Mandelbrot percolations

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From the dimension formula the following hold almost surely:

- If $p \leq 1/M^2$ then $\Lambda = \emptyset$.
- If $1/M^2 < p < 1/M$ then $\dim_{\mathrm{H}}(\Lambda) < 1$ (but $\Lambda \neq \emptyset$ with positive probability).
- If $p > \frac{1}{M}$ then either
  - (a) $\Lambda = \emptyset$ or
  - (b) $\dim_{\mathrm{H}}(\Lambda) > 1$.

Recall: 1

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \frac{\log(M^2 \cdot p)}{\log M} \text{ a.s.}$$

Marstrand Theorem

Theorem 1.1 (Marstrand)

Let $B \subset \mathbb{R}^2$ be a Borel set.

1. *If $\dim_{\rm H}(B) \leq 1$ then for $\mathcal{L}\mathrm{eb}$-a.e. θ, we have*

$$\dim_{\rm H}(\mathrm{proj}_\theta(B)) = \dim_{\rm H}(B)$$

2. *If $\dim_{\rm H}(B) > 1$ then for $\mathcal{L}\mathrm{eb}$-a.e. θ, we have*

$$\mathcal{L}\mathrm{eb}\left(\mathrm{proj}_\theta(B)\right) > 0.$$

The outline of the construction I

Given an integer $M \geq 2$ and a vector of probabilities

$$(p_o, p_1, \ldots, p_{M-1}) \in [0,1]^M.$$

Which is in general NOT a probability vector. We divide the unit interval $I = [0,1]$ into the M subintervals $I_k = \left[\frac{k-1}{M}, \frac{k}{M}\right]$, $k = 0, \ldots, M-1$. We keep I_k with probability p_k. For all intervals kept, repeat this algorithm infinitely many times. Whatever remains it is our random Cantor set Λ.

Motivation

$\Lambda_1, \Lambda_2 \subset \mathbb{R}$. The algebraic difference set

$$\Lambda_2 - \Lambda_1 := \{f_2 - f_1 : f_1 \in \Lambda_1,\ f_2 \in \Lambda_2\}.$$

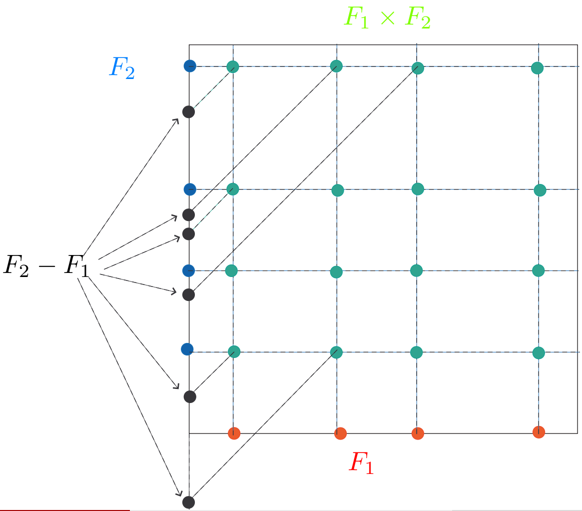
Motivation to study it comes from e.g. :

- Dynamical systems, unfolding of homoclinic tangency (Palis, Takens)
- Diophantine approximation (Moreira, Yoccoz).

Palis conjectured: For dynamically defined Cantor sets: "Generically" Either

- $\Lambda_2 - \Lambda_1$ is small: $\mathcal{L}\mathrm{eb}(\Lambda_2 - \Lambda_1) = 0$ or
- $\Lambda_2 - \Lambda_1$ is big: $\Lambda_2 - \Lambda_1$ contains some intervals.

The algebraic difference from geometric point of view II



The crosscorrelations

For $i \in \{0, \dots, M-1\}$ let

$$\gamma_i := \sum_{k=0}^{M-1} p_k p_{k+i \bmod M},$$

where p_i was the probability that we choose the interval i-th interval $I_i = \left[\frac{i-1}{M}, \frac{i}{M}\right]$.

$$\gamma_i := \sum_{k=0}^{M-1} p_k p_{k+i \bmod M}$$

Theorem 2.1 (Dekking, S.)

Assuming that $\Lambda_1, \Lambda_2 \neq \emptyset$, we have

(a) *If* $\forall i = 0, \ldots, M-1 : \gamma_i > 1$ *then almost surely* $\Lambda_2 - \Lambda_1$ *contains an interval .*

(b) *If* $\exists i \in \{0, \ldots, M-1\} : \gamma_i, \gamma_{i+1 \bmod M} < 1$ *then almost surely*

$\Lambda_2 - \Lambda_1$ *does not contain any interval .*

The Lebesgue measure of $\Lambda_2 - \Lambda_1$

We remind: $\gamma_i := \sum_{k=0}^{M-1} p_k p_{k+i \bmod M}$

Theorem 2.2 (Mora, S., Solomyak)

We assume that $p_0, \dots, p_{M-1} > 0$ *Moreover, we require that*

(A2) $$\Gamma := \gamma_0 \cdots \gamma_{M-1} > 1.$$

Then conditional on $\Lambda_1, \Lambda_2 \neq \emptyset$*, we have*

(1) $$\mathcal{L}eb(\Lambda_2 - \Lambda_1) > 0.$$

holds almost surely.

Positive Lebesgue measure with no intervals

Let $M = 3$ and

$$(p_0, p_1, p_2) = (0.52, 0.5, 0.72).$$

In this case we have

$$\gamma_0 = p_0^2 + p_1^2 + p_2^2 = 1.0388,$$

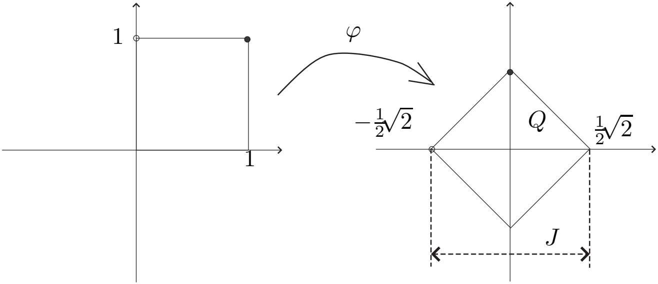
$$\gamma_1 = \gamma_2 = p_0 p_1 + p_1 p_2 + p_2 p_0 = 0.9944,$$

So, there is no interval.

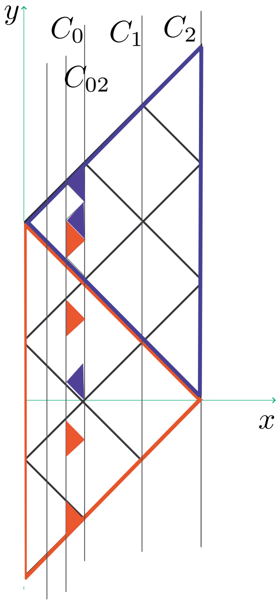
$$\gamma_0 \gamma_1 \gamma_2 = 1.0272 > 1$$

So, $\mathcal{L}eb(\Lambda_2 - \Lambda_1) > 0$.

