Mandelbrot percolations

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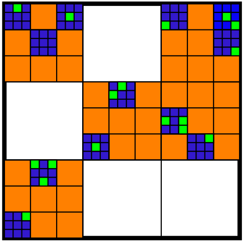
Fractal percolation, introduced by Mandelbrot early 1970's:

We partition the unit square into M^2 congruent sub squares each of them are independently retained with probability p and discarded with probability $1-p$. In the squares retained after the previous step we repeat the same process at infinitum.



Λ_3

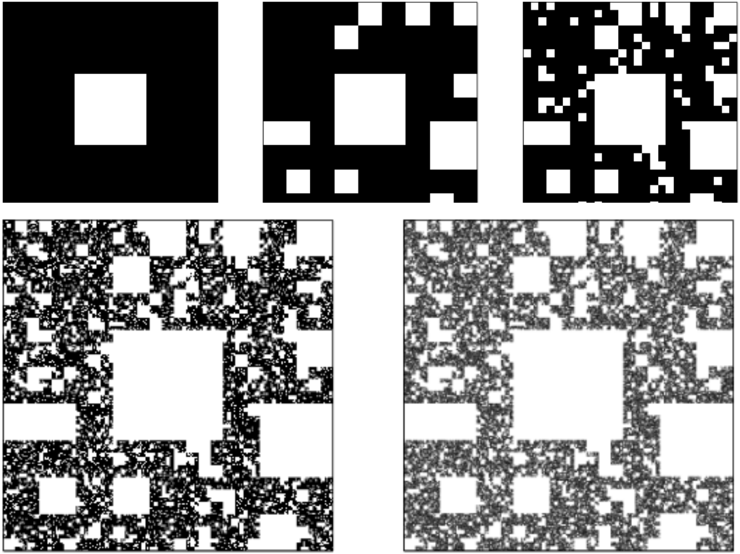


Figure: The Figure is from [3]. The first 5 approximation $M = 3$, $p = 0.85$

Let $\mathcal{E}_n$ be the set of retained level n squares. We write

$$\Lambda_n := \bigcup_{Q \in \mathcal{E}_n} Q.$$

Then the statistically self-similar set of interest is:

$$\Lambda := \bigcap_{n=1}^{\infty} \Lambda_n.$$

Definition 1.1

Officially: The process $\mathcal{E}_n$ is called Fractal percolation or Mandelbrot *percolation* (and it has a number of other names) and Λ is its limit sets. However, in practice many people call the limit set Λ Mandelbrot (or fractal) percolation.

The formal definition

The following formal definition of the Mandelbrot percolation in $\mathbb{R}^d$ was given by M. Dekking [2]. Let $\mathcal{I} := \left\{1, \dots, M^2\right\}$. We define $\mathcal{I}^0 := \emptyset$. Let $\mathcal{T} := \bigcup\limits_{n=0}^{\infty} \mathcal{I}^n$ be the M^d-array tree.

The probability space is $\Omega := \{0,1\}^{\mathcal{T}}$ the set of labeled M^d-array trees.

The probability measure $\mathbb{P}_{\mathbf{p}}$ on Ω is define in such a way that the family of labels $X_{\mathbf{i}} \in \{0,1\}$ of nodes $\mathbf{i} \in \mathcal{T}$ satisfy:

- $\mathbb{P}_{\mathbf{p}}(X_{\emptyset} = 1) = 1$
- $\mathbb{P}_{\mathbf{p}}(X_{i_1,\dots,i_n}) = p_{i_n}$
- $\{X_{\mathbf{i}}\}_{\mathbf{i}\in\mathcal{T}}$ are independent .

Following [2] we define the survival set of level n by

$$S_n := \left\{\mathbf{i} \in \mathcal{I}^n : X_{i_1\dots,i_k} = 1,\ \forall 1 \leq k \leq n\right\}.$$

Then

$$\Lambda_n = \bigcup_{\mathbf{i}\in S_n} Q_{\mathbf{i}}, \quad \Lambda = \bigcap_{n=1}^{\infty} \Lambda_n,$$

where $Q_{\mathbf{i}}$ are defined for finite words $\mathbf{i} \in$ by the following Figure:

