Quasicircles of dimension $1 + k^2$ do not exist

Oleg Ivrii

September 17, 2015

History

Integral means spectra of conformal maps ('85 – '91)

Makarov, Becker, Pommerenke,
Przytycki, Urbański, Zdunik, Bañuelos, Moore, Lyons...

Dimension of Quasicircles ('94 – early '00s)

Astala, Ransford, Smirnov

Weil-Petersson metric in complex dynamics ('08)

McMullen

Reminder: Quasiconformal maps

A *k-quasiconformal map* $f : \mathbb{C} \to \mathbb{C}$ is an o.p. homeo for which

$$\frac{\partial f}{\overline{\partial} z} = \mu(z) \frac{\partial f}{\partial z}$$

with

$$\|\mu\|_\infty \leq k, \qquad 0 \leq k < 1.$$

Converse: Measurable Riemann mapping theorem.

Allows one to embed q.c. maps into holomorphic families by solving

$$\frac{\partial f_t}{\overline{\partial} z} = \frac{t}{k} \mu(z) \frac{\partial f_t}{\partial z}, \qquad t \in \mathbb{D}.$$

The problem

90's formulation: Let $D(k)$, the maximal dimension of a k-quasicircle.

Theorem: (Smirnov) $D(k) \leq 1 + k^2$.

Is it sharp??

The problem

90's formulation: Let $D(k)$, the maximal dimension of a k-quasicircle.

Theorem: (Smirnov) $D(k) \leq 1 + k^2$.

Is it sharp??

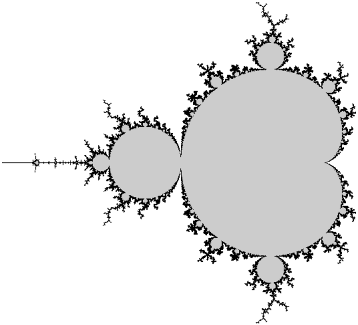
Modern formulation: Find the supremum of

$$\left(\frac{||\mu||_{\mathrm{WP}}}{||\mu||_T} \right)^2$$

over all tangent vectors in all generalized main cardioids in Poly_d, $d \geq 2$.

~~all Teichmüller spaces $\mathcal{T}_g$, $g \geq 2$.~~ ($D_{\mathrm{F}}(k) < 1 + (2/3)\, k^2$, k small)

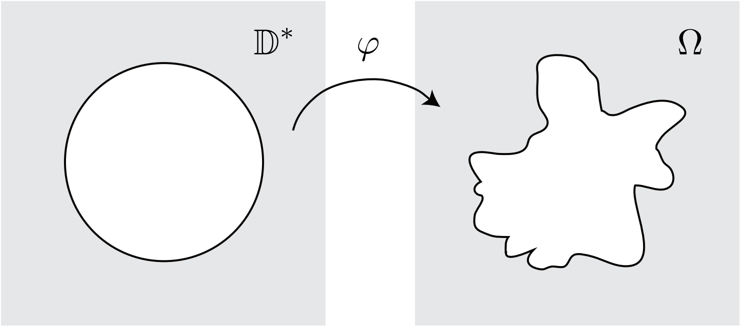
The Main Cardioid $\subset$ Mandelbrot Set $\subset$ Poly_d



Conjecture: The Weil-Petersson metric is incomplete; completion attaches a single point to geometrically finite parameters.

Riemann Mapping Theorem

Let $\mathbb{D}^* = \{z : |z| > 1\}$ be the exterior unit disk.



"Complexity of the boundary $\partial\Omega$" is manifested in the "complexity of the Riemann map".

