{-1, 0, 1\}$ and let $\mathbf{p} := \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$, Let η_λ be the corresponding self similar measure. That is the measure which corresponds to these probabilities and the IFS

$$\mathcal{F}_\lambda = \{\lambda x - 1, \lambda x, \lambda x + 1\}.$$

Observe that

(17) $$\eta_\lambda = \nu_\lambda^{1/2} * \nu_\lambda^{1/2},$$

where $\nu_\lambda^{1/2}$ was introduced on the slide # 5.4.

Using that $\prod_{i=1}^{3} p_i^{p_i} = \frac{1}{2\cdot\sqrt{2}}$ and for $q=2$
$\lambda_q^* := (2^{-q}+2\cdot 4^{-q})^{1/(1-q)} = \frac{3}{8}$ by Theorem 5.4 we have

(i) For $\lambda < \frac{1}{2\cdot\sqrt{2}}$ then $\eta_\lambda \perp \mathcal{L}\mathrm{eb}$.

(ii) For $\frac{1}{2\cdot\sqrt{2}} < \lambda < \frac{3}{8}$ then $\eta_\lambda \ll \mathcal{L}\mathrm{eb}$ but it has NOT L^2-density

(iii) For $\lambda > \frac{3}{8}$ $\eta_\lambda \ll \mathcal{L}\mathrm{eb}$ with L^2 density.

Application: the Schilling equation

Because of motivations from physics the functional equation called Schilling equation was intensively studied:

$$(18) \qquad y(\lambda t) = \frac{1}{4\lambda}\left[y(t+1)+y(t-1)+2y(t)\right],$$

where $0<\lambda<1$. With simple change of variables $t \mapsto \frac{t}{\lambda}$ we get

$$(19) \quad y(t) = \frac{1}{4\lambda}y\left(\frac{t}{\lambda}-1\right)+\frac{1}{2\lambda}y\left(\frac{t}{\lambda}\right)+\frac{1}{4\lambda}y\left(\frac{t}{\lambda}+1\right)$$

Equation (19) has a compactly supported solution y_λ in L^1 iff

$$(20) \qquad \mathcal{F}_\lambda := \{\lambda x - 1, \lambda x, \lambda x + 1\}$$

with probabilities $\mathbf{p} := \left(\frac{1}{4},\frac{1}{2},\frac{1}{4}\right)$ has an absolute continuous invariant measure. In this case the density function of ν_λ is y_λ. This is exactly the measure we considered previously. Derfel and Schilling [1] pointed out that for $\lambda > \frac{1}{2}$ the density is actually continuous.

On the exceptional parameters

Theorem 5.6 (Peres-Schlag 2000 [5])

Let $J \subset [\lambda_0, \lambda_0'] \left(\frac{1}{2},1\right)$ be an interval where the transversality condition holds for the Bernoulli convolution. Then the dimension of the exceptional parameters:

$$\dim_{\mathrm{H}}\left\{\lambda \in J : \frac{d\nu_\lambda}{dx} \notin L^2(\mathbb{R})\right\} \le 2 - \frac{\log 2}{\log(1/\lambda_0)}$$

Proof: Peres, Solomyak's Theorem I

We follow: Boris Solomyak, Notes on Bernoulli convolutions. `http://www.math.washington.edu/~solomyak/PREPRINTS/mandel2.pdf`
We apply the previous theorem for

$$\underline{D}_\lambda(x) := \underline{D}(\nu_\lambda, \mathcal{L}eb, x) = \liminf_{r\to 0}\frac{\nu_\lambda(x-r,x+r)}{2r}.$$

It is enough to prove that

$$(21) \qquad \mathcal{I} := \int_J \int_{\mathbb{R}} \underline{D}_\lambda(x)d\nu_\lambda(x)d\lambda < \infty.$$

Proof: Peres, Solomyak's Theorem II

For $\mathbf{i},\mathbf{j} \in \Sigma$ we define the function
$\Phi_{\mathbf{i},\mathbf{j}}(r) := \mathcal{L}eb\left\{\lambda \in J : |\Pi_\lambda(\mathbf{i}) - \Pi_\lambda(\mathbf{j})| < r\right\}$. Using Fatau Lemma and exchanging the order of integration yields that

$$\mathcal{I} \le \liminf_{r\to 0}\frac{1}{2r}\int_\Sigma\int_\Sigma \Phi_{\mathbf{i},\mathbf{j}}(r)d\mu(\mathbf{i})d\mu(\mathbf{j}).$$

Let $J = [\lambda_0, \lambda_1]$. From Transversality condition:

$$(22) \qquad \Phi_{\mathbf{i},\mathbf{j}}(r) \le \mathrm{const}\cdot\lambda_0^{-|\mathbf{i}\wedge\mathbf{j}|}\cdot r.$$

$\mathcal{I} \le \mathrm{const}\sum_{k=0}^{\infty}\lambda_0^{-k}\left(p_1^2+\cdots+p_m^2\right)^k < \infty$ holds since $\sum_{k=1}^{m} p_k^2 < \lambda_0$.