The formal definition

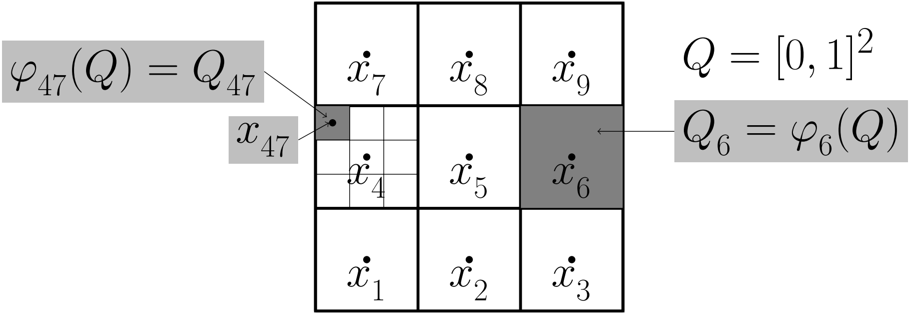


Figure: Definition of level *n* squares

It was proved by Falconer and independently Mauldin, Williams that conditioned on non-extinction:

$$(1) \qquad \dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \frac{\log(M^2 \cdot p)}{\log M} \text{ a.s.}$$

The meaning of the nominator of (1):

$$M^2 \cdot p = \mathbb{E}\left[\#\mathcal{E}_1\right].$$

Therefore

$$(2) \qquad M^2 \cdot p < 1 \Longrightarrow \Lambda = \emptyset \text{ a.s.}$$

Another consequence of (1) is:

(3) $\dim_{\mathrm{H}} \Lambda > 1$ a.s. conditioned on nonextinction

$$\Longleftrightarrow p > \frac{1}{M}.$$

Observe that $\#\mathcal{E}_n$ is a Galton-Watson Branching process with offspring distribution $\mathrm{Bin}(M^2, p)$. So, now we recall couple of things from Probability Theory about such processes.

Notation used throughout this chapter

- In this chapter we always assume that X is a r.v., whose values can only be non-negative integers.
- $\forall k \in \mathbb{N}$ let $p_k := \mathbb{P}(X = k)$.
- Probability Generator Function (p.g.f.) for r.v. X:

$$g_X(s) := \mathbb{E}\left[s^X\right] = \sum_{k=0}^{\infty} p_k \cdot s^k .$$

The probability generator function (in short p.g.f.) have the following three basic properties:

p.g.f.

(a) The p.g.f. uniquely defines the distribution function,

(b) p.g.f. of sums of independent, non-negative r.v. is equal to the product of their p.g.f.

(c) $\mathbb{E}\left[X(X-1)\cdots(X-k)\right] = g^{(k+1)}(1)$, where $g^{(k+1)}$ is the $k+1^{st}$ derivative of g.f. g. So (4)

$$\mathbb{E}\left[X\right] = g'(1) \text{ and } \mathbb{E}\left[X^2\right] = g''(1) + g'(1).$$

Lemma 2.1

Let X *and* N *be independent, non-negative r.v., the generator functions:* g_X *and* g_N*. Let* $\{X_i\}$ *be independent,* $X_i \stackrel{d}{=} X$.

$$R := X_1 + \cdots + X_N \text{ and then:}$$

$$g_R(s) = g_N(g_X(s)). \tag{5}$$

From here:

$$\mathbb{E}[R] = g'_R(1) = g'_N(\underbrace{g_X(1)}_{1}) \cdot g'_X(1) = \mathbb{E}[N] \cdot \mathbb{E}[X]. \tag{6}$$

Galton -Watson branching process $\{X_k\}_{k=0}^{\infty}$ is defined as follows: Given numbers $p_k \in [0,1]$ such that $\sum_{k=0}^{\infty} p_k = 1$.

We start with one individual. $X_0 := 1$. It has k offsprings with probability p_k. Then each of these offsprings (if there are any) has the same offspring distribution $\{p_k\}$. More formally: let Y be a r.v. with

$$\mathbb{P}(Y = k) = p_k, \quad k \geq 0.$$

Let

$$\left\{ Y_i^{(n)} \right\}_{n,i \geq 1} \quad \text{be independent copies of } Y.$$

The size of n-th generation is:

$$X_{n+1} := \sum_{i=1}^{X_n} Y_i^{(n+1)}. \tag{7}$$

That is $Y_i^{(n+1)}$ is the number of offsprings of the i-th individual on level n. Let

$$g(s) := g_1(s) =: \mathbb{E}\left[e^Y\right] = \sum_{n=0}^{\infty} p_n \cdot s^n.$$

Branching processes

Let

$$g_n := \mathbb{E}\left[s^{X_n}\right],$$

so g_n is the generator function of number X_n of individuals in the n^{th} generation. It follows from Lemma 2.1

$$g_{n+1} = g_n(g(s)).$$

From here we get by induction, that

$$g_n(s) = \underbrace{g \circ \cdots \circ g}_{n}(s) =: g^n(s). \tag{8}$$

Branching processes (cont.)

