# NON-HYPERBOLIC ERGODIC MEASURES

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Instead of the set of diffeomorphisms that preserve a Lebesgue (or some other fixed) measure, which is the standard setting in smooth ergodic theory, one can consider the set of all invariant ergodic measures for a given map. It is reasonable to expect that many of the properties of a map should be reflected in terms of the properties of its invariant measures. For example, a local diffeomorphism with positive Lyapunov exponents for every invariant ergodic measure must be uniformly expanding [AAS, C]. As another example, if for some diffeomorphism all the atomic measures of all its $C^1$-perturbations have only non-zero Lyapunov exponents, then the diffeomorphism must satisfy Axiom A [A, H].

In [DG] we conjectured that a generic diffeomorphism must either be uniformly hyperbolic or exhibit a non-hyperbolic (having some zero Lyapunov exponents) invariant ergodic measure. This question is closely related to the question on connectedness of the space of ergodic invariant measures of a given diffeomorphism.

Currently there are two methods of construction of ergodic non-hyperbolic measures in smooth dynamics.

First, the method based on periodic approximations was introduced in [GIKN] in the setting of skew-products, and later used to construct open sets of diffeomorphisms with such measures [KN] and applied to generic non-hyperbolic homoclinic classes of diffeomorphisms [DG, BDG]. This approach provides conditions for a sequence of atomic measures to converge to a non-trivial non-hyperbolic ergodic measure.

Second, in [BBD] some $C^1$-open conditions were stated that guarantee that a diffeomorphism possesses a nonhyperbolic ergodic measure with positive entropy. These conditions are satisfied for a large class of non-hyperbolic $C^1$ diffeomorphisms, and imply existence of a partially hyperbolic compact set with one-dimensional center direction and positive topological entropy on which the center Lyapunov exponent vanishes uniformly. The method uses a construction of a blender defined dynamically in terms of strict invariance of a family of discs, and allows to construct a $C^1$-open and dense subset of the set of non-Anosov robustly transitive diffeomorphisms consisting of systems with non-hyperbolic ergodic measures with positive entropy.

In the mini-course we intend to discuss both types of technics, and mention many related open questions.

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