Near-Parabolic renormalization; hyperbolicity and rigidity

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Lecture II:

Hyperbolicity of the near-parabolic renormalization operators

We are interested in the dynamics of the operators

$$\mathcal{R}_{\text{NP-t}}(\alpha \ltimes h) = \hat{\alpha}(\alpha \ltimes h) \ltimes \hat{h}(\alpha \ltimes h)$$

$$\mathcal{R}_{\text{NP-b}}(\alpha \ltimes h) = \check{\alpha}(\alpha \ltimes h) \ltimes \check{h}(\alpha \ltimes h)$$

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acting on $A(\rho) \ltimes \mathcal{F}$ with values in $\mathbb{C} \ltimes \mathcal{F}$.

Also recall that

$$\hat{\alpha}(\alpha \ltimes h) = \frac{-1}{\alpha} \bmod \mathbb{Z}, \qquad \check{\alpha}(\alpha \ltimes h) = \frac{-1}{\beta(\alpha \ltimes h)} \bmod \mathbb{Z}.$$

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$\mathcal{R}_{\text{NP-t}}$ preserves vertical fibers, while $\mathcal{R}_{\text{NP-b}}$ does not preserve them.

$\mathcal{F}$ is equipped with a Teichmüller metric:
for $f = P \circ \varphi^{-1}$ and $g = P \circ \psi^{-1}$ in $\mathcal{F}$,

$$\mathrm{d}_{\mathsf{Teich}}(f, g) = \inf \left\{ \log \mathrm{Dil}(\hat{\psi} \circ \hat{\varphi}^{-1}) \right\}$$

where $\inf$ is taken over all quasi-conformal extensions $\hat{\varphi}$ and $\hat{\psi}$ of φ and ψ onto $\mathbb{C}$.
Here,

$$\mathrm{Dil}(\eta) = \sup_{z \in \mathrm{Dom}\, \eta} \frac{|\eta_z| + |\eta_{\overline{z}}|}{|\eta_z| - |\eta_{\overline{z}}|}.$$

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$A(\rho)$ is equipped with the Euclidean metric.

We wish to understand

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$$D\,\mathcal{R}_{\text{NP-b}} = \begin{bmatrix} \frac{\partial \check{\alpha}}{\partial \alpha} & \frac{\partial \check{h}(\alpha \ltimes h)}{\partial \alpha} \\ \frac{\partial \check{\alpha}}{\partial h} & \frac{\partial \check{h}(\alpha \ltimes h)}{\partial h} \end{bmatrix}$$

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By Royden-Gardiner,

$$\mathrm{d}_{\mathsf{Teich}}(\hat{h}(\alpha \ltimes h_1), \hat{h}(\alpha \ltimes h_2) \leq 1 \cdot \mathrm{d}_{\mathsf{Teich}}(h_1, h_2),$$

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In my symbolic notations, these mean

$$|\frac{\partial \hat{h}(\alpha \ltimes h)}{\partial h}| \leq 1, \quad |\frac{\partial \check{h}(\alpha \ltimes h)}{\partial h}| \leq 1.$$

Recall that

$$\hat{\alpha}(\alpha \ltimes h) = \frac{-1}{\alpha} \bmod \mathbb{Z}$$

Then,

$$\frac{\partial \hat{\alpha}}{\partial \alpha} = \frac{1}{\alpha^2}$$

and

$$\frac{\partial \hat{\alpha}}{\partial h} = 0.$$

Recall that

$$\check{\alpha}(\alpha \ltimes h) = \frac{-1}{\beta(\alpha \ltimes h)} \mod \mathbb{Z}, \qquad (\alpha \ltimes h)'(\sigma) = e^{2\pi i\beta}.$$

we need

$$\frac{\partial \check{\alpha}(\alpha \ltimes h)}{\partial \alpha}, \quad \frac{\partial \check{\alpha}(\alpha \ltimes h)}{\partial h}$$

Proposition $\exists$ a Jordan domain $W \ni 0$, independence of α and h, such that every $\alpha \ltimes h \in A_\rho \ltimes \mathcal{F}$ has only two fixed points 0 and $\sigma(\alpha \ltimes h)$ in W.

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Then,

$$I(\alpha \ltimes h) := \frac{1}{2\pi i} \int_{\partial W} \frac{1}{z - (\alpha \ltimes h)(z)}\, dz = \frac{1}{1 - e^{2\pi i \alpha}} + \frac{1}{1 - e^{2\pi i \beta}}.$$

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$$B_3^{-1}|\alpha| \le |\beta(\alpha \ltimes h_1)| \le B_3|\alpha|, \quad B_4^{-1} \le |\frac{\partial\beta}{\partial\alpha}(\alpha \ltimes h_1)| \le B_4.$$

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and hence

$$\frac{\partial \check{\alpha}(\alpha \ltimes h)}{\partial \alpha} = \frac{1}{\beta^2} \cdot \frac{\partial \beta}{\partial \alpha} \simeq \frac{1}{\alpha^2}.$$

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$$|I(\alpha \ltimes h_1) - I(\alpha \ltimes h_2)| \leq B_5 \, \mathrm{d}_{\mathsf{Teich}}(h_1, h_2).$$

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Hence,

$$\begin{aligned} |\check{\alpha}(\alpha \ltimes h_1) - \check{\alpha}(\alpha \ltimes h_2)| &= \left| \frac{-1}{\beta(\alpha \ltimes h_1)} + \frac{1}{\beta(\alpha \ltimes h_2)} \right| \\ &\leq \frac{B_3^2}{|\alpha|^2} |\beta(\alpha \ltimes h_1) - \beta(\alpha \ltimes h_2)| \\ &\leq \frac{B_3^2}{|\alpha|^2} B_6 |\alpha^2| \, \mathrm{d}_{\mathsf{Teich}}(h_1, h_2) = B_3^2 B_6 \, \mathrm{d}_{\mathsf{Teich}}(h_1, h_2). \end{aligned}$$

In my symbolic notation, the previous bound means

$$\left|\frac{\partial\check{\alpha}(\alpha \ltimes h)}{\partial h}\right| \leq B_3^2 B_6$$

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To control these maps, one needs to understand how the Fatou coordinate $\Phi_{\alpha \ltimes h}$ depends on α, and how the renormalization is defined!