Applying this for $s = 0$:

$$\mathbb{P}(X_n = 0) = g^n(0). \tag{9}$$

Hence,

$$\mathbb{P}\,(\text{the probability of extinction}) = \lim_{n\to\infty} \mathbb{P}(X_n = 0).$$

The event:

$$\{\exists n,\ X_n = 0\} = \{X_n \to 0\}$$

is called extinction .

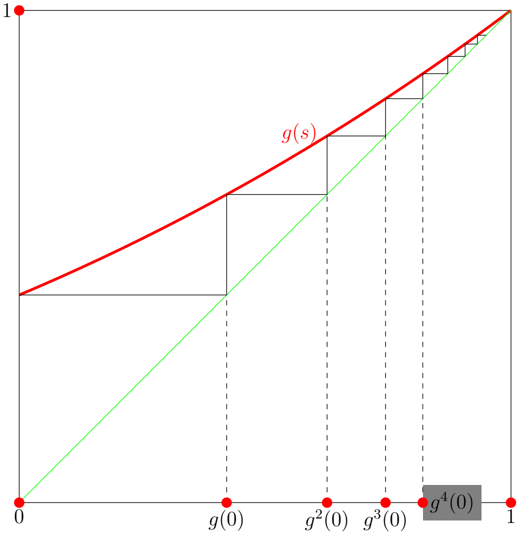
Proposition 2.2

Assume that $p_1 \neq 1$. Then on the event of nonextinction we have

$$Z_n \to \infty.$$

We write q for the extinction probability.

$\mathbb{E}[Y] = g'(1) < 1 \implies \lim_{n \to \infty} \mathbb{P}(X_n = 0) = 1$



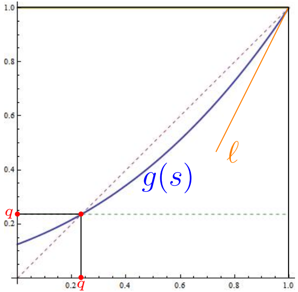
So, the probability of one individual has n children is p_n. Then the expected value of number of children is $m := \sum_{n=1}^{\infty} p_n \cdot n$. We have defined generator function

$$g(s) := \sum_{n=0}^{\infty} p_n \cdot s^n.$$

The function g goes over point $(1,1)$. Let ℓ be the tangent line of g in $s=1$. Then gradient of ℓ: $g'(1) = m$. If $m > 1$, then part of ℓ which goes in $[0,1]^2$ lies under $y = x$ so there exists "another" fix point $q < 1$ of g in $[0,1]$. From the graph: $0 \leq g'(q) < 1$. So if $g^n := \underbrace{g \circ \cdots \circ g}_{n}$, then

$$g^n(x) \to q, \ \forall x < 1.$$

From the previous slide and from the graph: q is the probability of extinction.



We say that the branching process is subcritical, critical or supercritical if the expected number of offsprings $m < 1$, $m = 1$ or $m > 1$ respectively.
Clearly

$$\mathbb{E}\left[X_n\right] = (g^n)'(1) = (g'(1))^n = m^n.$$

It is easy to prove that

Lemma 2.3

The sequence

$$\frac{X_n}{m^n}$$

is a martingale.

Since $\frac{X_n}{m^n}$ is non-negative, it converges. We call the limit W.

$$\lim_{n\to\infty} \frac{X_n}{m^n} =: W.$$

Theorem 2.4 (Kesten-Stigum)

Assume that $m > 1$. then the following are equivalent

(i) $\mathbb{P}(W = 0) = q$, *(probability of extinction).*

(ii) $\mathbb{E}[W] = 1$.

(iii) $\mathbb{E}[Y \cdot \log Y] < \infty$.

That is, if (iii) holds then $W > 0$ a.s. conditioned on nonextinction. This follows from a general zero-one property for the Galton-Watson branching process:

Definition 2.5 (inherited property)

We say that a property of trees is inherited if

- whenever the tree has this property so do all the descendent trees of the offspring of the root and
- every finite tree has this property.