Rotation



$$\Lambda := \varphi(\Lambda_1 \times \Lambda_2).$$


y
C_0
C_1
C_2
C_{02}
x

The expectation matrices $\mathcal{M}(k)$

In the k-th column $(k = 0, \dots, M-1)$ we consider the expectation **E** of the followings:

$$\underbrace{\begin{bmatrix} \mathbf{E}\{\#\text{small}\triangle \text{ in big}\triangle\} & \mathbf{E}\{\#\text{small}\triangle \text{ in big}\triangle\} \\ \mathbf{E}\{\#\text{small}\triangle \text{ in big}\triangle\} & \mathbf{E}\{\#\text{small}\triangle \text{ in big}\triangle\} \end{bmatrix}}_{\mathcal{M}(k)}$$

Then $\mathcal{M}(k_1 \dots k_n)$ is defined analogously for the n-th level column $\Lambda_{k_1 \dots k_n}$. Observe that

$$\mathcal{M}(k_1 \dots k_n) = \mathcal{M}(k_1) \cdots \mathcal{M}(k_n)$$

How does γ_i come into the picture?

$$\text{The first column sum of } \mathcal{M}(k) = \gamma_{k+1 \pmod 1}$$

$$\text{The second column sum of } \mathcal{M}(k) = \gamma_k$$

This implies that:

$$\text{If } \forall k, \gamma_k > s > 1$$

Then for every $k_1 \ldots k_n$:

$$\text{every column sums in } \mathcal{M}(k_1 \ldots k_n) > s^n.$$

$M = 4, (p_1, \dots, p_3) = (1, 0, 1, \rho)$

Consider the one-parameter ($0 \le \rho \le 1$) family of random Cantor sets: $M = 4$, $(p_1, \dots, p_3) = 1, 0, 1, \rho$. Then

$$\mathcal{M}(0) = \begin{bmatrix} \rho & 0 \\ \rho & 2 + \rho^2 \end{bmatrix}, \mathcal{M}(1) = \begin{bmatrix} 1 & \rho \\ 1 & \rho \end{bmatrix},$$

$$\mathcal{M}(2) = \begin{bmatrix} \rho & 1 \\ \rho & 1 \end{bmatrix}, \mathcal{M}(3) = \begin{bmatrix} 2 + \rho^2 & \rho \\ 0 & \rho \end{bmatrix}.$$

$$\gamma_0 = 2 + \rho^2,\ \gamma_1 = 2\rho,\ \gamma_2 = 2,\ \gamma_3 = 2\rho.$$

Application of Theorems DS

- For $\rho > \frac{1}{2}$:

$$\gamma_0, \dots, \gamma_3 > 1$$

  so, Theorem DS implies: $\exists$ an interval in $\Lambda_1 - \Lambda_2$ almost surely, conditioned on non-extinction.
- For $\rho < \frac{1}{2}$ Thm. DS gives (directly) nothing.

The order 2 Cantor set is the base M^2 Cantor set with vector

$$\left(p_0^{(2)}, \dots, p_{M^2-1}^{(2)}\right)$$

$$p_{Mi+j}^{(2)} = p_i p_j \quad \text{for} \quad i, j \in \{0, \dots, M-1\}.$$

We will denote the objects associated to $p^{(2)}$ all with a superindex (2), for instance $\Lambda^{(2)}$ is the random M^2-adic Cantor set generated by $p^{(2)}$, and $I_{k_1 \dots k_n}^{(2)}$ denotes an n-th level M^2-adic interval. Then
$i_1 \dots i_n, j_1 \dots j_n \in \{0, \dots, M-1\}^n$ one has

$$I_{Mi_1+j_1, \dots, Mi_n+j_n}^{(2)} = I_{i_1 j_1 \dots i_n j_n}.$$

Conclusion by higher order Cantor sets

Using higher order Cantor sets (up to order 324), and Matlab we obtained that the critical point ρ_c where $\Lambda_2 - \Lambda_1$ changes from empty to non empty interior, satisfies $0.3222 < \rho_c < 0.3226$. The fact that Theorem 2.1 can be applied not only the Mandelbnrot percolation sets but also for their iterates, was CLAIMED in [3] but proved carefully by Dekking and Don [2]. The most precise answer about whether or not we have an interval in the arithmetic difference was obtained by Dekking and Kuijvenhoven [1] in terms of lower spectral radius of a sequence of matrices.

Dekking Kuijvenhoven Theorem

Let $\mathbf{p} := \{p_0, \ldots p_{M-1}\}$ be a given list of probabilities. We consider two independent copies Λ_1, Λ_2 of the Mandelbrot percolation set on the line, corresponding to the parameters $\mathbf{p}$ and M.We construct the expectation matrices

$$\Gamma := \{\mathcal{M}(0), \ldots, \mathcal{M}(M-1)\}$$

as above.

Dekking Kuijvenhoven Theorem

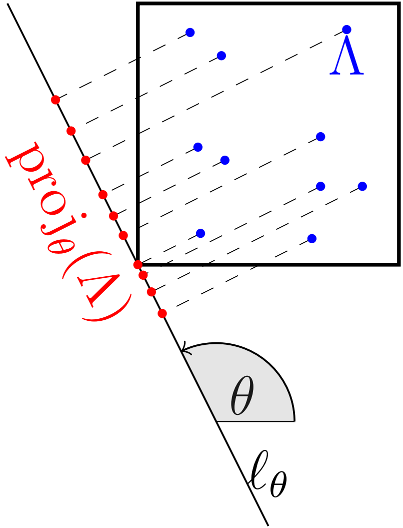
We consider the lower spectral radius

$$\underline{\rho}(\Gamma) := \liminf_{n\to\infty} \min_{i_1,\dots i_n} \|\mathcal{M}(i_1)\cdots\mathcal{M}(i_n)\|^{1/n}.$$

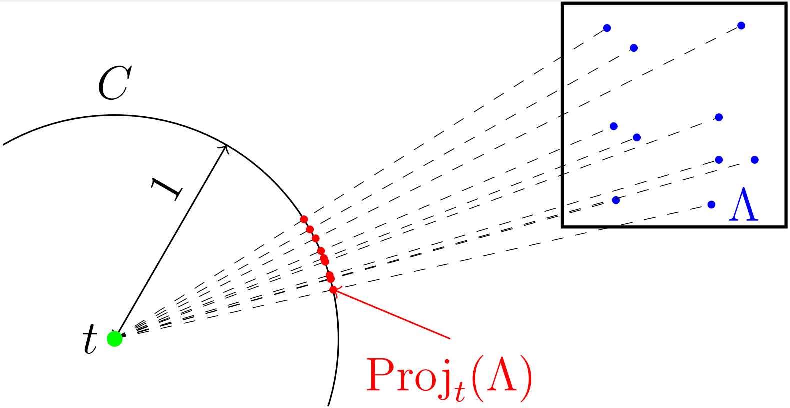
(i) If $\underline{\rho}(\Gamma) > 1$ then $\Lambda_1 - \Lambda_2$ contains some interval a.s. conditioned0 on $\Lambda_1 - \Lambda_2 \neq \emptyset$.

(ii) If $\underline{\rho}(\Gamma) < 1$ then $\Lambda_1 - \Lambda_2$ does not contain any intervals a.s..

Orthogonal projection to ℓ_θ

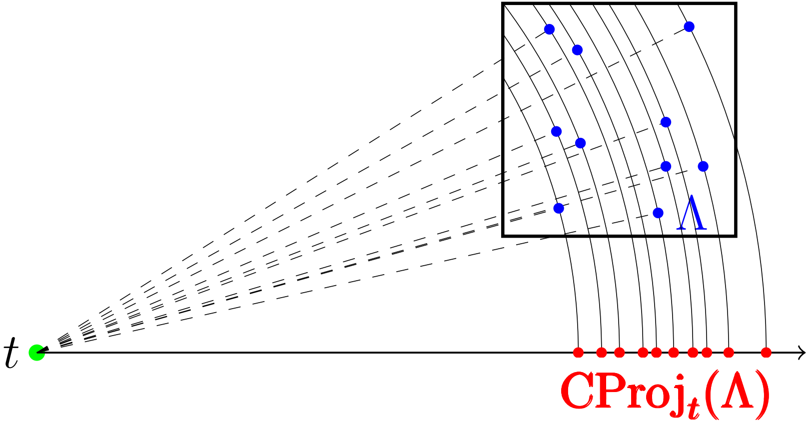


Radial and co-radial projections with center t



Let $\mathrm{CProj}_t(\Lambda) := \{\mathrm{dist}(t, x) : x \in \Lambda\}$ ($\mathrm{CProj}_t(\Lambda)$ is the set of the length of dashed lines above).

