Bedford-McMullen carpets - an example in dimension theory of self-affine sets

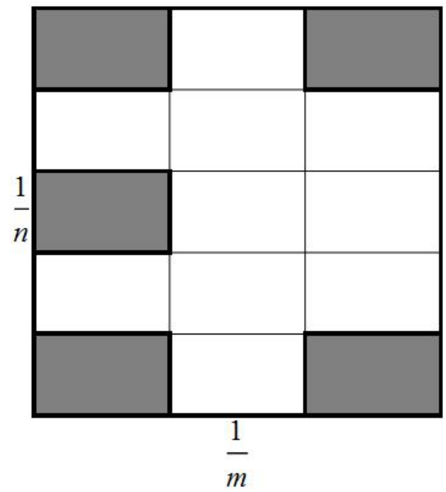
In this note we study the dimension of the classical Bedford-McMullen carpets. We apply the learned methods from the mini-course of Károly Simon on "Dimension Theory of self-affine and almost self-affine sets and measures" during Simons Semester in Banach Center, October 2015.

Let $n > m \geq 2$ be integers and let $A \subseteq \{0, \dots, m-1\} \times \{0, \dots, n-1\}$. We consider the following subset of $[0,1]^2$

$$\Lambda := \left\{ \begin{pmatrix} \sum_{k=1}^{\infty} \frac{i_k}{m^k} \\ \sum_{k=1}^{\infty} \frac{j_k}{n^k} \end{pmatrix} : (i_k, j_k) \in A \text{ for every } k \geq 1 \right\}.$$

It is easy to see that Λ is a self-affine set on the plane, i.e.

$$\Lambda = \bigcup_{(i,j)\in A} f_{(i,j)}(\Lambda), \text{ where } f_{(i,j)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{i}{m} \\ \frac{j}{n} \end{pmatrix}.$$



Introduce some notations:

$$\begin{aligned} M &:= \sharp A \\ Q &:= \{i : \exists j \text{ s.t. } (i,j) \in A\} \\ N &:= \sharp Q \\ T_i &:= \{j : (i,j) \in A\} \\ t_i &:= \sharp T_i \end{aligned}$$

First, let us observe that the projection of Λ to the vertical axes is a self-similar set on the line. Precisely, projΛ is the attractor of the IFS

$$\left\{ g_i : x \mapsto \frac{x+i}{m} \right\}_{i \in Q}, \tag{1}$$

where proj denotes the orthogonal projection to the x-axes.

Falconer showed that the Hausdorff and box dimension of any self-similar set are equal, moreover, since $\left\{g_i : x \mapsto \frac{x+i}{m}\right\}_{i\in Q}$ satisfies the open set condition w.r.t the interval $(0,1)$ we get

$$\dim_H \mathrm{proj}\Lambda = \dim_B \mathrm{proj}\Lambda = \frac{\log N}{\log m}. \tag{2}$$

Our first approach is to calculate the root of the subadditive pressure, i.e. the affinity dimension. Let us recall the definition of the pressure function. Let

$$\phi^s(A) := \begin{cases} \alpha_1(A)^s & \text{if } 0 \le s < 1, \\ \alpha_1(A)\alpha_2(A)^{s-1} & \text{if } 1 \le s < 2, \\ (\alpha_1(A)\alpha_2(A))^{s/2} & \text{if } s \ge 2, \end{cases}$$

where $\alpha_i(A)$ denotes the ith singular value of the matrix A. Then let s_0 be the unique root of the strictly monotone decreasing and continuous function $s \mapsto P(s)$, where

$$P(s) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{(i_1,j_1),\dots,(i_n,j_n)\in A} \phi^s\left(f'_{(i_1,j_1)} \cdots f'_{(i_n,j_n)}\right)$$

the subadditive pressure function. Because of the special form of our matrices, one can show that

$$s_0 = \min\left\{\frac{\log M}{\log m}, 1 + \frac{\log M - \log m}{\log n}\right\}. \tag{3}$$

Falconer showed that this is an upper bound for the box counting dimension for every self-affine set.