The proof of the following Proposition is available in [4, Proposition 5.6].

Proposition 2.6

Every inherited property has probability either 0 or 1 conditioned on nonextinction.

Let $\tau(d)$ be the probability that a Galton- Watson tree contain a d-ary sub-tree begging at the root (initial individual). (In particular $\tau(1)$ is the survival probability.) Dekking and Pakes [5] proved that

Proposition 2.7 (Dekking, Pakes)

Let g be the p.g.f. of a super critical Galton-Watson process. Put

$$D_d(s) := \sum_{j=1}^{d-1} (1-s)^j \frac{(D^j g)(s)}{j!}.$$

Then $1-\tau(d)$ is the smallest fixed point of G_d in $[0,1]$.

Using the previous proposition and the fact that for a $\mathrm{Bin}\,(N,p)$ distribution the p.g.f. is:

$$g(s) = (p \cdot s + (1-p))^N . \tag{10}$$

one can show as a homework exercise that

Corollary 2.8

If the offspring distribution in a Galton-Watson tree is $\mathrm{Bin}(d+1,p)$, then for $p<1$ large enough we have $\tau(d)>0$.

In the case of Mandelbrot percolation with parameters (M,p), the offspring distribution is $\mathrm{Bin}\left(M^2,p\right)$.

Definition 2.9

We say that a deterministic set $E \subset [0,1]^2$ is SC-like if it can be presented as

$$E := \bigcap_{n=1}^{\infty} E_n,$$

where E_n is the union of $(M^2-1)^n$ level n squares in such a uniform way that any $Q \subset E_n$ level n square contains exactly M^2-1 level $n+1$ squares which are contained in E_{n+1}. Similarly to the Sierṕinski Carpet we deleted one level n square in every step of the construction. However, as oppose to the Sierpiński-Carpet we do not require that the only deleted square assume in each step the same position.

Corollary 2.10

If we choose $p < 1$ sufficiently close to 1 then $\Lambda(p)$ contains an SC-like set with positive probability.

The following theorem is due to Dvoretzky, Erdős , Kakutani, Taylor. Consider the d-dimensional Brownian trace:

$$[B_d] := \{B_d(t) : t \in [0,1]\}$$

where $B_d(t)$ is the d-dimensional Brownian motion started from a point in $\mathbb{R}^d$ or the distribution of the initial point has bounded density on $[0,1]^d$.

Theorem 3.1 (Intersection of Brownian traces)

(i) $d \geq 4$: *Two independent Brownian traces which started from different points are disjoint.*

(ii) $d = 3$:

1. *Two independent Brownian traces intersect a.s.*
2. *Three independent Brownian traces started from different points, have no mutual points of intersection.*

Theorem 3.1 cont.

(iii) $d = 2$: any finite number of Brownian motions have mutual points of intersections.

Actually more is true. Hawkes proved in 1971 that for every k, if we consider k independent Brownian traces on the plane and $H \subset \mathbb{R}^2$ is an arbitrary Borel set then the Hausdorff dimension of those points of H which are mutual intersection points of these k Brownian traces is gual to the Hausdorff dimension of H.

Definition

Consider the Mandelbrot Percolation on $\mathbb{R}^d$ for $M = 2$ and for a given $p \in (0, 1)$. Since we always choose here $M = 2$ the resulted random set is denoted by $\Lambda_d(p)$.

Yuval Peres provided a simpler proof in [6] using Mandelbrot percolations. We sketch some ideas of this way of verifying Theorem 3.1 above.

Preliminaries

The following Theorem were proved by Hawkes 1981 and Lyons 1990.

Theorem 3.2

Let $p = 2^{-\beta} < 1$. *For any set* $H \subset [0,1]^d$ *we have*

(i) *If* $\dim_{\mathrm{H}}(H) < \beta$ *then* $H \cap \Lambda_d(p) = \emptyset$ *a.s..*

(ii) *If* $\dim_{\mathrm{H}}(H) > \beta$ *then* $H \cap \Lambda_d(p) \neq \emptyset$ *with positive probability.*

An important tool

Let μ be a Borel measure on $\mathbb{R}^d$. The β-energy of μ is

$$\mathcal{E}_\beta := \int \int |x - y|^{-\beta} d\mu(x) d\mu(y).$$

Given a Borel set $H \subset \mathbb{R}^d$. We define

$$\mathrm{Cap}_\beta(H) := \left[\inf_{\mathrm{spt}(\mu) \subset H} \mathcal{E}_\beta(\mu) \right]^{-1},$$

where the infinum is taken for probability measures with the convention $1/\infty = 0$.