The co-radial projection



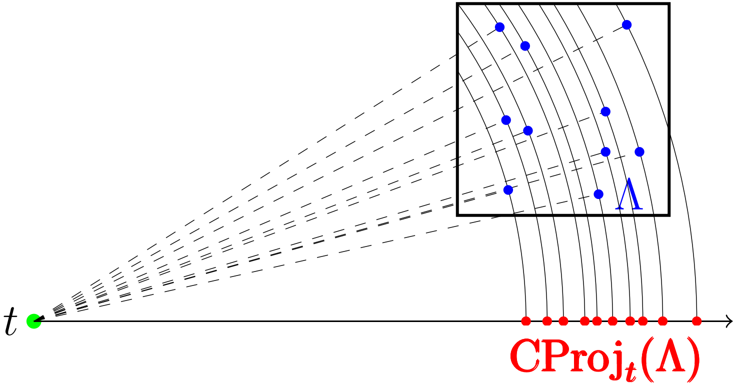


Figure: The orthogonal proj_α, radial Proj_t, co-radial CProj_t projections and the auxiliary projections Π_α, R_t, and $\tilde{R}_t$.

Theorem 4.1 (Falconer and Grimmett)

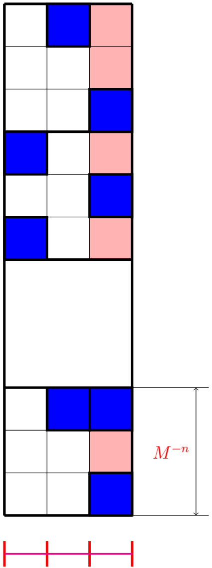
Assume that all $p_{i,j} \equiv p$ and

$$p > \frac{1}{M} \tag{2}$$

Then the orthogonal projection to the x-axis and to the y-axis of Λ contain an interval almost surely, conditioned on non-extinction.

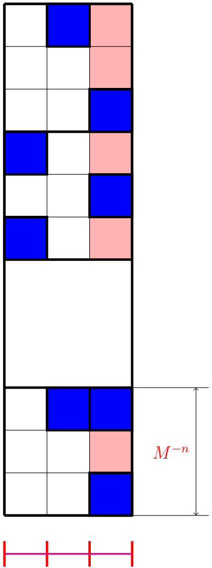
Our research was inspired by this paper. The idea of the proof: use large deviation theory for the INDEPENDENT number of level n successors of squares which are in the same vertical column.

Namely, $\dim_{\mathrm{H}} \Lambda > 1 \Longrightarrow \exists n, \exists$ a level n column with exponentially many retained level n squares. This column is the biggest column on the next figure. So, there are exponentially many (this is 3 on the figure) level n squares in this column.

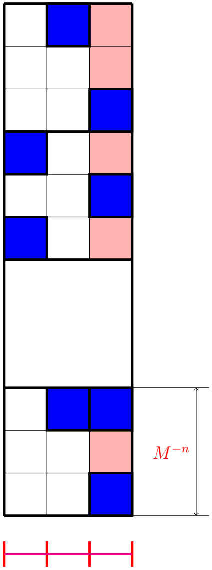


Now we move from level n to level $n+1$. We focus on any of the level $n+1$ columns (red column on the Figure). Independently each of the exponentially many (3) level-n retained square, gives birth an expected number of $pM > 1$ number of level $n+1$ squares in itself.

By Large Deviation Thm there exists an $\alpha > 1$ s.t. apart from a supper exponentially small probability, the number of retained level $n+1$ squares is at least α times the retained level n squares in the red column. This holds for all the other exponentially many level $n+1$ columns.



This implies that in each column on the figure there will be $\alpha > 1$ times more squares of level $n+1$ than of level n except with a super exponentially small probability.
Then we proceed to level $n+k$ squares of the level n-column. And similarly we get that there are exponenetially many level $n+k$ retained squares in each level $n+k$ column, apart from a supperexponentially small probability.



Theorem [R., S.] (When $p > \frac{1}{M}$)

We assume that

$$p > \frac{1}{M}.$$

Then the following statements hold almost surely conditioned on $\Lambda \neq \emptyset$:

$$\forall \theta \in [0, \pi], \ \mathrm{proj}_\theta(\Lambda) \text{ contains an interval }.$$

Further,

$$\forall t \in \mathbb{R}^2, \ \mathrm{Proj}_t(\Lambda) \text{ and } \mathrm{CProj}_t(\Lambda) \text{ contain an interval }.$$

Theorem [M. Rams, S.] (general case)

We partition the unit square into M^2 congruent sub squares the (i,j)-th one is retained with probability $p_{i,j}$ and discarded with probability $1-p_{i,j}$ independently. In the squares retained after the previous step we repeat the same process at infinitum.

$p_{0,2}$	$p_{1,2}$	$p_{2,2}$
$p_{0,1}$	$p_{1,1}$	$p_{2,1}$
$p_{0,0}$	$p_{1,0}$	$p_{2,0}$

Theorem Rams, S.

Assume that

1. $\forall k$: $\sum_{i=0}^{M-1} p_{i,k} > 1$ and $\sum_{j=0}^{M-1} p_{k,j} > 1$ and
2. $\forall \alpha \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$, α is good.

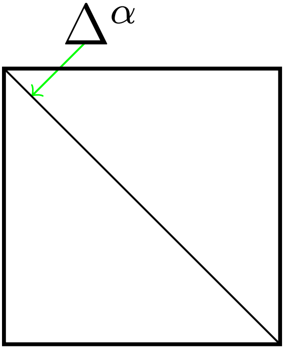
Then the following statements hold almost surely conditioned on $\Lambda \neq \emptyset$:

$$\forall \theta \in [0, \pi], \ \mathrm{proj}_\theta(\Lambda) \text{ containes an interval } .$$

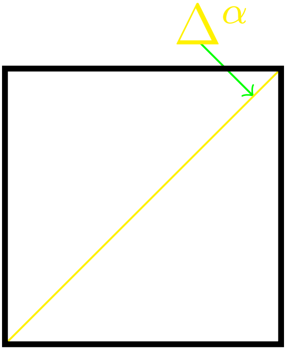
Further,

$$\forall t \in \mathbb{R}^2, \ \mathrm{Proj}_t(\Lambda) \text{ containes an interval } .$$