The class B_γ

The methods below are due to Peres and Solomyak [12], [7] and [8]. Let $\gamma > 0$. Peres Solomyak introduced:

$$(23) \quad B_\gamma := \left\{ g(x) = 1 + \sum_{n=1}^{\infty} a_n x^n : |a_n| \le \gamma,\ n \ge 1 \right\}.$$

Let J be a closed sub-interval of $[0,1]$ and let $\gamma, \delta > 0$. We say that a B_γ satisfies that **δ-transversality condition** on J if:

(24)

$$\forall g \in B_\gamma : \quad (\lambda \in J \text{ and } g(\lambda) < \delta) \Longrightarrow g'(\lambda) < -\delta.$$

That is all $\forall g \in B_\gamma$ whenever the graph of g meets a horizontal line below the height of δ, it crosses it with a slope at most $-\delta$.

Definition 6.1 ($*$-functions)

Let $\gamma > 0$. we say that $h(x)$ is a $*$-function for B_γ if for some $k \geq 1$ and $a_k \in \mathbb{R}$ we have

$$h(x) = 1 - \gamma \sum_{i=1}^{k-1} x^i + a_k x^k + \gamma \sum_{i=k+1}^{\infty} x^i. \tag{25}$$

Lemma 6.2

Assume that $h(x)$ is a $*$-function for B_γ and there exists $x_0 \in (0,1)$ and $\delta \in (0,\gamma)$ such that $h(x)$ satisfies:

$$h(x_0) > \delta \text{ and } h'(x_0) < -\delta. \tag{26}$$

Then the δ-transversality holds for B_γ on the interval $[0, x_0]$.

We write

$$\mathcal{B}_{m,\mathcal{I}} := \left\{ 1 + \sum_{i \in \mathcal{I} \setminus \{0\}} a_i x^i : \ |a_i| \leq m-1 \right\}.$$

If $\mathcal{I} = \mathbb{N}$ then we suppress it. Let $J \subset (0,1)$ be a closed interval and $\delta > 0$.

Definition 6.3

We say that the δ-transversality condition holds for $\mathcal{B}_{m,\mathcal{I}}$ on J if

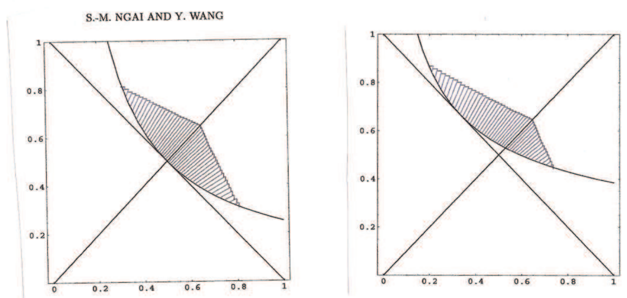
$$\forall k \in \mathcal{I},\ k < n, \forall g \in \mathcal{B}_{m,\sigma^k \mathcal{I}}, \forall \lambda \in J, \quad g(\lambda) < \delta \Longrightarrow g'(\lambda) < -\delta. \tag{27}$$

Further generalization of Solomyak Theorem II

Theorem 6.4 (S.M. Ngai, Y. Wang)

Let $\mu_{\rho_1,\rho_2,p_1,p_2}$ be the self-simlar measure for the IFS (we are on $\mathbb{R}$) $S_1(x) := \rho_1 x \quad S_2(x) := \rho_2 x + 1$, which corresponds to the probabilities p_1, p_2. That is for $\mu := \mu_{\rho_1,\rho_2,p_1,p_2}$, $\mu(A) = p_1 \mu(S_1^{-1}A) + p_2 \mu(S_2^{-1}(A))$ for a Borel set $A \subset \mathbb{R}$. Then the regions of singularity and verified absolute continuity are shown or the next slide. On the figure on the left hand side we assumed that $p_1 = p_2 = \frac{1}{2}$. On the figure on the right hand side we assumed that $p_1 = \frac{1}{3}$ and $p_2 = \frac{2}{3}$.

Further generalization of Solomyak's Theorem III



A Sinai's problem I

Consider the random series

$$X := 1 + Z_1 + Z_1 Z_2 + \cdots + Z_1 Z_2 \cdots Z_n + \cdots$$

where Z_i are i.i.d. taking values in $\{1-a, 1+a\}$ for a fixed $0 < a < 1$ with probabilities $\left(\frac{1}{2}, \frac{1}{2}\right)$. The series converges almost surely since the Lyaponov exponent:

$$\chi := \mathbb{E}[\log Z] = \frac{1}{2} \log(1-a^2) < 0.$$

Let ν^a be the distribution of X.

A Sinai's problem II

Problem 7.1 (Sinai)

For which $a \in (0,1)$ is the measure ν^a absolute continuous w.r.t. $\mathcal{L}eb$?

This question was motivated by a statistical version of the famous $3n+1$ problem.