Frostman 1935

Theorem 3.3

For $K \subset \mathbb{R}^d$ we have

$$\dim_{\mathrm{H}}(K) = \inf\left\{\beta > 0 : \mathrm{Cap}_{\beta}(K) = 0\right\}. \tag{11}$$

The following Proposition was stated in [6, Corllary 4.3] and it follows from theorems due to Benjamini, Lyons, Pemantle, Peres

Proposition 3.4

Let $\beta \geq 0$ and $d \geq 1$. then for any closed $K \subset [0,1]^d$ we have

$$\mathbb{P}\left(Q_d\left(2^{-\beta}\right) \cap K \neq \emptyset\right) \asymp \mathrm{Cap}_\beta(K).$$

Definition 3.5

Two random Borel set A and B in $\mathbb{R}^d$ are intersection-equivalent $A \sim_i B$ in the open set U, if for any closed set $H \subset U$, we have

$$\mathbb{P}(A \cap H \neq \emptyset) \asymp \mathbb{P}(B \cap H \neq \emptyset), \tag{12}$$

where $\asymp$ means that the ratio of the two sides are in between two positive constants. In this case (3.5) holds for every Borel set H.

This section is about the intersection equivalence between some Mandelbrot percolation sets and Brownian traces.

The following theorem was proved by Yuval Peres [6] in 1996.

Theorem 3.6

(i) *If* $d \geq 3$ *then* $[B_d]$ *is intersection-equivalent to* $\Lambda_d\left(2^{2-d}\right)$ *in the unit cube.*

(ii) *Let* $d = 2$. *For any Borel set* H

$$\exists p < 1 \text{ s.t. } \mathbb{P}\left(\Lambda_2(p) \cap H\right) > 0 \Longrightarrow \mathbb{P}\left([B_2] \cap H\right) = 1.$$

Lemma 3.7

Let $A_1, \dots, A_k$ and $F_1, \dots, F_k$ be random Borel sets in $\mathbb{R}^d$ for some d, s.t. $A_j \sim_i B_j$ for all $j = 1, \dots, k$. Then

$$A_1 \cap \cdots \cap A_k \sim_i F_1 \cap \cdots F_k. \tag{13}$$

proof

It is enough to prove for $k = 2$ (induction). Further, it is enough to prove that

$$A_1 \cap A_2 \sim_i F_1 \cap A_2. \tag{14}$$

This is done by conditioning on A_2:

proof cont.

$$\mathbb{P}\left(A_1 \cap A_2 \cap H \neq \emptyset\right) = \mathbb{E}\left[\mathbb{P}\left(A_1 \cap A_2 \cap H \neq \emptyset | A_2\right)\right]$$
$$\asymp \mathbb{E}\left[\mathbb{P}\left(F_1 \cap A_2 \cap H \neq \emptyset | A_2\right)\right]$$
$$\mathbb{P}\left(F_1 \cap A_2 \cap H \neq \emptyset\right). \square$$

Lemma 3.8

For any $0 < p, g < 1$, *if* $\Lambda_d(p)$ *and* $\Lambda'_d(q)$ *are independent, then*

$$\Lambda_d(p) \cap \Lambda'_d(q) \stackrel{d}{=} \Lambda_d(pq). \tag{15}$$

The proof is immediate from the construction.

Proof of Thm 3.1 (i)

Let $d = 4$. Then

$$\Lambda_4\left(\frac{1}{4}\right) \sim_i \{B_4(t) : t \geq \varepsilon\},$$

and

$$\Lambda_4\left(\frac{1}{4}\right) \sim_i \{B_4'(s) : s \geq \varepsilon\}$$

So, by Lemma 3.7, we have

Proof of Thm 3.1 (i) cont.