if $\alpha \in (0, \pi/2)$ if $\alpha \in (\pi/2, \pi)$

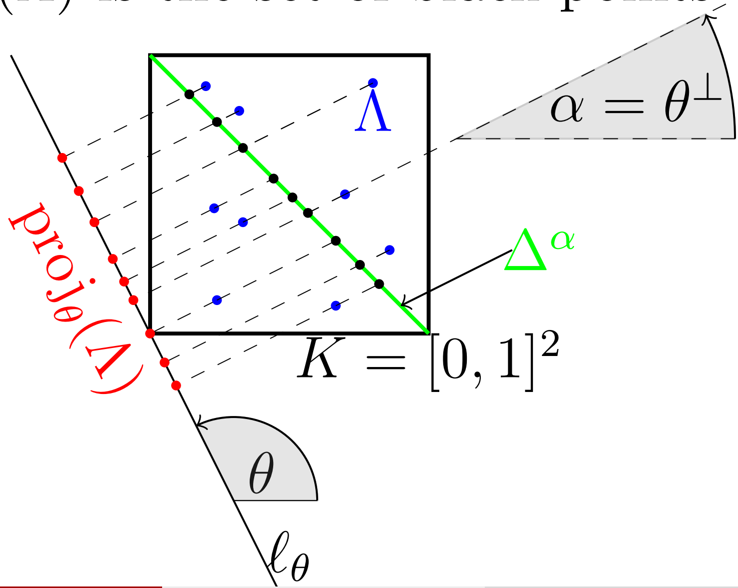


$K = [0, 1]^2$

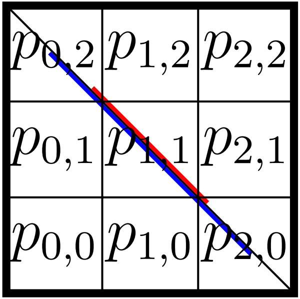


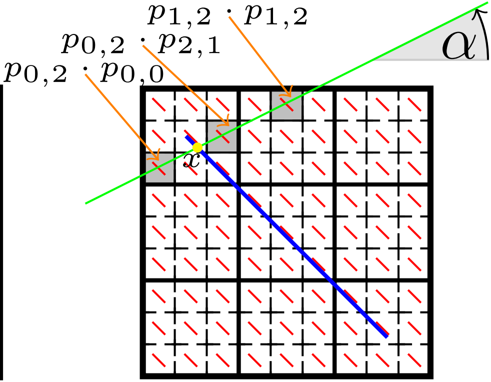
$K = [0, 1]^2$

$\Pi_\alpha(\Lambda)$ is the set of black points



α is good if $\exists \Delta_1^\alpha, \Delta_2^\alpha \subset \Delta^\alpha$, and $\exists r_\alpha \in \mathbb{N}$ such that $\Delta_1^\alpha \subset \mathrm{int}(\Delta_2^\alpha)$ and $\forall x \in \Delta_2^\alpha$ the sum of the probabilities of the gray squares > 2.



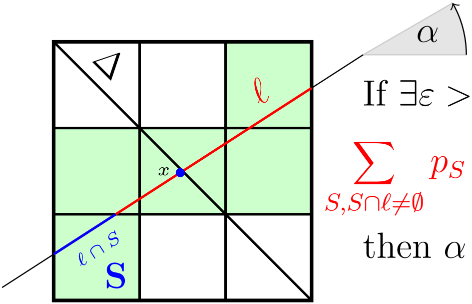


$$p_{0,2} \cdot p_{0,0} + p_{0,2} \cdot p_{2,1} + p_{1,2} \cdot p_{1,2} > 2$$

Remarks

The gray sum is equal to the expected number of level r_α red diagonals whose Π_α-projection covers x.

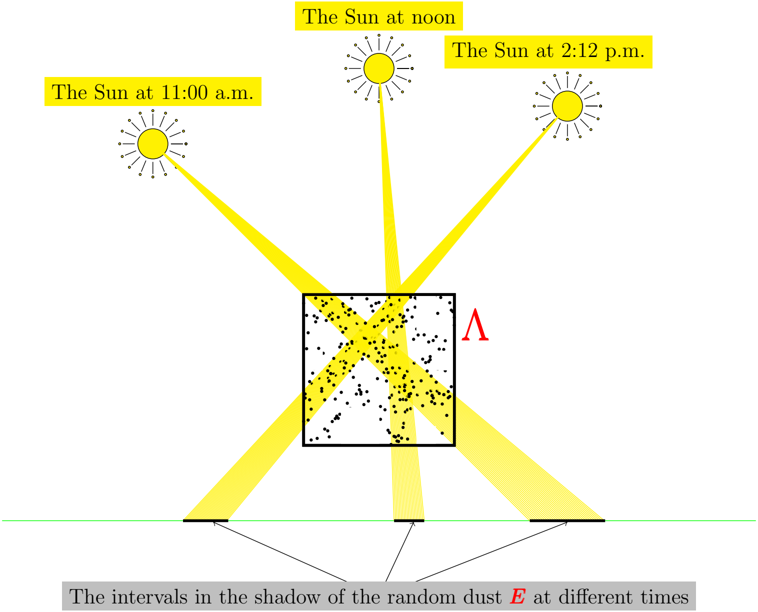
How to find our if α is a good angle?



If $\exists \varepsilon > 0$ s.t. $\forall x \in \Delta$:

$$\sum_{S, S \cap \ell \neq \emptyset} p_S \cdot \frac{1}{M} \cdot |\ell \cap S| \geq (1+\varepsilon) \cdot |\ell|$$

then α is a good angle.


The Sun at noon
The Sun at 2:12 p.m.
The Sun at 11:00 a.m.
Λ
The intervals in the shadow of the random dust E at different times

What happens in dimension higher than 2

Theorem 6.1 (Vagó and S.)

The same happens in dimension higher than 2 as on the plane.

The method of the proofs is the same in higher dimension. However, there are some technical difficulties that appear in higher dimension which are not present when we work on the plane.

Theorem [Rams, S.] If $\frac{1}{M^2} < p \leq \frac{1}{M}$

Theorem 7.1

Let $\ell \subset \mathbb{R}^2$ be a straight line and let Λ_ℓ be the orthogonal projection of Λ to ℓ.

*Then for almost all realizations of Λ (conditioned on $\Lambda \neq \emptyset$) and for **all** straight lines ℓ we have:*

$$\dim_{\mathrm{H}}(\Lambda_\ell) = \dim_{\mathrm{H}}(\Lambda). \tag{3}$$

Actually much more is true:

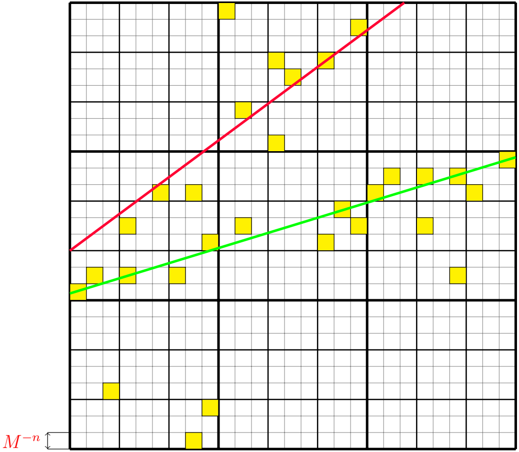
Lines intersect $\leq c \cdot n$ squares of level n

Theorem 7.2 (Rams, S.)

*If $\frac{1}{M^2} < p \leq \frac{1}{M}$ then for almost all realizations of Λ (conditioned on $\Lambda \neq \emptyset$) and for **all** straight lines ℓ : there exists a constant C such that **the number of level n squares having nonempty intersection with Λ is at most $c \cdot n$**.*

*On the other hand, almost surely for n big enough, we can find **some** line of 45° angle which intersects $\mathtt{const} \cdot n$ level n squares.*

First I draw the theorem and then I state it more precisely.



Recall: 2

$\frac{1}{M^2} < p \leq \frac{1}{M}$ $\Rightarrow$ Then every line ℓ intersects at most $\texttt{const} \cdot n$ level n squares.

Previous theorem stated more precisely I

Recall that Λ_n is the union of retained level-n squares. Let Δ be the decreasing diagonal of the unit square K (the diagonal connecting points $(0,1)$ and $(1,0)$).