Remarks

1. ν^a is the invariant measure for the IFS
$$\{1 + (1-a)x, 1 + (1+a)x\},$$
with prob. $(1/2, 1/2)$.
2. $\text{supp}\nu^a = [Fix(1 + (1-a)x), \infty)$,
3. If $a > \frac{\sqrt{3}}{2}$ then $\log 2 < -\frac{1}{2}\log(1-a^2)$. Thus for the entropy h_ν of the measure ν we obtain: $h_\nu < -\chi$. This implies that: $\dim_{\mathrm{H}} \nu^a < 1$. Therefore $\nu^a \perp \mathcal{L}eb$.
4. **Conjecture:**
$$\nu^a \ll \mathcal{L}eb \text{ for a.e. } 0 < a < \frac{\sqrt{3}}{2}. \tag{28}$$

We did not managed to solve this problem but we answered positively the corresponding problem in the randomly perturbed case. Namely, Let

$$Z_i := \lambda_i Y,$$

where $\lambda_i \in \{1-a, 1+a\}$ with probability $(1/2, 1/2)$ and the error Y has absolute continuous distribution on $(1-\varepsilon_1, 1+\varepsilon_2)$ for small $\varepsilon_1, \varepsilon_2 > 0$ with bounded density and we assume that $\mathbb{E}[\log Y] = 0$. The error y_i at every steps are i.i.d. with distribution Y and independent on everything else.

The randomly perturbed case I

Theorem 7.2 (Peres, S.,Solomyak)

Let $\nu^a_{\mathbf{y}}$ be the conditional distribution for a given sequence of errors $\mathbf{y} = (y_1, y_2, \dots)$. Then

1. *If $0 < a < \frac{\sqrt{3}}{2}$ then for a.a. $\mathbf{y}$ we have $\nu^a_{\mathbf{y}} \ll \mathcal{L}eb$;*
2. *If $a \geq \frac{\sqrt{3}}{2}$ then for a.a. $\mathbf{y}$ we have $\dim_{\mathrm{H}} \nu^a_{\mathbf{y}} = \frac{2\log 2}{\log \frac{1}{1-a^2}}$*

The randomly perturbed case II

Given $\{S_i(x) = \lambda_i x + d_i\}_{i=1}^m$ on $\mathbb{R}$. We assume that $\lambda_i > 0$ but some λ_i may be greater than 1.
Let Y be a random variable with an absolute continuous distribution η on $(0, \infty)$, such that

$$(29) \qquad \exists C_1 > 0 : \quad \frac{d\eta}{dx} \leq C_1 x^{-1}, \ \forall x > 0.$$

Let μ be an ergodic invariant measure on $\Sigma := \{1, \dots, m\}^{\mathbb{N}}$. The Lyapunov exponent is

$$\chi(\mu, \eta) := \mathbb{E}[\log \lambda Y] = \mathbb{E}[\log Y] + \int_{\Sigma} \log \lambda_{i_1} d\mu(\mathbf{i}).$$

The randomly perturbed case III

We assume that our IFS is contracting on average. That is

$$(30) \qquad \chi(\mu, \chi) < 0.$$

The natural projection $\Pi : \Sigma \times \mathbb{R}^{\mathbb{N}} \to \mathbf{R}$ is:

$$\Pi(\mathbf{i}, \mathbf{y}) := d_{i_1} + \cdots + d_{i_{n+1}} \lambda_{i_1 \dots i_n} y_{1 \dots n} + \cdots$$

where $y_{1\dots n} := y_1 \cdots y_n$ and $\lambda_{i_1 \dots i_n} := \lambda_{i_1} \cdots \lambda_{i_n}$.

$$\Pi_{\mathbf{y}}(\mathbf{i}) := \Pi(\mathbf{i}, \mathbf{y}) \text{ and } \nu_{\mathbf{y}} := (\Pi_{\mathbf{y}})_* \mu.$$

The randomly perturbed case IV

Theorem 7.3 (Peres, S., Solomyak)

If one of the following two conditions is satisfied:

(a) *$d_i \neq d_j$ for all $i \neq j$*
(b) *$d_i = 1$ and $\lambda_i \neq \lambda_j$ for all $i \neq j$*

then for η_∞ a.a. $\mathbf{y}$ we have

1. $$\frac{h_\mu}{|\chi(\mu, \eta)|} > 1 \Longrightarrow \nu_{\mathbf{y}} \ll \mathcal{L}eb,$$
2. $$\frac{h_\mu}{|\chi(\mu, \eta)|} \leq 1 \Longrightarrow \dim_{\mathrm{H}}(\nu_{\mathbf{y}}) = \frac{h_\mu}{|\chi(\mu, \eta)|}.$$