$$\{B_4(t) : t \geq \varepsilon\} \cap \{B_4'(s) : s \geq \varepsilon\} \cap [0,1]^4$$
$$\sim_i \Lambda_4\left(\frac{1}{4}\right) \cap \widetilde{\Lambda}_4\left(\frac{1}{4}\right) \cap [0,1]^4.$$

$$\mathbb{P}\left(\{B_4(t) : t \geq \varepsilon\} \cap \{B_4'(s) : s \geq \varepsilon\} \cap [0,1]^4 \neq \emptyset\right)$$
$$\asymp \mathbb{P}\left(\Lambda_4\left(\frac{1}{4}\right) \cap \widetilde{\Lambda}_4\left(\frac{1}{4}\right) \neq \emptyset\right)$$
$$= \mathbb{P}\left(\Lambda_4\left(\frac{1}{16}\right) \neq \emptyset\right) = 0.$$

Hence, there are no intersections apart from possibly the initial points.

Proof of Thm 3.1 (ii)

$$\{B_3(t) : t \geq \varepsilon\} \sim_i \Lambda_3\left(\frac{1}{2}\right)$$

$$\{B'_3(s) : s \geq \varepsilon\} \sim_i \Lambda'_3\left(\frac{1}{2}\right)$$

Hence

(16)

$$\left(\{B_3(t) : t \geq \varepsilon\} \cap \{B'_3(s) : s \geq \varepsilon\} \cap [0,1]^3\right) \sim_i \Lambda_3\left(\frac{1}{4}\right).$$

Proof of Thm 3.1 (ii) cont.

In case of $\Lambda_3\left(\frac{1}{4}\right)$ every individual has maximum 8 children independently each with probability $1/4$ that is

$$\text{expected number of offsprings is } = 8 \cdot \frac{1}{4} = 2 > 1$$

This implies that with positive probability $\Lambda_3\left(\frac{1}{4}\right) \neq \emptyset$. So, from (16) we obtain that two independent copies of Brownian traces in $\mathbb{R}^3$ intersect with positive probability.

Proof of Thm 3.1 (ii) cont.

The mutual intersection of three independent Brownian traces is intersection equivalent to

$$\Lambda_3\left(\frac{1}{2}\right) \cap \tilde{\Lambda}_3\left(\frac{1}{2}\right) \cap \widehat{\Lambda}_3\left(\frac{1}{2}\right) \sim_i \Lambda_3\left(\frac{1}{8}\right).$$

(processes of different color are independent). Then

$$\text{expected number of offsprings is } = 8 \cdot \frac{1}{8} = 1.$$

It is well known from the theory of Branching processes that this implies that $\Lambda_3\left(\frac{1}{8}\right) = \emptyset$ a.s.. So, the same holds for the mutual intersection of three Borwnian traces started from different points.

Λ percolates

Let $\Lambda(\omega)$ be a realization of the Mandelbrot percolation random Cantor set. We say that $\Lambda(\omega)$ percolates if there is a connected component of $\Lambda(\omega)$ which connects the left and the right walls of the square $[0,1]^2$.

Let us write $\Lambda_{|\leftrightsquigarrow|}$ for the event that the random self-similar set Λ percolates.

Theorem [J.T Chayes, L. Chayes, R. Durrett] [1]

Let TD be the event that Λ is totally disconnected. That is all connected components are singletons. Let

$$p_c := \inf \left\{ p : \mathbb{P}_p \left(E_{|\leftrightsquigarrow|} \right) > 0 \right\}$$

Then $0 < p_c < 1$ and

$$p_c = \sup \left\{ p : \mathbb{P}_p \left(TD \right) = 1 \right\}.$$

If $p < p_c < 1$ then all connected components of Λ are singletons. If $p \geq p_c$ then Λ percolates with positive probability. (As opposed to the usual percolation, which percolates at p_c with zero probability.)

Henk Don [3] in 2013 published some ne bounds on the crfitical probability for different values of M.

lower bounds: $p_c(2) > 0.881$ and $p_c(3) > 0.784$.

upper bounds : $p_c(2) < 0.993$, $p_c(3) < 0.94$ and
$p_c(4) < 0.972$.

The weaker assertion that we prove

In what follows we prove that following weaker assertion: Let $M \geq 3$ be fixed. We consider the Mandelbrot percolation on the plane which corresponds to probability p. Let us denote it by $\Lambda(p)$. We prove that

$$(17) \quad \exists p_c < 1, \text{ s.t. } p > p_c, \quad \mathbb{P}(\Lambda(p) \text{ percolates.}) > 0.$$

That is $\Lambda(p)$ contains a continuous path $t \mapsto (x(t), y(t))$, $t \in [0,1]$ such that $x(0) = 0$ and $x(1) = 1$. Sometimes we express this as $\Lambda(p)$ has a left-right crossing.