Definition 7.3 (Slices of Λ)

Consider the family of all lines with argument between 0 and $\pi/2$ having non-empty intersection with $\mathrm{int}(\Delta)$. The unit square K cuts out a line segment from each of these lines. Let $\mathfrak{L}$ be the set of all line segments obtained in this way. The sets of the form $\Lambda \cap \ell, \ell \in \mathcal{L}$ are the slices of Λ.

Let $L_n(\ell) := |\Lambda_n \cap \ell|, \quad \ell \in \mathfrak{L}.$

Previous theorem stated more precisely II

Clearly, $\mathfrak{L}$ can be presented as a countable union of families of lines segments $\mathfrak{L}^\theta$ whose angles $\mathrm{Arg}(\ell)$ are θ-separated from both 0 and $\pi/2$:

$$\mathfrak{L}^\theta := \left\{ \ell \in \mathfrak{L} : \min \left\{ \mathrm{Arg}(\ell), \frac{\pi}{2} - \mathrm{Arg}(\ell) \right\} > \theta \right\}, 0 < \theta < \frac{\pi}{4}.$$

Previous theorem stated more precisely II

Corollary 7.4

For almost all realizations of E we have

(4)

$$\forall\theta \in \left(0, \frac{\pi}{4}\right), \exists N, \forall n \geq N, \forall \ell \in \mathfrak{L}^{\theta}; \# \mathcal{E}_n(\ell) \leq \mathrm{const} \cdot n,$$

where $\mathcal{E}_n(\ell)$ is the collection of selected level n squares that intersects Λ.

Large deviation estimate for $L_n(\ell)$ I

Theorem 7.5 (Hoeffding)

Let $X_1, \dots, X_m$ be independent bounded random variables with $a_i \leq X_i \leq b_i$, $(i = 1, \dots, m)$. Then for any $t > 0$:

$$\mathbb{P}\left(X_1 + \cdots + X_m - \mathbb{E}\left[X_1 + \cdots + X_m\right] \geq t\right) \leq \exp\left(\frac{-2t^2}{\sum_{i=1}^{m}(b_i - a_i)^2}\right).$$

Large deviation estimate for $L_n(\ell)$ II

We apply this to prove:

Lemma 7.6

For every $u > 1$ *there is a constant* $r = r(u) > 0$ *such that for every* $n \geq 1$, $\ell \in \mathfrak{L}$ *and* $0 < R < |\ell|$,
(5)

$$\mathbb{P}\left(L_n(\ell) > pL_{n-1}(\ell) \cdot u | L_{n-1}(\ell) \geq R\right) < \exp\left(-rM^{(n-1)}R\right)$$

Recall: 3

$$L_n(\ell) := |\Lambda_n \cap \ell|, \quad \ell \in \mathfrak{L}.$$

Summary

1. If $0 < p \leq 1/M^2$ then Λ dies out in finitely many steps almost surely.
2. If $\frac{1}{M^2} < p < \frac{1}{M}$ The $\Lambda \neq \emptyset$ with positive probability but $\dim_{\mathrm{H}}(\Lambda) = \frac{\log(M^2 p)}{M} < 1$. For almost all non-empty realizations, for all projections (all radial, co-radial and all orthogonal projections) the dimension of Λ does not decrease under the projection .
3. If $\frac{1}{M} < p < p_c$. Conditioned on non-extinction, almost surely: all projections of Λ contain some intervals but Λ is totally disconnected .
4. If $p \geq p_c$ then Λ percolates.

Definition 7.7

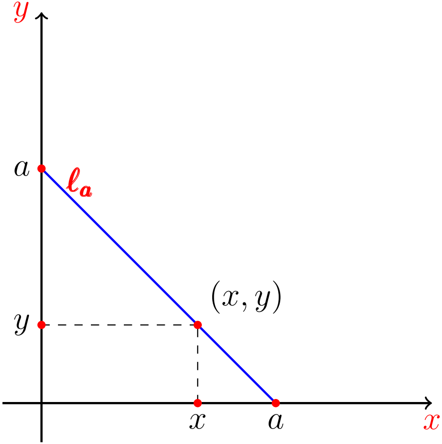
We say that $f[0,1]^2 \to \mathbb{R}$ is a strictly monotonic smooth function if $f \in \mathcal{C}^2[0,1]$ and $f'_x \neq 0$, $f'_y \neq 0$.

Theorem 7.8 (Rams, S.)

If $p > \frac{1}{M}$ $(\dim_{\mathrm{H}} \Lambda > 1)$ then for every strictly monotonic smooth function f, $f(\Lambda)$ contains an interval, almost surely conditioned on non-extinction.

Examples:

- $\{x + y : (x,y) \in \Lambda\} \supset$ interval .
- $\{x \cdot y : (x,y) \in \Lambda\} \supset$ interval .



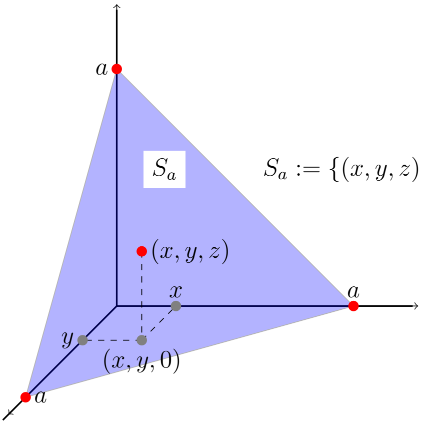
The arithmetic sum of the sets Λ_1, Λ_2 is:

$$\Lambda_1 + \Lambda_2 := \{x + y : x \in \Lambda_1,\ y \in \Lambda_2\}$$

The geometric interpretation of the arithmetic sum is:

$$\Lambda_1 + \Lambda_2 := \{a : \ell_a \cap \Lambda_1 \times \Lambda_2 \neq \emptyset\}.$$

So, $\Lambda_1 + \Lambda_2$ is the 45° projection of $\Lambda_1 \times \Lambda_2$.



$$S_a := \{(x, y, z) : x + y + z = a\}$$

$$a = x + y + z \iff (x, y, z) \in S_a$$

$$\Lambda_1 + \Lambda_2 + \Lambda_3 = \{a : S_a \cap \Lambda_1 \times \Lambda_2 \times \Lambda_3 \neq \emptyset\}.$$

Recall: 4

If $\frac{1}{M^2} < p \leq \frac{1}{M}$ then for almost all realizations of Λ (conditioned on $\Lambda \neq \emptyset$) and for **all** straight lines ℓ : there exists a constant C such that **the number of level *n* squares having nonempty intersection with Λ is at most $c \cdot n$**.

The same theorem holds if we substitute the two-dimensional Mandelbrot percolation Cantor set with the product of two independent one dimensional Cantor sets having the same M and probabilities p_1, p_2 such that $p = p_1 \cdot p_2$.

Let $\Lambda_1, \Lambda_2, \Lambda_3$ be one dimensional Mandelbrot percolation fractals constructed with the same M but with may be different probabilities p_1, p_2, p_3. Let Λ be the three dimensional Mandelbrot percolation with the same M and

$$p := p_1 p_2 p_3$$

The random Cantor sets

$$\Lambda_1 \times \Lambda_2 \times \Lambda_3 \text{ and } \Lambda$$

share many common features:

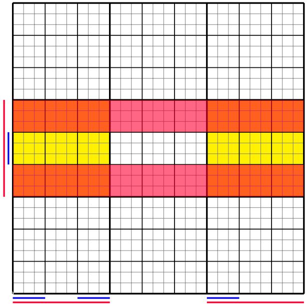
$$\dim \Lambda_1 \times \Lambda_2 \times \Lambda_3 = \dim \Lambda = \frac{\log M^3 p}{\log M}.$$

conditioned on non-extinction.