Consider the self similar IFS on $\mathbb{R}$

$$(31) \qquad \mathcal{F} := \{\varphi_i(x) = r_i \cdot x + a_i\},$$

$r_i \in (-1, 1) \setminus \{0\}$, $a_i \in \mathbb{R}$. Let Λ be the attractor of $\mathcal{F}$ and $s(\mathcal{F})$ be the similarity dimension of $\mathcal{F}$. For a $\mathbf{p} = (p_1, \dots p_m)$ probability vector let $\nu = \nu_{\mathbf{p}}$ the corresponding self similar measure and let

$$\dim_{\mathrm{S}}(\mu) := \frac{\sum_{i=1}^m p_i \log p_i}{\sum_{i=1}^m p_i \log |r_i|}$$

For an $\mathbf{i}, \mathbf{j} \in \{1, \dots, m\}^n$ we introduce the distance

$$(32) \qquad d(\mathbf{i}, \mathbf{j}) := \begin{cases} \infty, & \text{if } r_{\mathbf{i}} \neq r_{\mathbf{j}}; \\ |\varphi_{\mathbf{i}}(0) - \varphi_{\mathbf{j}}(0)|, & \text{if } r_{\mathbf{i}} = r_{\mathbf{j}}. \end{cases}$$

$$\Delta_n := \min\{d(\mathbf{i}, \mathbf{j}) : |\mathbf{i}| = |\mathbf{j}| = n, \mathbf{i} \neq \mathbf{j}\}$$

- Exact overlap $\longrightarrow \Delta_n = 0$
- $\Delta_n \to 0$ exponentially. Namely: $\#\{|\mathbf{i}| = n\} = m^n$. On the other hand: $\#\{r_{\mathbf{i}} : |\mathbf{i}| = n\}$ is polynomially many. So, there exists distinct $\mathbf{i}, \mathbf{j}$ of length n with $r_{\mathbf{i}} = r_{\mathbf{j}}$ with exponentially small $|\varphi_{\mathbf{i}}(0) - \varphi_{\mathbf{j}}(0)|$. In case the OSC holds, we have $\Delta_n \to 0$ exponentially.

Main Theorem of Hochman

For any probability vector $\mathbf{p}$
(33)
$$\dim_{\mathrm{H}}(\mu) < \min\{1, \dim_{\mathrm{S}}(\mu)\} \Rightarrow \lim_{n\to\infty} -\frac{1}{n}\log\Delta_n = \infty$$

That is Δ_n tends to 0 super-exponentially.

IFS with algebraic parameters

Theorem 8.1 (Hochman)

For an IFS with algebraic parameters we have

- *Either there are exact overlaps, or*
- $\dim_{\mathrm{H}}\Lambda = \min\{1, \dim_{\mathrm{S}}\Lambda\}$

Proof

In the proof we assume that $f_i(x) = rx + a_i$, $i = 1, \dots, m$ with $r_i \in (0,1)$. Then
$$f_{\mathbf{i}} = r^n x + f_{\mathbf{i}}(0).$$

Proof (Cont.)

Let
$$r = \frac{p}{q} \text{ and } a_i = \frac{p_i}{q_i}.$$
Let
$$Q := \prod_{i=1}^{m} q_i$$
Then for every $\mathbf{i} \in \{1, \dots, m\}^n$ exists $N(\mathbf{i}) \in \mathbb{N}$ s.t.
$$f_{\mathbf{i}}(0) = \sum_{k=1}^{n} a_{i_k} r^{n-k} = \frac{N(\mathbf{i})}{Q \cdot q^n} \in \mathbb{Q}.$$

Proof (Cont.)

Suppose that for $\forall n$, we have $\Delta_n > 0$. Then chose $\mathbf{i}, \mathbf{j} \in \{1, \dots, m\}^n$ s.t.
$$\Delta_n = f_{\mathbf{i}}(0) - f_{\mathbf{j}}(0) = \frac{N(\mathbf{i}) - N(\mathbf{j})}{Q \cdot q^n} > 0.$$
Then
$$\Delta_n \geq \frac{1}{Q \cdot q^n}.$$
So, $\Delta_n \to 0$ exponentially fast, so there is no dimension drop.

Right angle Sierpinski triangle with contraction 1/3

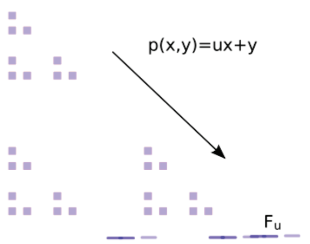


Figure: Figure is stolen from a talk of Hocham

$$\mathcal{F} := \left\{ \sum_{n=1}^{\infty} (i_n, j_n) \cdot 3^{-n} : (i_n, j_n) \in \{(0,0), (1,0), (0,1)\} \right\}$$

The orthogonal projection to a line with slope $-1/t$ is up to a linear coordinate change is
$$p_t(x,y) = tx + y$$
Under this projection the projected IFS on the line is
$$\mathcal{F}_t := \left\{ f_1(x) = \tfrac{1}{3}x, f_2(x) = \tfrac{1}{3}(x+1), f_3(x) = \tfrac{1}{3}(x+t). \right\}$$
Let Λ_t be the attractor of $\mathcal{F}_t$.