Box counting dimension. First, let us define the lower and upper box counting dimension of a bounded set $A \subset \mathbb{R}^2$.

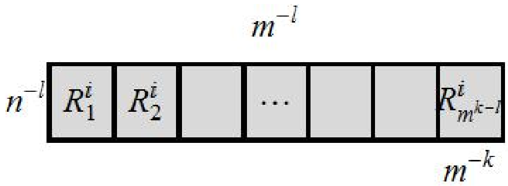
$$\underline{\dim}_B A = \liminf_{\delta\to 0+} \frac{\log N_\delta(A)}{-\log\delta} \text{ and } \overline{\dim}_B A = \limsup_{\delta\to 0+} \frac{\log N_\delta(A)}{-\log\delta},$$

where $N_\delta(A) = \min\left\{N : \exists \underline{x}_1, \dots, \underline{x}_N \in \mathbb{R}^2 \text{ s.t. } A \subseteq \bigcup_{i=1}^N B_\delta(\underline{x}_i)\right\}$ and $B_\delta(\underline{x})$ denotes the closed ball with radius δ and centered at $\underline{x}$. If the limit exists then we denote it by $\dim_B A$.

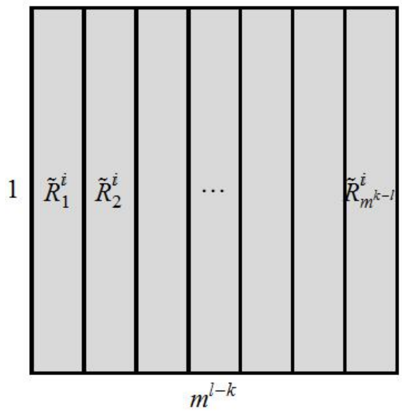
Let us observe that in the liminf and limsup δ can be replaced by any exponential sequence. That is,

$$\underline{\dim}_B A = \liminf_{l\to\infty} \frac{\log N_{1/n^l}(A)}{l\log n} \text{ and } \overline{\dim}_B A = \limsup_{l\to\infty} \frac{\log N_{1/n^l}(A)}{l\log n}.$$

Now, we construct an optimal cover for Λ. Note that for every $l \ge 1$ and every $(i_1,j_1),\dots,(i_l,j_l)$ the rectangles $f_{(i_1,j_1)} \circ \cdots \circ f_{(i_l,j_l)}((0,1)^2)$ are disjoint. Let $k \ge 1$ the smallest integer such that $1/m^k \le 1/n^l$, i.e. $k = \lceil l\frac{\log n}{\log m}\rceil$. Divide the horizontal side (which has length m^{-l}) of the rectangle $f_{(i_1,j_1)} \circ \cdots \circ f_{(i_l,j_l)}((0,1)^2)$ into m^{k-l} equal parts. Denote these "approximate squares" from left to right by $R_1^{\mathbf{i}}, R_2^{\mathbf{i}}, \dots, R_{m^{k-l}}^{\mathbf{i}}$, where $\mathbf{i} = (i_1,j_1),\dots,(i_l,j_l)$. Thus, any of the rectangles has vertical side length n^{-l} and horizontal side length m^{-k}.



In the general case, for the definition of subadditive pressure we used all of the approximate squares to cover the set. In the case of Bedford-McMullen carpet, because of the special structure, we do not need all of them. By applying the inverse function $f^{-1}_{(i_l,j_l)} \circ \cdots \circ f^{-1}_{(i_1,j_1)}$ for the rectangle $f_{(i_1,j_1)} \circ \cdots \circ f_{(i_l,j_l)}((0,1)^2)$ we get the following picture:



where $\widetilde{R}^{\mathbf{i}}_k = f^{-1}_{(i_l,j_l)} \circ \cdots \circ f^{-1}_{(i_1,j_1)}(R^{\mathbf{i}}_k)$. So we see that we may choose only those $\widetilde{R}^{\mathbf{i}}_k$ columns for which $\widetilde{R}^{\mathbf{i}}_k \cap \Lambda \neq \emptyset$. The number of such $\widetilde{R}^{\mathbf{i}}_k$ columns is equal to the number of non-empty columns in the $k-l$th iteration. Or in other words, the number of intervals with lenght m^{l-k} needed to cover projΛ. So by (2), for every $\mathbf{i} \in A^l$

$$\sharp \left\{ R^{\mathbf{i}}_k : R^{\mathbf{i}}_k \cap \Lambda \neq \emptyset \text{ for } k = 1, \ldots, m^{k-l} \right\} = N^{k-l}.$$

Therefore

$$\widetilde{N}_l := \left\{ R^{\mathbf{i}}_k : \mathbf{i} \in A^l \ \& \ R^{\mathbf{i}}_k \cap \Lambda \neq \emptyset \text{ for } k = 1, \ldots, m^{k-l} \right\} \text{ and } \sharp \widetilde{N}_l = M^l N^{k-l}.$$

Since every rectangle $R^{\mathbf{i}}_k \in \widetilde{N}_l$ can be extended to a ball with radius $1/n^l$. Thus, $\sharp \widetilde{N}_l \geq N_{1/n^l}(\Lambda)$ and

$$\overline{\dim}_B \Lambda \leq \limsup_{l\to\infty} \frac{\log M^l N^{k-l}}{l \log n} =$$

$$\limsup_{l\to\infty} \frac{\log M}{\log n} + \left(\frac{\lceil l \frac{\log n}{\log m} \rceil}{l} - 1 \right) \frac{\log N}{\log n} = \frac{\log M}{\log n} + \left(1 - \frac{\log m}{\log n} \right) \frac{\log N}{\log m}.$$

On the other hand, let $\mathcal{B}_l := \left\{ B_{1/n^l}(\underline{x}_1), \ldots, B_{1/n^l}(\underline{x}_{N_{1/n^l}(\Lambda)}) \right\}$ be the set of balls which covers optimally the set Λ. Then for every $R^{\mathbf{i}}_k \in \widetilde{N}_l$ intersects at least one $B \in \mathcal{B}_l$, moreover, any ball $B \in \mathcal{B}_l$ may intersect at most $3m$ approximate squares from $\widetilde{N}_l$. Hence,

$$\sharp \widetilde{N}_l \leq 3m N_{1/n^l}(\Lambda)$$

and therefore,

$$\underline{\dim}_B \Lambda \geq \liminf_{l\to\infty} \frac{\log (3m)^{-1} M^l N^{k-l}}{l \log n} = \frac{\log M}{\log n} + \left(1 - \frac{\log m}{\log n} \right) \frac{\log N}{\log m}.$$

In summary, we get that the box counting dimension exists and

$$\dim_B \Lambda = \frac{\log M}{\log n} + \left(1 - \frac{\log m}{\log n} \right) \frac{\log N}{\log m}. \tag{4}$$

Remark 1. *By using the formulas* (3) *and* (4)*, simple algebraic calculations show that*

$$s_0 = \dim_B \Lambda \ \Leftrightarrow\ N = M \ \text{or}\ N = m.$$

In other words, by (2)*, the box dimension is equal to the affinity dimension if and only if*

$$\dim_B \mathrm{proj}\Lambda = \min\{1, \dim_B \Lambda\}.$$

Lower bound for Hausdorff dimension. Now, we turn to the case of Hausdorff dimension. Let μ be a self-affine measure with probability vector $\underline{p} = (p_{(i,j)})_{(i,j)\in A}$. That is, μ is the unique compactly supported measure, for which

$$\int h(x)d\mu(x) = \sum_{(i,j)\in A} p_{(i,j)} \int h(f_{(i,j)}(x))d\mu(x),$$

for any continuous test function h on Λ.