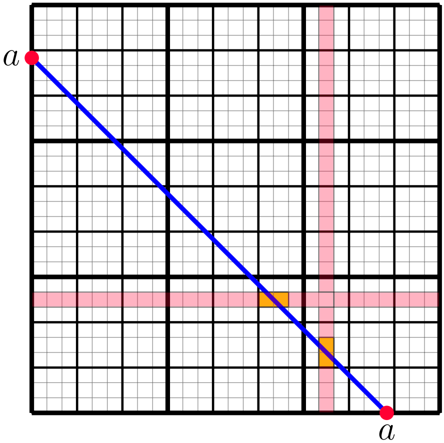
Dependency in the product set

$$\Lambda_{123} := \Lambda_1 \times \Lambda_2 \times \Lambda_3, \ \Lambda_{12} := \Lambda_1 \times \Lambda_2.$$

In Λ_{123} and in Λ_{12} there is NO independence between the successors of two cubes having one side common.



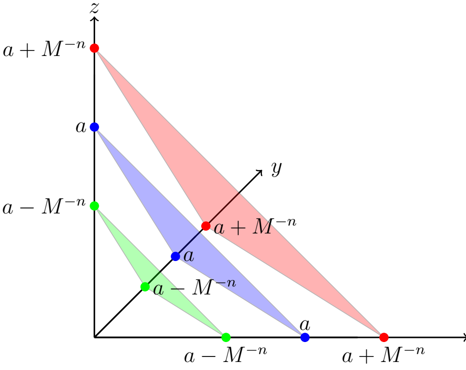
Λ and Λ_{12} are a little bit different from the point of 45° projection



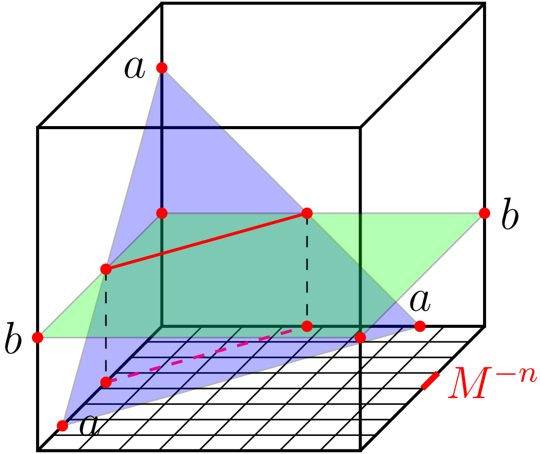
Let $\mathcal{E}^n$ be the set of selected level n cubes in $\Lambda_{1,2,3}^n$.
Since $\dim_{\mathrm{B}} \Lambda_{123} > 1$ so for a $\tau > 0$:

$$\#\mathcal{E}^n \approx M^n \cdot M^{\tau \cdot n}.$$

The colored planes: $3M^n$ planes that are orthogonal to $(1,1,1)$ and the consecutive ones are separated by M^{-n}. By pigeon hole principle one of the planes intersects $\mathrm{const} \cdot M^{\tau n}$ selected level n cubes. Assume that this is the **blue plane**.

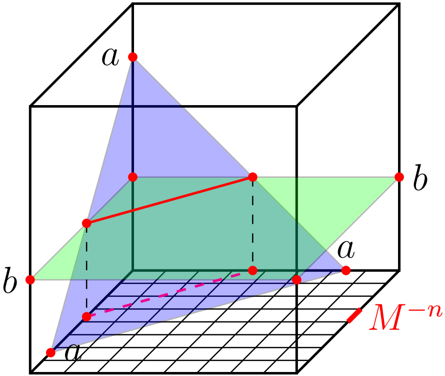


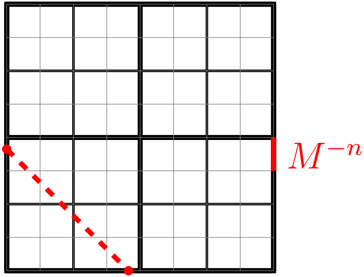
Among the $M^{\tau n}$ cubes which intersect the blue plane the ones sharing one common side are NOT independent.
For example those who intersect the red line are NOT independent.



;

$\dim_{\mathrm{H}} \Lambda_{123} > 1$ but $\dim_{\mathrm{H}} \Lambda_{12}, \dim_{\mathrm{H}} \Lambda_{23}, \dim_{\mathrm{H}} \Lambda_{31} < 1$.

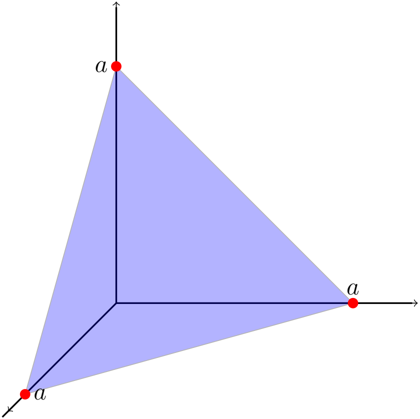




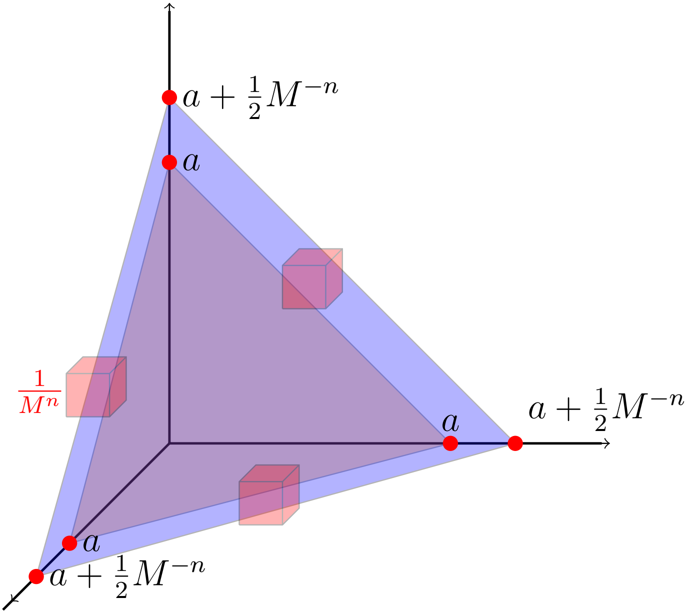
The point is that on the red dashed line there could be potentially M^n selected level n squares but in reality there will be only $c \cdot n$ selected squares.

An easy combinatorial Lemma shows that for a $t > 0$ constant there are M^{nt} selected level n squares that have

- no common sides (so what ever happens in these cubes in the future is independent)
- such that they all intersect the blue plane.



Then we use Large deviation theory similarly to Falconer Grimett to get intervals in the projection.