Clearly the similarity dimension $s(\mathcal{F}_t) = 1$. By a Theorem of Marstrand $\dim_{\mathrm{H}}(\Lambda_t) = 1$ holds for Lebesgue almost all t. Kenyon proved that the same holds for a G_δ and dense subset of t and also described the set of rational t for which $\dim_{\mathrm{H}}(\Lambda_t) = 1$.
It has been an open conjecture of Frurstenberg sinse 1970s if
$$t \text{ irrational } \Rightarrow \dim_{\mathrm{H}}(\Lambda_t) = 1?$$
Using his theorem above Hochman proved this conjecture.

Hochman I

Let $I \subset \mathbb{R}$ be a compact parameter interval and $m \geq 2$. For every parameter $t \in I$ given a self-similar IFS on the line:
$$\Phi_t := \{\varphi_{i,t}(x) = r_i(t) \cdot (x - a_i(t))\}_{i=1}^{m},$$
where
$$r_i : I \to (-1,1) \setminus \{0\} \text{ and } a_i : I \to \mathbb{R}$$
are real analytic functions. Let Π_t be the natural projection from $\Sigma := \{1, \dots, m\}^{\mathbb{N}}$ to the attractor Λ_t of Φ_t.

Hochman II

For every probability vector $\mathbf{p} := (p_1, \dots, p_m)$ the associated self-similar measure is

$$\nu_{\mathbf{p},t} := (\Pi_t)_*(\mathbf{p}^{\mathbb{N}}).$$

Its similarity dimension is defined by

$$\dim_{\mathrm{S}}(\nu_{\mathbf{p},t}) := \frac{\sum_{i=1}^{m} p_i \log p_i}{\sum_{i=1}^{m} p_i \log r_i(t)}$$

Hochman III

The similarity dimension of Λ_t is the solution $s(t)$ of

$$r_1^{s(t)}(t) + \cdots + r_m^{s(t)}(t) = 1.$$

We say that a parameter $t \in I$ is exceptional if either $\dim_{\mathrm{H}} \Lambda_t < \min\{1, s(t)\}$ or there exists a probability vector $\mathbf{p} := (p_1, \dots, p_m)$ such that $\dim_{\mathrm{H}}(\nu_{\mathbf{p},t}) < \min\{1, \dim_{\mathrm{S}}(\nu_{\mathbf{p},t})\}$

Hochman IV

Theorem 8.2 (Hochman)

Assume that

$$\textit{if } \Pi_t(\mathbf{i}) = \Pi_t(\mathbf{j}) \textit{ holds for } \textbf{\textit{all}}\ t \in I \textit{ then } \mathbf{i} = \mathbf{j}.$$

Then both the Hausdorff and the packing dimension of the set of exceptional parameters are equal to 0.

Built on Hochman's theorem Pablo Shmerkin has proved very recently a theorem which implies that

Theorem 8.3 (Shmerkin)

The set of exceptional parameters in Solomyak's theorem is has Hausdorff dimension zero.

I will give the sketch of the proof below.

Notation

Let $\mathcal{P}$ be the set of probability measures on $\mathbb{R}$. We write

$$\mathbb{P}_m := \left\{ (p_1, \dots, p_m) : p_i > 0, \sum_{i=1}^{m} p_i = 1 \right\}.$$

Given a self-similar IFS $\mathcal{F} = \{f_1, \dots, f_m\}$ on $\mathbb{R}$. The contraction ratios are $r_1, \dots, r_m$. We write $\Lambda = \Lambda(F)$ for the attractor. We know that

$$\forall \mathbf{p} \in \mathbb{P}_m,\ \exists! \mu = \mu(\mathcal{F}, \mathbf{p}) \text{ s.t. } \mu = \sum_{i=1}^{m} p_i \cdot (f_i)_* \mu,$$

where $(f_i)_* \mu(B) := \mu(f_i^{-1}(B))$.

Notation (Cont.)

We have defined the similarity dimension $s(\mathcal{F})$ of $\mathcal{F}$ as the solution of $\sum_{i=1}^{m} r_i^s = 1$. The similarity dimension of the measure $\mu = \mu(\mathcal{F}, \mathbf{p})$ is defined by

$$s(\mathcal{F}, \mathbf{p}) := \frac{\sum_{i=1}^{m} p_i \log p_i}{\sum_{i=1}^{m} p_i \log r_i}.$$

The lower Hausdorff dimension of the measure μ

$$(34) \quad \dim_{\mathrm{H}} \mu := \underline{\dim}_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}}(B) : \mu(B) > 0\}$$
$$= \operatorname{essinf}_{x \sim \mu} \liminf_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Notation (Cont.)

