Near-Parabolic renormalization; hyperbolicity and rigidity

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There are several notions of renormalization in complex dynamics:

- Polynomial-like renormalization
- Commuting-pair renormalization
- Cylinder renormalization
- Sector renormalization
- Near-parabolic renormalization

On circle:

- renormalization of critical circle maps
- renormalization of critical circle covers
- renormalization of Henon maps

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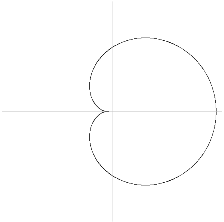
Here, we focus on near-parabolic renormalizations!

There is an explicit Jordan domain $U \subset \mathbb{C}$ bounded by an analytic curve:



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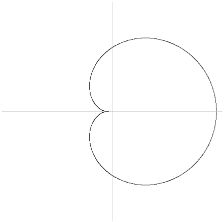

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Let

$$P(z) = z(1+z)^2.$$

- $P(0) = 0$ and $P'(0) = 1$,
- $P'(-1) = P'(-1/3) = 0$; $P(-1) = 0$ and $P(-1/3) = -4/27 \in U$.

$P : U \to P(U)$ has a particular covering structure.

Let $\mathcal{F}$ be the set of maps

$$h = P \circ \psi^{-1}$$

where

- $\psi : U \to \mathbb{C}$ is univalent and has quasi-conformal extension onto $\mathbb{C}$,
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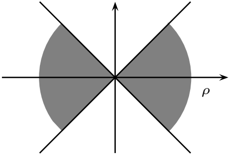
It follows that

- h is defined on $\psi(U)$,
- $h(0) = 0$, $h'(0) = 1$,
- h has a critical point at c.p. $= \psi(-1/3)$ which is mapped to $-4/27$,
- $h : \psi(U) \to P(U)$ has the same covering structure as the one of P.

Let $A_\rho = \{\alpha \in \mathbb{C} \mid 0 < |\alpha| \le \rho, |\operatorname{Im}\alpha| \le |\operatorname{Re}\alpha|\}$,



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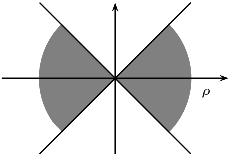

For $\alpha \in A_\rho$ and $h \in \mathcal{F}$, let

$$(\alpha \ltimes h)(z) = h(e^{2\pi i \alpha} z).$$

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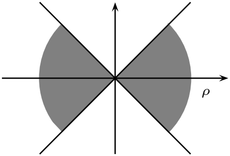

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Set

$$A_\rho \ltimes \mathcal{F} = \{(\alpha \ltimes h) \mid \alpha \in A_\rho, h \in \mathcal{F}\}$$

We equip $A_\rho \ltimes \mathcal{F}$ with the topology of uniform convergence on compact sets.

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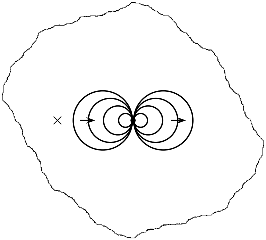
Since

$$\mathcal{F} \hookrightarrow \{\phi : \mathbb{D} \to \mathbb{C} \mid \phi(0) = 0, \phi'(0) = 1\}$$

by Koebe distortion theorem, $\mathcal{F}$ forms a pre-compact class of maps.

Dynamics of a map $h \in \mathcal{F}$;

h has a parabolic fixed point at 0; the orbit of c.p. tends to 0.



If ρ is small enough, $\alpha \ltimes h$ has two preferred fixed points at 0 and $\sigma = \sigma(\alpha \ltimes h)$. $|\sigma| = O(|\alpha|)$.

We have

$$(\alpha \ltimes h)'(0) = e^{2\pi i\alpha}, \qquad (\alpha \ltimes h)'(\sigma) = e^{2\pi i\beta}$$

where β is a complex number with $-1/2 < \operatorname{Re}\beta \leq 1/2$.

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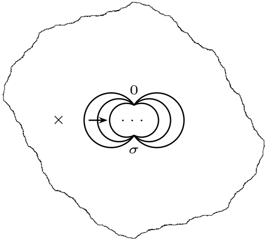


There is a simply connected region

$$\mathcal{P}_{\alpha\ltimes h} \subset \mathrm{Dom}(h)$$

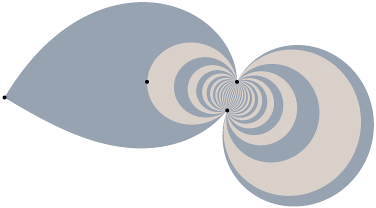
which is bounded by analytic curves landing at 0, σ, and c.p.,

as well as a univalent map

$$\Phi_{\alpha\ltimes h} : \mathcal{P}_{\alpha\ltimes h} \to \mathbb{C}$$

such that

$$\Phi_{\alpha\ltimes h}((\alpha\ltimes h)(z)) = \Phi_{\alpha\ltimes h}(z) + 1, \text{ on } \mathcal{P}_{\alpha\ltimes h}, \quad \Phi_{\alpha\ltimes h}(\text{c.p.}) = 0.$$



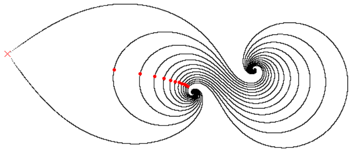
Proposition (Ch. 2009)

One may choose $\mathcal{P}_{\alpha\ltimes h}$ and $\Phi_{\alpha\ltimes h}$ such that

$$\Phi_{\alpha\ltimes h}(\mathcal{P}_{\alpha\ltimes h}) = \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z \le \operatorname{Re} \frac{1}{\alpha} - k_1\}$$

and for $y \ge 0$,

$$\arg \Phi_{\alpha\ltimes h}^{-1}(iy) \simeq -2\pi y \operatorname{Im} \alpha + \arg \sigma + C_{\alpha\ltimes h}.$$



We drop the subscripts $\alpha \ltimes h$ from $\mathcal{P}_{\alpha \ltimes h}$ and $\Phi_{\alpha \ltimes h}$, ...