$a + \frac{1}{2}M^{-n}$
a
$\frac{1}{M^n}$
a
$a + \frac{1}{2}M^{-n}$
a
$a + \frac{1}{2}M^{-n}$

Here we always assume that we are in the homogeneous case and the dimesion (in case of non-extinction) is greater than 1. That is

$$p_{i,j} \equiv p > \frac{1}{M}. \tag{6}$$

It is well know from the theory of Branching processes that for

$$\lim_{n\to\infty} \frac{\#\mathcal{E}_n}{(M^2 \cdot p)^n} = W > 0, \text{ a.s.} \tag{7}$$

That is

$$\lim_{n\to\infty} \frac{\#\mathcal{E}_n \cdot M^{-2n}}{W \cdot p^n} = 1, \text{ a.s..} \tag{8}$$

We write $\mathcal{E}_n$ for the collection of retained level-n squares. Let $\mathcal{L}eb$ be the two dimensional Lebesgue measure. Then the natural measure on Λ is:

$$(9) \quad \mu := \lim_{n\to\infty} \frac{\mathcal{L}eb|_{\Lambda_n}}{\mathcal{L}eb(\Lambda_n)} = \lim_{n\to\infty} \frac{\mathcal{L}eb|_{\Lambda_n}}{\#\mathcal{E}_n \cdot M^{-2n}} = \lim_{n\to\infty} \frac{\mathcal{L}eb|_{\Lambda_n}}{p^n \cdot W},$$

where in the last step we used (8) and the limit is meant as a weak limit. It was proved by Mauldin Williams [4] that this limit exists. Y. Peres and M. Rams investigated the θ-angle orthogonal projection of the natural measure $\mu_\theta := (\mathrm{proj}_\theta)_*\mu$ .They proved that

Theorem 9.1 (Peres, Rams [5])

Assume that $Mp > 1$ *(this equivalent with* $\dim_{\mathrm{H}} \Lambda > 1$ *a.s. conditioned on non-extinction.) Then conditioned on non-extinction, for almost all realization the following holds: for all* θ *the projected measure* μ_θ *is absolute continuous. Moreover, if* $\theta \neq 0, \pi/2$ *then the density is Hölder continuous. For the verital and horizontal directions the density in not defined at the M-adic points. Apart from them the density is Hölder cont. for a specially chosen metric.*

One important idea of the proof is that instead of the natural measure μ it is enough to verify the statement for the measure

$$\widetilde{\mu} := W \cdot \mu = \lim_{n \to \infty} \underbrace{\frac{\mathcal{L}eb|_{\Lambda_n}}{p^n}}_{\widetilde{\mu}_n}$$

This is so, because as we discussed in Theorem 2.4, in File A, the r.v. $W > 0$ a.s. conditioned on non-extinction. Now, the measure $\{\widetilde{\mu}_n\}_{n=1}^{\infty}$ is a martingale:

$$\mathbb{E}\left[\widetilde{\mu}_{n+1} | \mathcal{E}_n\right] = \widetilde{\mu}_n. \tag{10}$$

Beside this $\widetilde{\mu}_n$ has another impostant property: If we take the projection proj_θ of the measure $\widetilde{\mu}_n$ to the line of angle angle θ we obtain the measure $\widetilde{\mu}_{n,\theta}$. Observe that this measure has a geometric meaning. Namely, $\widetilde{\mu}_{n,\theta}$ is absolute continuous and its density $\frac{d\widetilde{\mu}_{n,\theta}}{dx}(z)$ at $z \in \ell_\theta$ (the line of angle θ) is

$$\frac{d\widetilde{\mu}_{n,\theta}}{dx}(z) = \frac{\left|\ell_\theta^\perp(z) \cap \Lambda_n\right|}{p^n}$$

This method of Peres and Rams [5] was used by Shmerkin and Suomala [6] (2015) to obtain similar results for many general families of random fractals, where the natural measure (or its rescalled version) is a martingale. The Shmerkin and Soumala paper [6] is a very long paper with lots of applications about the slices and the projections (not only linear ones) of random measures. Here I mention only one example.

Let $r > 0$ be a positive number and $Q(\cdot)$ be the measure $\mathbb{R}^2 \times \left(0, \frac{1}{2}\right)$ defined by $r \cdot s^{-1} d\mathbf{x} ds$. The inhomogeneous Poisson point process with intensity Q is a random countable set $X := \{\mathbf{x}_i, r_i\}$ satisfying:

- For every Borel set $B \subset \mathbb{R} \times \left(0, \frac{1}{2}\right)$

$$\# (X \cap B) \sim \mathrm{Poi}\,(Q(B))$$

- If $B_i \subset \mathbb{R}^2 \times \left(0, \frac{1}{2}\right)$ are pairwise disjoint then the random variables

$$\# \{X \cap B_i\}$$

are independent .

The random cut-out set is

$$A := \overline{B(0,1) \setminus \bigcup_j B(x_j, r_j)} \tag{11}$$

$$A_n := B(0,1) \setminus \bigcup_j \left\{ B(x_j, r_j) : r_j > 2^{-n} \right\}$$

Let $\alpha := c \cdot r$, where c is a constant. Let $d\mu_n(x) := 2^{\alpha n} \mathbb{1}_{A_n}(x)$. The natural measure is

$$\mu_\infty := \lim_{n \to \infty} \mu_n,$$

where the lim is the weak limit.

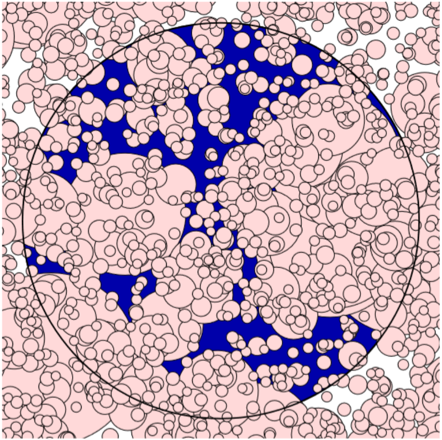


Figure: Figure is from Smerkin Suomala paper

Shmerkin, Suomala Theorem:
The projection of μ_∞ is absolute continuous with Hölder continuous density almost surely whenever the attractor has Hausdorr dimension greater than 1 .

I learned that proof of the dimension formula which is presented here from Michel Dekking.

In this section we are on $\mathbb{R}$. Recall the dimension formula for the homogeneous Mandelbrot percolation with parameters M, p on the line was:

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \frac{\log(M \cdot p)}{\log M} \text{ a.s.} \tag{12}$$

Recall also that the meaning of the nominator of (12):

$$M \cdot p = \mathbb{E}\left[\#\mathcal{E}_1\right].$$

Now we prove formula (12).

$$q := \mathbb{P}\left(\Lambda = \emptyset\right), \quad I_k := \left[\tfrac{k}{M}, \tfrac{k+1}{M}\right], k = 0, \dots, M-1$$

In this one-dimensional setting, the dimension of the Mandelbrot percolation set Λ is $\frac{\log Mp}{M}$. We always assume that

$$p > \frac{1}{M} \tag{13}$$

otherwise $\Lambda = \emptyset$ a.s..

Lemma 10.1

For every $\alpha > 0$ either $\mathcal{H}^{\alpha}(\Lambda) = 0$ holds a.s. or $\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda) = 0\right) = q$. In formula:

$$\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda) = 0\right) \in \{q, 1\}. \tag{14}$$

proof

Let

$$Z_n := \#\mathcal{E}_n$$

We write

$$g(s) := \mathbb{E}\left[s^{Z_1}\right]$$

for the p.g.f. of Z_1.
On the next slide we prove that
$\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda) = 0\right)$ is a fixed point of g.