Let us observe again, that the projection of the measure μ onto the x-axes is a self-similar measure (like in the case of the set). That is,

$$\mathrm{proj}_*\mu = \sum_{i\in Q}\left(\sum_{j\in T_i} p_{i,j}\right)\mathrm{proj}_*\mu\circ g_i^{-1},$$

where $\mathrm{proj}_*\mu = \mu\circ\mathrm{proj}^{-1}$ and g_is are from the IFS (1).

By the Feng-Hu formula, we are able to calculate the Hausdorff dimension of the measure μ, i.e.

$$\dim_H \mu = \frac{-\sum_{(i,j)\in A} p_{i,j}\log p_{i,j}}{\log n} + \left(1 - \frac{\log m}{\log n}\right)\frac{-\sum_{i\in Q}\sum_{j\in T_i} p_{i,j}\log\sum_{j\in T_i} p_{i,j}}{\log m}. \tag{5}$$

For simplicity, let us denote $\sum_{j\in T_i} p_{i,j}$ by q_i.

Remark 2. *The formulas* (4) *and* (5) *are very similar to each other. By the definition of the entropy,*

$$\log M = -\sum_{(i,j)\in A} p_{i,j}\log p_{i,j} \Leftrightarrow p_{i,j} = \frac{1}{M}$$

and

$$\log N = -\sum_{i\in Q} q_i \log q_i \Leftrightarrow q_i = \frac{1}{N}.$$

Thus, $\dim_B \Lambda = \dim_H \mu$ *for a probability vector* $\underline{p} = (p_{(i,j)})_{(i,j)\in A}$ *if and only if* $p_{i,j} = 1/M$ *for every* $(i,j)\in A$ *and* $t_i = M/N$ *for every* $i\in Q$*.*

By definition, $\dim_H \Lambda \geq \dim_H \mu$ therefore to get a lower bound, we maximize the value of (5). Use the method of Lagrange-multipliers! That is, we maximize the function

$$d(\underline{p},\lambda) = \dim_H \mu + \lambda(\sum_{(i,j)\in A} p_{i,j} - 1).$$

It is easy to see that $d(\underline{p},\lambda)$ is concave. By taking the derivative w.r.t $p_{i,j}$ we get

$$\frac{-\log p_{i,j} - 1}{\log n} + \left(1 - \frac{\log m}{\log n}\right)\frac{-\log\sum_{j\in T_i} p_{i,j} - 1}{\log m} + \lambda = 0.$$

Thus, $p_{i,j} = q_i/t_i$ for every $(i,j)\in A$ (for fixed i it the value is independent of j.) Thus, it is enough to maximize the function

$$(\underline{q},\lambda) \mapsto \frac{-\sum_{i\in Q} q_i\log q_i/t_i}{\log n} + \left(1 - \frac{\log m}{\log n}\right)\frac{-\sum_{i\in Q} q_i\log q_i}{\log m} + \lambda(\sum_{i\in Q} q_i - 1).$$

By taking the derivative w.r.t q_i we get

$$\frac{\log q_i/t_i - 1}{\log n} + \left(1 - \frac{\log m}{\log n}\right)\frac{-\log q_i - 1}{\log m} + \lambda = 0,$$

Thus,

$$q_i = \frac{t_i^{\frac{\log m}{\log n}}}{\sum_{i'\in Q} t_{i'}^{\frac{\log m}{\log n}}} \text{ and } p_{i,j} = \frac{t_i^{\frac{\log m}{\log n}-1}}{\sum_{i'\in Q} t_{i'}^{\frac{\log m}{\log n}}}. \tag{6}$$

Hence, we get

$$\dim_H \Lambda \geq \frac{\sum_{i\in Q} t_i^{\frac{\log m}{\log n}}}{\log m}. \tag{7}$$

Upper bound for Hausdorff dimension. Our claim is that the lower bound in (7) is sharp. One way to show that is to find an optimal cover for the set Λ. However, our natural cover, which was constructed to calculate the box dimension, is not optimal if there is an $i \in Q$ such that $t_i \neq M/N$, see Remark 2. Therefore, we use here a mass distribution principle.