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Theorem (Ch. 2015)

There is $L > 0$ such that for every $h \in \mathcal{F}$ the maps

$$\alpha \mapsto \hat{h}(\alpha, h) \text{ and } \alpha \mapsto \check{h}(\alpha, h)$$

are L-Lipschitz with respect to d_{Eucl} on $A(\rho)$ and d_{Teich} on $\mathcal{F}$.

Recall

$$D\,\mathcal{R}_{\text{NP-t}} = \begin{bmatrix} \frac{\partial \hat{\alpha}}{\partial \alpha} & \frac{\partial \hat{h}(\alpha \ltimes h)}{\partial \alpha} \\ \frac{\partial \hat{\alpha}}{\partial h} & \frac{\partial \hat{h}(\alpha \ltimes h)}{\partial h} \end{bmatrix} \qquad D\,\mathcal{R}_{\text{NP-b}} = \begin{bmatrix} \frac{\partial \check{\alpha}}{\partial \alpha} & \frac{\partial \check{h}(\alpha \ltimes h)}{\partial \alpha} \\ \frac{\partial \check{\alpha}}{\partial h} & \frac{\partial \check{h}(\alpha \ltimes h)}{\partial h} \end{bmatrix}$$

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Combining the previous bounds:

$$\left| D\,\mathcal{R}_{\text{NP-t}} \right| \simeq \begin{bmatrix} \frac{1}{\alpha^2} & L \\ 0 & 1 \end{bmatrix} \qquad \left| D\,\mathcal{R}_{\text{NP-b}} \right| \simeq \begin{bmatrix} \frac{1}{\alpha^2} & L \\ C & 1 \end{bmatrix}$$

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What does this imply?

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For $k > 0$, we say that Υ is k-*horizontal*, if Υ is continuous on Δ, and for all $s_1, s_2 \in \Delta$ we have

$$\mathrm{d}_{\mathsf{Teich}}(h(s_1), h(s_2)) \leq k|\alpha(s_1) - \alpha(s_2)|.$$

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Theorem

There are $\rho' > 0$ and $k > 0$ such that for every k-horizontal curve Υ in $A_{\rho'} \ltimes \mathcal{F}$, the curves $\mathcal{R}_{\text{NP-t}}(\Upsilon)$ and $\mathcal{R}_{\text{NP-b}}(\Upsilon)$ are k-horizontal in $A_\infty \ltimes \mathcal{F}$.

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In other words, $\mathcal{R}_{\text{NP-t}}$ and $\mathcal{R}_{\text{NP-b}}$ map cone-fields of k_1-horizontal curves into themselves.

Let $\kappa = (\kappa_1, \kappa_2, \kappa_3, \dots) \in \{t, b\}^{\mathbb{N}}$. For $n \geq 1$, consider

$$\Lambda(\langle \kappa_i \rangle_{i=1}^{n}) = \left\{ \alpha \ltimes h \;\middle|\; \mathcal{R}_{\mathrm{NP}\text{-}\kappa_n} \circ \cdots \circ \mathcal{R}_{\mathrm{NP}\text{-}\kappa_1}(\alpha \ltimes h) \text{ is defined} \right\}.$$

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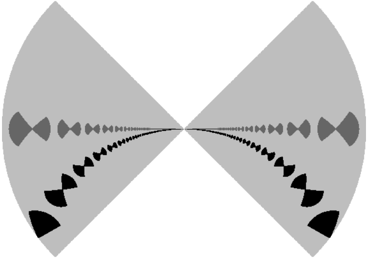
$\Lambda(t, \kappa_2)=$ "dark grey region" $\ltimes \mathcal{F}$; $\Lambda(b, \kappa_2) \simeq$ "black region" $\ltimes \mathcal{F}$:



The invariance of k-horizontal curves implies that

Theorem (Ch. Shishikura 2015)

For all k-horizontal family of maps $\Upsilon : A_{\rho'} \to A_{\rho'} \ltimes \mathcal{F}$ and all $\kappa \in \{t, b\}^{\mathbb{N}}$, every connected component of the set $\Lambda(\kappa) \cap \Upsilon(A_{\rho'})$ is a single point.

It follows from the above Theorem and some more work:

Theorem (Ch., Shishikura 2015)

The renormalizations operators $\mathcal{R}_{\text{NP-t}}$ *and* $\mathcal{R}_{\text{NP-b}}$ *are uniformly hyperbolic on* $A_{\rho'} \ltimes \mathcal{F}_0$.
Moreover, $D\,\mathcal{R}_{\text{NP-t}}$ *and* $D\,\mathcal{R}_{\text{NP-b}}$ *at each point in* $A_{\rho'} \ltimes \mathcal{F}_0$ *have an invariant one-dimensional expanding direction and an invariant uniformly contracting co-dimension-one direction.*

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- the Feigenbaum-Coullet-Tresser *universality of the scaling laws*,
- the geometry of the Mandelbrot set (local-connectivity),
- dynamics of infinitely polynomial-like renormalizable quadratic polynomials with degenerating geometries,