Using that the set of fixed points of g consists of 1 and q, this will complete the proof. So the calculation is as follows:

proof cont.

$$\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda)=0\right)=\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda_0)=0,\dots,\mathcal{H}^{\alpha}(\Lambda_{M-1})=0\right)$$

$$=\sum_{k=0}^{M}\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda_i)=0,\forall i=0,\dots,M-1|Z_1=k\right)\cdot\mathbb{P}\left(Z_1=k\right)$$

$$=\sum_{k=0}^{M}\left[\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda_0)=0|Z_1=k\right)\right]^k\cdot\mathbb{P}\left(Z_1=k\right)$$

$$=\sum_{k=0}^{M}\left[\mathbb{P}\left(\left(\frac{1}{M}\right)^{\alpha}\mathcal{H}^{\alpha}(\Lambda)=0\right)\right]^k\cdot\mathbb{P}\left(Z_1=k\right)$$

$$=\sum_{k=0}^{M}\left[\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda)=0\right)\right]^k\cdot\mathbb{P}\left(Z_1=k\right).\square$$

The upper bound

Λ_n consists of Z_1 intervals of length M^{-n}. This implies that

$$\mathcal{H}^{\alpha}_{M^{-n}}(\Lambda) \leq Z_n \cdot \left(M^{-n}\right)^{\alpha}. \tag{15}$$

Using this and the Markov inequality:

$$\begin{aligned}\mathbb{P}\left(\mathcal{H}^{\alpha}_{M^{-n}}(\Lambda) \geq \varepsilon\right) &\leq \frac{\mathbb{E}\left[\mathcal{H}^{\alpha}_{M^{-n}}(\Lambda)\right]}{\varepsilon} \\ &\leq \frac{\mathbb{E}\left[Z_n\right]}{\varepsilon M^{n\alpha}} = \frac{\mathbb{E}\left[Z_1\right]^n}{\varepsilon M^{n\alpha}} = \frac{1}{\varepsilon}\left(\frac{\mathbb{E}\left[Z_1\right]}{M^{\alpha}}\right)^n\end{aligned} \tag{16}$$

The upper bound cont.

Let

$$\alpha > \frac{\log \mathbb{E}\left[Z_1\right]}{\log M} = \frac{\log(Mp)}{\log M}.$$

Then

$$\mathbb{E}\left[Z_1\right] < M^{\alpha}.$$

Using Borel Cantelli and (16) this means that

$$\mathbb{P}\left(\mathcal{H}^{\alpha}(\Lambda) = 0\right) = 1,$$

since $\lim_{n\to\infty} \mathcal{H}^{\alpha}_{M^{-n}} = \mathcal{H}^{\alpha}(\Lambda)$. That is $\dim_{\mathrm{H}} \Lambda \leq \alpha$ a.s.$\square$

The following Lemma is a corollary of Lemma 10.1 by an immediate case analysis:

Lemma 10.2

The random variable $\dim_{\mathrm{H}} \Lambda$ *is almost surely* *constant* *on the event* $\{\Lambda \neq \emptyset\}$.

The lower bound

Let

$$s := \frac{\log(Mp)}{\log M}.$$

We want to prove that

(17) $\dim_{\mathrm{H}} \Lambda \geq s$ a.s. conditioned on nonextinction.

First we prove that

Lemma 10.3

If $B \subset [0,1]$ has the property that $\mathbb{P}(\Lambda \cap B \neq 0) > 0$ then this implies that $\dim_{\mathrm{H}} B \geq \frac{-\log p}{M}$.

The lower bound cont.

Proof of the Lemma

Recall that in the definition of the Hausdorff dimension we can restrict ourselves to covers by M-adic intervals like $I := \left[\frac{k-1}{M^n}, \frac{k}{M^n}\right]$. If I is such an interval then

$$\mathbb{P}\left(\Lambda \cap I \neq \emptyset\right) \leq \mathbb{P}\left(I_{\mathrm{left}} \cup I \cup I_{\mathrm{right}} \text{ selected }\right) = 3p^n$$

Using that the solution of the equation $p^n = \left(M^{-n}\right)^x$ is $x = \frac{-\log p}{\log M}$, from the previous formula we get that

$$\mathbb{P}\left(\Lambda \cap I \neq \emptyset\right) \leq 3|I|^{\frac{-\log p}{\log M}}. \tag{18}$$

The lower bound cont.

Proof of the Lemma cont.

To prove that $\dim_{\mathrm{H}} B \geq \frac{-\log p}{\log M}$ it is enough to verify that there exists a constant $C > 0$ such that for an arbitrary covering $\{I_k\}$ of Λ by M-adic intervals (not necessarily of the same length) we have

$$\sum_k |I_k|^{\frac{-\log p}{\log M}} > C > 0. \tag{19}$$

To see this, we define $C := \mathbb{P}(\Lambda \cap B)$. By assumption $C > 0$. Using that $\{I_k\}$ is a cover of B we have:

The lower bound cont.

Proof of the Lemma cont.

$$0 < C = \mathbb{P}(\Lambda \cap B \neq \emptyset) \leq \mathbb{P}\left(\Lambda \cap \bigcup_k I_k \neq \emptyset\right) \leq \sum_k 3|I_k|^{\frac{-\log p}{\log M}}.$$

This completes the proof of the Lemma.

The lower bound cont.

Now we consider three Mandelbrot percolation sets $\Lambda, \widetilde{\Lambda}$ and $\widehat{\Lambda}$ on the line. One of the parameters for all of them is the same M. The other parameters are $p, \widetilde{p}$ and $\widehat{p}$ respectively. We assume that

$$\widehat{p} = p \cdot \widetilde{p}. \tag{20}$$

We have already discussed that

$$\widehat{\Lambda} \stackrel{d}{=} \Lambda \cap \widetilde{\Lambda}. \tag{21}$$

In particular,

The lower bound cont.

$$\mathbb{P}_{\widehat{p}}\left(\widehat{\Lambda} \neq \emptyset\right) = \left(\mathbb{P}_p \times \mathbb{P}_{\widetilde{p}}\right)\left(\Lambda \cap \widetilde{\Lambda} \neq \emptyset\right) \tag{22}$$

Let

$$V_{p,\widetilde{p}} := \left\{\omega_p \in \Omega_p : \mathbb{P}_{\widetilde{p}}\left(\widetilde{\omega_p} \in \Omega_{\widetilde{p}} : \Lambda(\omega_p) \cap \widetilde{\Lambda}(\omega_{\widetilde{p}}) \neq \emptyset\right) > 0\right\}$$

The lower bound cont.

Lemma 10.4

Assume that $\widehat{p} > \frac{1}{M}$. *We choose* $p, \widetilde{p}$ *such that (as always)* $\widehat{p} = p \cdot \widetilde{p}$. *Then*

$$\mathbb{P}_p \left(V_{p,\widetilde{p}} \right) > 0 \tag{23}$$

Proof.

By assumption $\mathbb{P}_{\widehat{p}} \left(\widehat{\Lambda} \neq \emptyset \right) > 0$. Then the assertion of the Lemma follows from (22) and Fubini Theorem. □

The lower bound cont.

Here we use the notation and assumption of Lemma 10.4. Now we fix an $\omega_p \in V_{p,\widetilde{p}}$. Let $B := \Lambda(\omega_p)$. Then by the definition of $V_{p,\widetilde{p}}$ we have

$$\mathbb{P}_{\widetilde{p}}\left(\omega_{\widetilde{p}} \in \Omega_{\widetilde{p}} : \Lambda(\omega_{\widetilde{p}}) \cap B \neq \emptyset\right) > 0.$$

This implies by Lemma 10.3 that

$$\dim_{\mathrm{H}} \Lambda(\omega_p) \geq \frac{-\log \widetilde{p}}{\log M}.$$

The lower bound cont.

We have assumed that

$$\frac{1}{M} < \widehat{p} = p \cdot \widetilde{p}.$$

That is $\widetilde{p} > \frac{1}{Mp}$ and $\widetilde{p}$ can be as close to $\frac{1}{Mp}$ as we want. So on a set of positive $\mathcal{P}$-measure of $\omega \in V_{p,\widetilde{p}}$, we have

$$\frac{-\log \widetilde{p}}{\log M} \leq \dim_{\mathrm{H}} \Lambda(\omega_p) \leq \frac{\log(Mp)}{\log M}. \tag{24}$$

But we know that $\dim_{\mathrm{H}} \Lambda_p(\omega_p)$ is constant on $\Lambda_p \neq \emptyset$ this completes the proof.□

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