Lemma 1. *Let ν be a probability measure on a set $B \subset \mathbb{R}^d$ such that $\nu(B) = 1$ and*

$$\liminf_{r\to 0+} \frac{\log \nu(B_r(x))}{\log r} \leq \alpha \textit{ for } \textbf{\textit{every}} \ x \in B,$$

where $B_r(x)$ denotes the ball centered at x with radius r. Then $\dim_H B \leq \alpha$.

Proof. Let us recall here the definition of Hausdorff measure, i.e.

$$\mathcal{H}^s_\delta(B) = \inf\left\{\sum_i |U_i|^s : B \subseteq \bigcup_i U_i \ \& \ |U_i| < \delta\right\} \text{ and } \mathcal{H}^s(B) = \sup_{\delta>0} \mathcal{H}^s_\delta(B).$$

By our assumption, for every $\varepsilon, \delta > 0$ and every $x \in B$ there exists $\delta > R(x) > 0$ such that

$$\nu(B_{R(x)}(x)) \geq R(x)^{\alpha+\varepsilon}.$$

Since $\bigcup_{x\in B} B_{R(x)}(x)$ is a cover of B, by Besicovitch's covering theorem we get that there exists a $c > 0$ and countable subsets $\mathcal{B}_j$, $j = 1, \dots, c$, of the family of balls $\left\{B_{R(x)}(x)\right\}_{x\in B}$ such that

$$\bigcup_{j=1}^{c} \bigcup_{U\in\mathcal{B}_j} U \supseteq B \text{ and } U \cap U' = \emptyset \text{ for every } U \neq U' \in \mathcal{B}_j.$$

Thus,

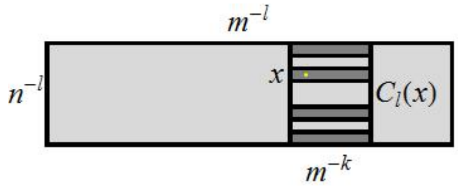
$$\mathcal{H}^{\alpha+\varepsilon}_\delta(B) \leq \sum_{j=1}^{c} \sum_{U\in\mathcal{B}_j} |U|^{\alpha+\varepsilon} \leq \sum_{j=1}^{c} \sum_{U\in\mathcal{B}_j} \nu(U) = c \text{ and therefore } \mathcal{H}^{\alpha+\varepsilon}(B) \leq c.$$

Since $\varepsilon > 0$ was arbitrary, the statement follows. ☐

We apply Lemma 1 for the self-affine measure μ with probability vector defined in (6). Let $x = \left(\sum_{k=1}^\infty \frac{i_k}{m^k}, \sum_{k=1}^\infty \frac{j_k}{n^k}\right)^T \in \Lambda$ and let $l \geq 1$ integer. Denote by $C_l(x)$ the following approximate square

$$C_l(x) = \left\{\begin{pmatrix} \sum_{r=1}^\infty \frac{i'_r}{m^r} \\ \sum_{r=1}^\infty \frac{j'_r}{n^r} \end{pmatrix} \in \Lambda : i_p = i'_p \text{ for } p = 1, \dots, k \text{ and } j_q = j'_q \text{ for } q = 1, \dots, l\right\},$$

where $k=\lceil l\frac{\log n}{\log m}\rceil$. In other words, $C_l(x)$ is the union of all kth level cylinder sets $\mathcal{C}_k$ such that $\mathrm{proj}(x)\in\mathrm{proj}\left(f_{(i_1,j_1)}\circ\cdots\circ f_{(i_l,j_l)}(\Lambda)\cap\mathcal{C}_k\right)$.



