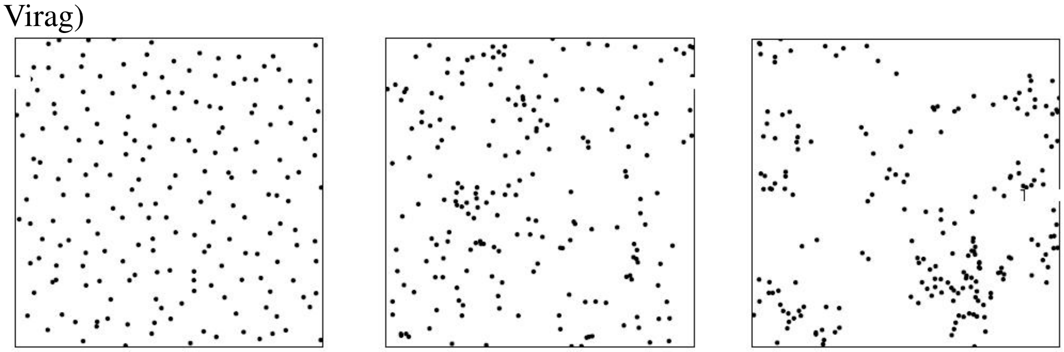
Program of minicourse in Warsaw

The theory of Determinantal Point Processes has recently become an extremely active area of research. This theory was initially introduced by Odile Macchi in mathematic physics for modelizing the random fermion field. After 90's, the determinantal point processes had become an important tool in many areas and combining the research fields like probabilities, combinatorics, representations etc.

Comparing to related point processes as Possion point processes and permenantal point processes, determinantal processes exhibit repulsion between particles. This can be seen from the following figure.

Figure 1: Samples of translation invariant point processes in the plane: determinantal (left), Poisson (center) and permanental (right). (Hough, Krishnapur, Peres, Virag)



Here we would like to mention that the determinantal point processes arise naturally in the following fields:

(1) Asymptotic behavior of the spectrum of large random matrices from GUE (Gaussian Unitary Ensembles): Assume that $X^{(N)} = (X_{ij}^{(N)})_{1\leq i,j\leq N}$ is a random Hermitian matrix whose coefficients are random variables, such that $\{X_{ii}, \sqrt{2}\Re X_{ij}, \sqrt{2}\Im X_{ij} : 1 \leq i < j \leq N\}$ are i.i.d. standard real Gaussian

random variables. A classical theorem in random matrix theory says that the spectrum of $X^{(N)}$ is an N-point determinantal point process on $\mathbb{R}$. The spectrum, after right scaling, converges in distribution to the Dyson sine kernel determinantal point process.

(2) Representation theory, random Young diagrams: Let $\mathbb{Y}_n$ be the set of Young diagrams of size n. Any diagram in $\mathbb{Y}_n$ is parametrized by a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ of n, so that $|\lambda| := \sum_i \lambda_i = n$. Given a random partition chosen from $\mathbb{Y} = \bigcup_{n\in\mathbb{N}} \mathbb{Y}_n$ with respect to the Poissonized Plancherel measure

$$\mathrm{Prob}\{\lambda\} = e^{-\theta^2} \left(\frac{\theta^{|\lambda|} \dim \lambda}{|\lambda|!} \right)^2,$$

where $\dim \lambda$ stands for the dimension of the irreducible representation of the symmetric group $S(n)$ parametrized by λ. Then the modified Frobenius coordinates $\mathrm{Fr}(\lambda) := \{\lambda_i - i + \frac{1}{2}\}$ is a determinantal point process on $\mathbb{Z} + \frac{1}{2}$.

(3) Uniform spanning trees in graphs: Consider $G = (V, E)$ a finite undirected connected graph. Let T be the random spanning tree uniformly chosen from the set of spanning trees of of G. Then by identifying any spanning tree with its set of edges, T becomes a determinantal point process on E.

(4) Zero set of Gaussian holomorphic function defined on hyperbolic disc: Let $(g_n)_{n\geq 0}$ be i.i.d. complex standard normal random variable, then almost surely, we can define a holomorphic function on the unit disc $\mathbb{D}$ by

$$f(z) = \sum_{n=0}^{\infty} g_n z^n.$$

By a famous result of Peres-Virág, the set of zeros of f form a determinantal point process governed by the Bergman kernel.

We plan to give an elementary introduction to determinantal point processes and introduce some recent developpements in this area. In particular, during our lectures, we will talk about the quasi-invariant action of the infinite symmetric group $S(\infty)$ on the determinantal point process on the lattice $\mathbb{Z}$.