Clearly,

$$\dim_{\mathrm{H}} \Lambda(\mathcal{F}) \leq s(\mathcal{F}) \text{ and } \dim_{\mathrm{H}} \mu(\mathcal{F}, \mathbf{p}) \leq s(\mathcal{F}, \mathbf{p}).$$

with equality under SSC. The lower correlation dimension of μ is

$$\dim_2 \mu := \liminf_{r \downarrow 0} \frac{\log \int \mu(B(x,r)) d\mu(x)}{\log r}$$

It was proved by Yorke that

Notation (Cont.)

$$(35) \qquad \dim_2 \mu = \sup\{s > 0 : I_s(\mu) < \infty\},$$

where we remind that the s-energy $I_s(\mu)$ was defined as

$$(36) \qquad I_s(\mu) := \iint |x - y|^{-s} d\mu(x) d\mu(y)$$

We can express $I_s(\mu)$ with the Fourier transform

$$(37) \qquad \hat{\mu}(\xi) := \int e^{i\xi x} d\mu(x)$$

of the measure μ as follows:

Notation (Cont.)

(38) $$I_s(\mu) = C(s) \cdot \int |\xi|^{s-1} |\hat{\mu}(\xi)|^2 d\xi.$$

(39)
If $s < \dim_2 \mu$, $\frac{s}{2} < \beta$ then $|\hat{\mu}(\xi)| < |\xi|^{-\beta}$, at "average".

The following Shmerkin Theorem is an improvement of Solomyak's Theorem and it is a very nice application of Hochman's Theorem.

Theorem 8.4 (Shmerkin 2013)

Let $a_1, \dots, a_m$ be distinct numbers and for a $\lambda \in (0,1)$ let
$$\mathcal{F}_\lambda := \{\lambda x + a_1, \dots, \lambda x + a_m\}.$$
then there exists an exceptional set E s.t.

- $\dim_{\mathrm{H}}(E) = 0$ *and*
- *for every $\lambda \in (0,1) \setminus E$ and for every $\mathbf{p} \in \mathbb{P}_m$:*

$$s(\mathcal{F}_\lambda, \mathbf{p}) > 0 \Longrightarrow \mu(\mathcal{F}_\lambda, \mathbf{p}) \ll \mathcal{L}\mathrm{eb}.$$

Note that the exceptional set of λ is the same for all probability vector $\mathbf{p}$.

Definition 8.5 (Power decay of the Fourier transform)

Let

(40) $$\mathcal{D} := \left\{\nu : |\hat{\nu}(\xi)| \le C \cdot |\xi|^{-s} \text{ for some } C, s > 0\right\}.$$

If $\nu \in \mathcal{D}$ then we say that the Fourier transform of μ has a power decay at infinity.

Lemma 8.6

Let $\nu \in \mathcal{D}$ and $\mu \in \mathcal{P}$.

(a) *If $\dim_2 \mu = 1$ then $\nu * \mu \ll \mathcal{L}\mathrm{eb}$ with L^2-density.*

(b) *If $\dim_{\mathrm{H}} \mu = 1$ then $\nu * \mu \ll \mathcal{L}\mathrm{eb}$.*

Proof.

Proof of the Lemma Part (a) By assumption there is an $s > 0$ such that

(41) $$\hat{\nu}(\xi) = \mathcal{O}\left(|\xi|^{-s}\right).$$

Using that $\dim_2 \mu = 1$ we get by (38)

(42) $$1 = \sup\{t \ge 0 : I_t(\mu) < \infty\} = \sup\left\{t \ge 0 : \int |\xi|^{t-1} \cdot |\hat{\mu}|^2 d\xi < \infty\right\}.$$

Let s be as in (41). Chose $1 - \frac{s}{2} < t < 1$. That is $-\frac{s}{2} < t - 1$. Using this and (42) we get □

Proof of the Lemma Part (a) (Cont.)

$$\int |\xi|^{-s/2} \cdot |\hat{\mu}(\xi)|^2 \, d\xi < \infty.$$

We apply this and (41) to get that $\exists K >$ s.t.

(43)
$$\int |\xi|^{s/2} \cdot |\widehat{\nu * \mu}(\xi)|^2 d\xi = \int \underbrace{|\xi|^s \cdot |\hat{\nu}(\xi)|^2}_{\le K \text{ by } (41)} \cdot |\hat{\mu}(\xi)|^2 \cdot |\xi|^{-s/2} dx \le K \cdot \int |\hat{\mu}(\xi)|^2 \cdot |\xi|^{-s/2} d\xi < \infty.$$

That is $\widehat{\nu * \mu} \in L^2(\mathbb{R})$ that is $\nu * \mu \ll \mathcal{L}\mathrm{eb}$ with L^2 density. This completes the proof of part (a).

Proof of the Lemma Part (b)

We use Egorov Theorem for the second line of (34). This yields that $\forall \varepsilon > 0$, $\exists$ a constant $C_\varepsilon > 0$ and set A_ε with $\mu(A_\varepsilon) > 1 - \varepsilon$ s.t. for

$$\mu_\varepsilon := \frac{\mu|_{A_\varepsilon}}{\mu(A_\varepsilon)}$$

we have

$$\mu_\varepsilon(B(x,r)) \le C_\varepsilon \cdot r^{1-s/4}, \quad \forall x \in A_\varepsilon.$$

In this way $\dim_2 \ge 1 - \frac{s}{4}$. ($s$ is from (41)). Then the same argument as above shows that $\nu * \mu_\varepsilon \ll \mathcal{L}\mathrm{eb}$. Letting $\varepsilon \downarrow 0$ finishes the proof of part (b).