Like during the calculations of box dimension, $f^{-1}_{(i_l,j_l)}\circ\cdots\circ f^{-1}_{(i_1,j_1)}(C_l(x))$ is the $k-l$th level cylinder set of the IFS $\left\{g_i : x\mapsto\frac{x+i}{m}\right\}_{i\in Q}$, which contains $\mathrm{proj}(f^{-1}_{(i_l,j_l)}\circ\cdots\circ f^{-1}_{(i_1,j_1)}(x))=\sum_{r=1}^{\infty}\frac{i_{l+r}}{m^r}$. That is,

$$f_{(i_1,j_1)}\circ\cdots\circ f_{(i_l,j_l)}\left(\mathrm{proj}^{-1}\left[\sum_{r=1}^{k-l}\frac{i_{l+r}}{m^r},\sum_{r=1}^{k-l}\frac{i_{l+r}}{m^r}+\frac{1}{m^{k-l}}\right]\right)=C_l(x).$$

Therefore, by using the definition of μ

$$\mu(B_{\sqrt{2}/n^l}(x))\geq\mu(C_l(x))=\frac{t_{i_1}^{\frac{\log m}{\log n}-1}\cdots t_{i_l}^{\frac{\log m}{\log n}-1}\cdot t_{i_{l+1}}^{\frac{\log m}{\log n}}\cdots t_{i_k}^{\frac{\log m}{\log n}}}{\left(\sum_{i'\in Q}t_{i'}^{\frac{\log m}{\log n}}\right)^k}$$

Thus,

$$\frac{\log\mu(B_{\sqrt{2}/n^l}(x))}{-l\log n}\leq\frac{\lceil l\frac{\log n}{\log m}\rceil\log m}{l\log n}\cdot\frac{\log\sum_{i'\in Q}t_{i'}^{\frac{\log m}{\log n}}}{\log m}+\frac{-1}{\log n}\left(\frac{1}{\lceil l\frac{\log n}{\log m}\rceil}\sum_{r=1}^{\lceil l\frac{\log n}{\log m}\rceil}\log t_{i_r}-\frac{1}{l}\sum_{r=1}^{l}\log t_{i_r}\right).$$

Hence, if

$$\limsup_{l\to\infty}\frac{\left(\prod_{r=1}^{\lceil l\frac{\log n}{\log m}\rceil}t_{i_r}\right)^{1/\lceil l\frac{\log n}{\log m}\rceil}}{\left(\prod_{r=1}^{l}t_{i_r}\right)^{1/l}}\geq 1 \tag{8}$$

then

$$\liminf_{l\to\infty}\frac{\log\mu(B_{\sqrt{2}/n^l}(x))}{-l\log n}\leq\frac{\log\sum_{i\in Q}t_i^{\frac{\log m}{\log n}}}{\log m}\quad\text{for every } x\in\Lambda$$

and by Lemma 1

$$\dim_H\Lambda\leq\frac{\log\sum_{i\in Q}t_i^{\frac{\log m}{\log n}}}{\log m}. \tag{9}$$

To show (8) holds, we need the following simple lemma:

Lemma 2. *Let $\{a_n\}$ be a sequence of positive real numbers and let $c>1$. If $\limsup_{n\to\infty}\frac{a_{\lceil cn\rceil}}{a_n}<1$ then $\liminf_{n\to\infty}a_n=0$.*

But for every sequence $i_1,i_2,\dots,i_r,\dots$ and every $l\geq 1$

$$\left(\prod_{r=1}^{l}t_{i_r}\right)^{1/l}\geq 1,$$

thus (9) holds.

Remark 3. *Since we have shown that the self-affine measure μ with probabilities defined in* (6) *has maximal dimension, i.e.*

$$\dim_H \mu = \dim_H \Lambda,$$

by Remark 2 we get

$$\dim_B \Lambda = \dim_H \Lambda \Leftrightarrow t_i = \frac{M}{N} \text{ for every } i \in Q.$$

In particular, in that case the $\frac{\log \sum_{i \in Q} t_i^{\frac{\log m}{\log n}}}{\log m}$*-dimensional Hausdorff measure is positive and finite.*