Lemma 4.1

Assume that $M \geq 3$. Assume that each level n square gives birth to at least $M^2 - 1$ level $n+1$ squares. Then there is a left-to-right crossing at all levels n. More precisely:

For every n, there is a sequence

$$Q_1, \dots, Q_k \in \mathcal{E}_n$$

of level n retained squares such that any two consecutive squares share a common side and Q_1 has a common side with the Eastern and Q_k has a common side with the Western wall of $[0,1]^2$.

Proof

Recall that we write $\mathcal{E}_n$ for the collection of retained level n squares. Observe that

$$(18)\quad \forall Q \in \mathcal{E}_n,\ \forall C, D \in \mathcal{E}_{n+1}, C, D \subset Q, \qquad C \text{ side connected to } D.$$

Assume that there is a left-to-right crossing

$$Q_1, \dots, Q_r \in \mathcal{E}_n.$$

We define

$$Q_0 = Q_{r+1} := [0,1]^2.$$

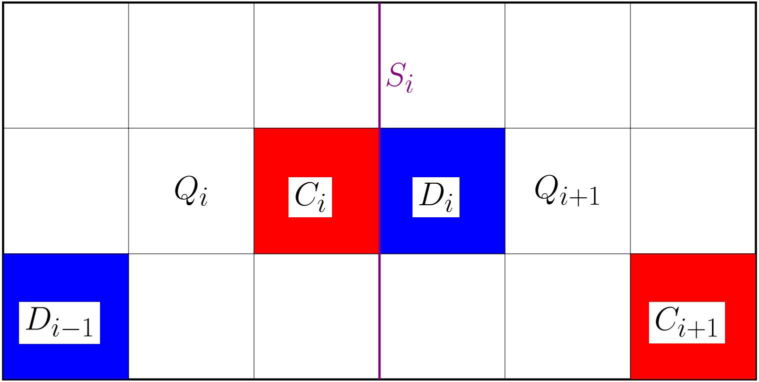


Figure: Figure for the proof of Lemma 4.1

Proof cont.

For every $i = 0, \dots r$, let S_i be the common side of Q_i and Q_{i+1} for $i = 0, \dots r$. We define

$$C_i, D_i \in \mathcal{E}_{n+1}, C_i \subset Q_i, \text{ and } D_i \subset Q_{i+1} \text{ share a side.}$$

By (18) D_i is side connected to C_{i+1}. This implies that there is a left-to-right crossing from D_0 to C_r. □

So, what we prove it the following special case of Theorem 49

Theorem 4.2

For $M \geq 3$ the left-right crossing probability $\theta_\infty(p)$ is positive.

The same is true for $M = 2$ but it requires extra steps in the proof.

Proof.

By Corollary 2.10 $\Lambda(p)$ contains an SC-like set with positive probability. However, in Lemma 4.1 we have proved that in an SC-like set we can always find a left-to-right crossing. □

Preparation for the case of $M = 2$

Let $\Lambda_{n,M}(p)$ be the n-th approximation of the Mandelbrot percolation set on the plane when we divide the square into M^2 congruent sub-squares of size M^{-1} and the probability of retaining them is p. We need the definition of stochastic ordering:

Definition 4.3 (Stochastic domination)

Let X, Y be r.v. not necessarily living on the same probability space. We say that Y stochastically dominates X, $(Y \succeq X)$ if

$$\forall x,\ \mathbb{P}(X \leq x) \geq \mathbb{P}(Y \leq x). \tag{19}$$

The case of $M = 2$

Our aim is the sketch why Lemma 4.1 holds for $M = 2$.

- For every $p \in (0,1)$ there is a $q \in (0,1)$ such that if $Y \sim \mathrm{Bernoulli}(q)$ and $X_1, \dots, X_{M^2}$ are independent and $X_i \sim \mathrm{Bernoulli}(\sqrt{p})$ then

$$Y \succeq \max\{X_1, \dots, X_{M^2}\}$$

- Then $\mathcal{E}_{2,M}(q) \succeq \mathcal{E}_{1,M^2}(p)$
- In case of $M = 2$ we can apply Lemma 4.1 for $\mathcal{E}_{1,4}(p)$ which completes the proof.

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