It was known known already by Erdős and Kahane that the Bernoulli convolutions are in $\mathcal{D}$ apart from a zero-dimensional set of parameters. Now we prove a little bit more than that. First we start with a proposition which is proved in [6, Proposition 6.1]

Proposition 8.7

Let

(44) $$G_\ell := \left\{\theta > 1 : \liminf_{N \to \infty} \frac{1}{N} \min_{t \in [1,\theta]} \left|\left\{n \in \{0, \dots, N-1\} : \|t\theta^n\| \ge \frac{1}{\ell}\right\}\right| > \frac{1}{\ell}\right\},$$

where $\|x\|$ is the distance of x from the closest integer. Then for any $1 < \Theta_1 < \Theta_2 < \infty$ there is a $C = C(\Theta_1, \Theta_2) > 0$ s.t.

(45) $$\dim_{\mathrm{H}}([\Theta_1, \Theta_2] \setminus G_\ell) \le \frac{C \log(C\ell)}{\ell}.$$

The following result is due to T. Watenabe:

Proposition 8.8

$\exists E \subset (0,1)$, with $\dim_{\mathrm{H}} E = 0$ s.t.

$$\forall \lambda \in (0,1) \setminus E, \ \forall \mathbf{p} \in \mathbb{P}_m, \ \forall \text{ distinct } a_1, \dots, a_m \in \mathbb{R}$$

if $\mathcal{F} := (\lambda x + a_1, \dots, \lambda x + a_m)$ then $\mu(\mathcal{F}, \mathbf{p}) \in \mathcal{D}$.

Proof of the Proposition 8.8 .

Let G_ℓ be as in formula (44). We write

$$E := \left\{ \lambda : \lambda^{-1} \in \left((1,\infty) \setminus \bigcup_{\ell \in \mathbb{N}} G_\ell \right) \right\}.$$

Proof of the Proposition 8.8 (Cont.)

Then by Proposition 8.7 we have $\dim_{\mathrm{H}} E = 0$. Fix an $\lambda \in (0,1) \setminus E$ and we also fix distinct $a_1, \dots, a_m \in \mathbb{R}$ and a $\mathbf{p} \in \mathbb{P}_m$. WLOG we may assume that $a_1 = 0$ and $a_2 = 1$. Let $\mathcal{F} := (\lambda x + a_1, \dots, \lambda x + a_m)$ and $\mu = \mu(\mathcal{F}, \mathbf{p})$.

It is easy to see that

$$\hat{\mu}(\xi) = \prod_{n=0}^{\infty} \Phi(\lambda^n \xi),$$

where

$$\Phi(\zeta) = \sum_{i=1}^{m} p_j \cdot \exp(i\pi a_j \zeta).$$

Proof of the Proposition 8.8 (Cont.)

By assumption $\exists \ell$ s.t.

(46)

$$\liminf_{N\to\infty} \frac{1}{N} \min_{t \in [1,\lambda^{-1}]} \left| \left\{ n \in \{0, \dots, N-1\} : \left\| \frac{t}{\lambda^n} \right\| \geq \frac{1}{\ell} \right\} \right| > \frac{1}{\ell}.$$

Using the definition of Φ and the normalization ($a_1 = 0, a_2 = 1$) we obtain that there is $\delta > 0$ s.t.

$$\|\zeta\| > \frac{1}{\ell} \Longrightarrow |\Phi(\zeta)| \leq 1 - \delta.$$

Proof of the Proposition 8.8 (Cont.)

For $\xi = \frac{t}{\lambda^N}$ and N large enough, for $s := \frac{\log(1-\delta)}{\ell \log \lambda} > 0$ we have

$$|\hat{\mu}(\xi)| \leq \prod_{i=1}^{N-1} \left| \Phi\left(\frac{t}{\lambda^n}\right) \right| \leq (1-\delta)^{N/\ell} = \mathcal{O}(|\xi|^{-s}). \ \blacksquare$$

Now we are ready to prove Theorem 8.4. Recall that by Hochman Theorem:

(47) $$\dim_{\mathrm{H}} \mu(\mathcal{F}_\lambda, \mathbf{p}) = \min\{1, s(\mathcal{F}_\lambda, \mathbf{p})\}$$

The attractor of $\mathcal{F}_\lambda$ is

(48) $$\Lambda_\lambda = \left\{ \sum_{i=0}^{\infty} a_i \lambda^i, \ a_i \in \{1, \dots, m\} \right\}.$$

We can think of this for a moment as a formal collection of countably many infinite sums. Assume that we cancel every k-th term of all of these sums.