Define

$$A = \{z \in \mathcal{P} : 1/2 \leq \mathrm{Re}(\Phi(z)) \leq 3/2\,,\ 2 \leq \mathrm{Im}\,\Phi(z)\}$$

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It follows from the work of Inou-Shishikura that there are chains

$$A^k \xrightarrow[1-1]{\alpha\ltimes h} A^{k-1} \xrightarrow[1-1]{\alpha\ltimes h} \dots \xrightarrow[1-1]{\alpha\ltimes h} A^1 \xrightarrow[1-1]{\alpha\ltimes h} A$$

and

$$C^k \xrightarrow[1-1]{\alpha\ltimes h} C^{k-1} \xrightarrow[1-1]{\alpha\ltimes h} \dots \xrightarrow[1-1]{\alpha\ltimes h} C^1 \xrightarrow[2-1]{\alpha\ltimes h} C$$

where A^k and C^k are contained in $\mathcal{P}$.

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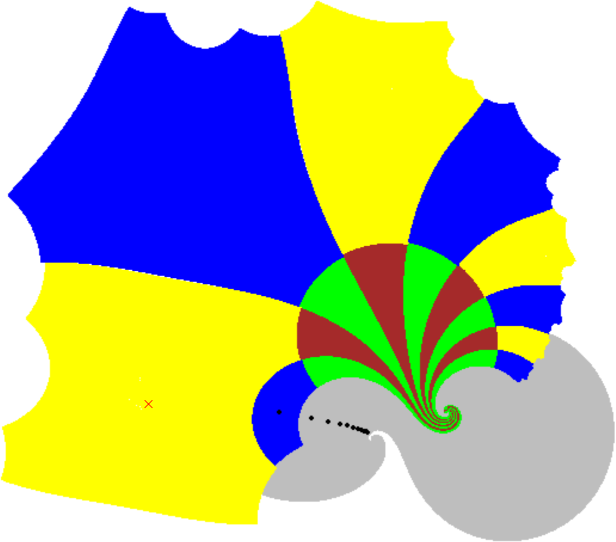
Prop. (Ch.) k is uniformly bounded from above independent of α and h.



Let

$$E = \Phi \circ (\alpha \ltimes h)^{\circ k} \circ \Phi^{-1} : \Phi(A^k \cup C^k) \to \Phi(A \cup C).$$

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E projects under $\mathbb{E}\mathrm{xp}(\zeta) = \frac{-4}{27} e^{2\pi i \zeta}$ to a holomorphic map defined on a punctured neighborhood of 0. That is, there is a map $\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h)$ with

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h) \circ \mathbb{E}\mathrm{xp}(\zeta) = \mathbb{E}\mathrm{xp} \circ E(\zeta)$$

It follows that

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h)(z) \simeq e^{-2\pi i \frac{-1}{\alpha}} z + a_2 z^2 + \dots$$

The above map is called the top near-parabolic renormalization of $\alpha \ltimes h$.

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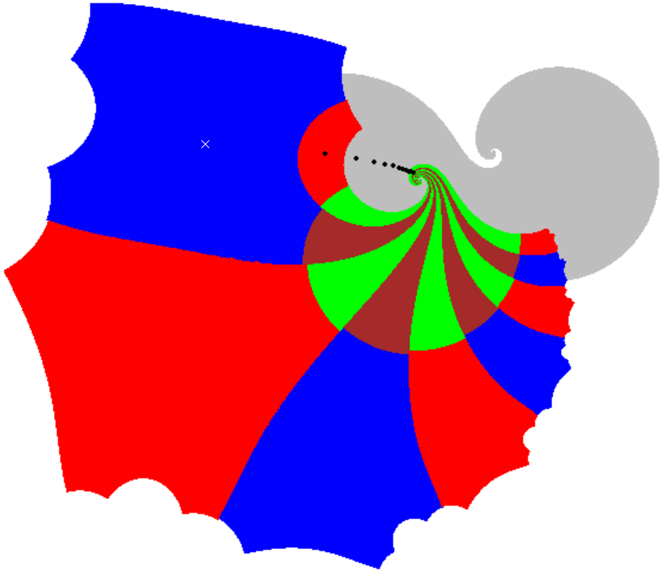
Key point: while the return map may require large number of iterates, renormalization is defined using the composition of $k+2$ maps?

Inou-Shishikura: The above map has the same covering structure as the one of P on U! That is,

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h) \in \{\frac{-1}{\alpha} \mod \mathbb{Z}\} \ltimes \mathcal{F}.$$

There is a similar process to define a "return map" near σ-fixed point: It gives us

$$\mathcal{R}_{\text{NP-b}}(\alpha \ltimes h) \in \{\frac{-1}{\beta} \bmod \mathbb{Z}\} \ltimes \mathcal{F}.$$



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$\alpha \ltimes Q_0$ does not belong to $\alpha \ltimes \mathcal{F}$!

However, $\mathcal{R}_{\text{NP-t}}(\alpha \ltimes Q_0)$ and $\mathcal{R}_{\text{NP-b}}(\alpha \ltimes Q_0)$ are defined in the same fashion, and

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes Q_0) \in \{\frac{-1}{\alpha} \bmod \mathbb{Z}\} \ltimes \mathcal{F},$$

$$\mathcal{R}_{\text{NP-b}}(\alpha \ltimes Q_0) \in \{\frac{-1}{\beta} \bmod \mathbb{Z}\} \ltimes \mathcal{F}.$$