Then we get a collection of infinite sums which corresponds in the same way to anther IFS. Namely it corresponds to

(49) $$\mathcal{F}_\lambda^{(k)} := \left\{ \lambda^k x + \sum_{j=0}^{k-2} a_{i_{j+1}} \lambda^j \right\}_{(i_1,\dots,i_{k-1}) \in \{1,\dots,m\}^{k-1}}.$$

The corresponding probability vector is

(50) $$\mathbf{p}^{(k)} = (p_{i_1} \cdots p_{i_{k-1}})_{(i_1,\dots,i_{k-1}) \in \{1,\dots,m\}^{k-1}}.$$

The weighted IFS $\left(\mathcal{F}^{(k)}, \mathbf{p}^{(k)}\right)$ is called "skipping every k-th digit IFS".

Properties of $\left(\mathcal{F}^{(k)}, \mathbf{p}^{(k)}\right)$

(a) $s\left(\mathcal{F}^{(k)}, \mathbf{p}^{(k)}\right) = \left(1 - \frac{1}{k}\right) s(\mathcal{F}, \mathbf{p})$.

(b) The family $\left\{\mathcal{F}_\lambda^{(k)}\right\}$ satisfies the non-degeneracy condition of Hochman's theorem. This is so because for $\mathbf{i}, \mathbf{j} \in \Sigma$, $\mathbf{i} \neq \mathbf{j}$ we have:

$$\Pi^{(k)}(\mathbf{i}) - \Pi^{(k)}(\mathbf{j})$$

is a non-trivial power series with bounded coefficients.

Properties of $\left(\mathcal{F}^{(k)}, \mathbf{p}^{(k)}\right)$ (Cont.)

(c)

$$\mu(\mathcal{F}_\lambda, \mathbf{p}) = \mu(\mathcal{F}_{\lambda^k}, \mathbf{p}) * \mu\left(\mathcal{F}_\lambda^{(k)}, \mathbf{p}^{(k)}\right).$$

This follows from the fact that the power series which appear in (48) consist of summands corresponding to i which are divisible with k and i which are not divisible with k. The sum can be considered as the sum of independent andom variables and therefore the distribution of the sum is the convolution of tghe distributions.

It follows from (a) and (b) above and from Hochman Theorem that $\exists E_k$ with $\dim_{\mathrm{H}} E_k = 0$, s.t. if $\lambda \in (0,1) \setminus E_k$ and $s(\mathcal{F}_\lambda, \mathbf{p}) > \frac{k}{k-1}$ (so by (a), $s\left(\mathcal{F}^{(k)}, \mathbf{p}^{(k)}\right) > 1$) then

$$\dim_{\mathrm{H}} \mu\left(\mathcal{F}_\lambda^{(k)}, \mathbf{p}^{(k)}\right) = 1. \quad (51)$$

Let $\widetilde{E}$ be the exceptional set in Proposition 8.8. Put

$$E'_k := \left\{\lambda : \lambda^k \in \widetilde{E}\right\}.$$

Clearly, $\dim_{\mathrm{H}} E'_k = 0$.

From (c) above and Lema 8.6 we obtain that

$$\lambda \in ((0,1) \setminus (E'_k \cup E_k)) \,\&\, s(\mathcal{F}_\lambda, \mathbf{p}) > 1 + \frac{1}{k} \Longrightarrow \mu(\mathcal{F}_\lambda, \mathbf{p}) \ll \mathcal{L}\mathrm{eb}.$$

This yields the assertion of Shmerkin theorem, where the exceptional set is

$$E := \bigcup_{k=1}^{\infty} (E_k \cup E'_k).$$

Shmerkin-Solomyak Theorem (2014)

Let $\mathbf{u} \mapsto (\Lambda_{\mathbf{u}}, a_{\mathbf{u}})$ be real-analitic from $\mathbb{R}^\ell \supset U \to (0,1) \times \mathbb{R}^m$. such that the following non-degeneracy condition holds:

$$\forall \mathbf{i} \neq \mathbf{j}, \mathbf{i}, \mathbf{j} \in \Sigma \ \exists u, \text{ s.t. } \Pi^{\mathbf{u}}(\mathbf{i}) \neq \Pi^{\mathbf{u}}(\mathbf{j}),$$

where $\Pi^{\mathbf{u}}$ is the natural proj. that corresponds to $\mathcal{F}_{\mathbf{u}} := (\lambda_{\mathbf{u}} x + a_{\mathbf{u},i})_{i=1,\dots,m}$. Assume that $\mathbf{p} = (p_1, \dots, p_m)$ is a probability measure such that the similarity dimension is grater than 1. Then for all but a set Hausdorff dimension zero parameters the self-similar meausre associated to $(\mathcal{F}_{\mathbf{u}}, \mathbf{p})$ is absolute continuous w.r.t. the Lebesgue measure with L^q , $q = q(\mathbf{u}, \mathbf{p}) > 1$ density.

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