Integral means spectra

For a conformal map $\varphi : \mathbb{D}^* \to \Omega$, the integral means spectrum is given by

$$\beta_\varphi(p) = \limsup_{R \to 1^+} \frac{\log \int_{|z|=R} |\varphi'(z)|^p \, |dz|}{\log \frac{1}{R-1}}, \qquad p \in \mathbb{R}.$$

Integral means spectra

For a conformal map $\varphi : \mathbb{D}^* \to \Omega$, the integral means spectrum is given by

$$\beta_\varphi(p) = \limsup_{R \to 1^+} \frac{\log \int_{|z|=R} |\varphi'(z)|^p \, |dz|}{\log \frac{1}{R-1}}, \qquad p \in \mathbb{R}.$$

Problem: Find the **universal** integral means spectrum

$$B(p) := \sup_{\varphi \in \mathcal{T}(\mathbb{D}^*)} \beta_\varphi(p),$$

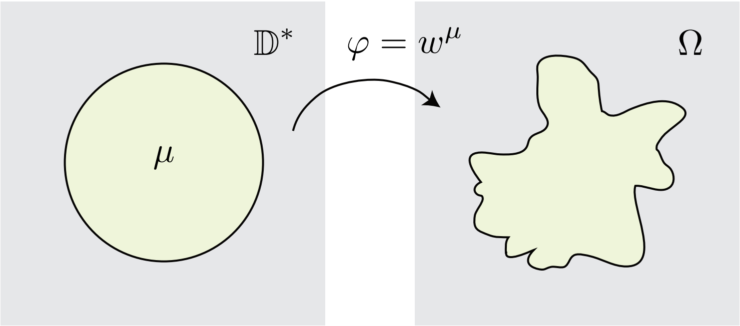
where $\mathcal{T}(\mathbb{D}^*)$ is the universal Teichmüller space.

Universal Teichmüller space

By definition,

$$\mathcal{T}(\mathbb{D}^*) := \bigcup_{0 \leq k < 1} \Sigma_k,$$

where $\Sigma_k = \{\varphi :$ admit a k-quasiconformal extension to $\mathbb{C}\}$.



Relation to "Dimensions of Quasicircles"

In view of Royden's theorem,

$$\text{Teichmüller metric} = \text{Kobayashi metric},$$

it is also natural to consider

$$B_k(p) := \sup_{\varphi \in \Sigma_k} \beta_\varphi(p).$$

Relation to “Dimensions of Quasicircles”

In view of Royden’s theorem,

$$\text{Teichmüller metric} = \text{Kobayashi metric},$$

it is also natural to consider

$$B_k(p) := \sup_{\varphi \in \Sigma_k} \beta_\varphi(p).$$

Also, if Ω is a quasidisk, then

$$\beta_\varphi(p) = p - 1 \quad \Longleftrightarrow \quad p = \mathrm{M.dim}\, \partial\Omega.$$

Growth of Bloch functions

A holomorphic function $b(z)$ on $\mathbb{D}^*$ is Bloch if

$$\|b\|_{\mathcal{B}^*} := \sup_{z\in\mathbb{D}^*} \left| b'(z)(|z|^2 - 1) \right| < \infty.$$

Growth of Bloch functions

A holomorphic function $b(z)$ on $\mathbb{D}^*$ is Bloch if

$$\|b\|_{\mathcal{B}^*} := \sup_{z\in\mathbb{D}^*} \left| b'(z)(|z|^2 - 1) \right| < \infty.$$

Bers embedding: $\mathcal{T}(\mathbb{D}^*) \to \mathcal{B}^*$ given by $\varphi \to b_\varphi := \log \varphi'$.

Growth of Bloch functions

A holomorphic function $b(z)$ on $\mathbb{D}^*$ is Bloch if

$$\|b\|_{\mathcal{B}^*} := \sup_{z\in\mathbb{D}^*} \left|b'(z)(|z|^2-1)\right| < \infty.$$

Bers embedding: $\mathcal{T}(\mathbb{D}^*) \to \mathcal{B}^*$ given by $\varphi \to b_\varphi := \log \varphi'$.

For such a function, one can define its asymptotic variance by

$$\sigma^2(b) = \limsup_{R\to 1^+} \frac{1}{2\pi|\log(R-1)|} \int_{|z|=R} |b(z)|^2 \, |dz|.$$

(The asymptotic variance is finite.)

Equality of Characteristics

Przytycki, Urbański, Zdunik, Makarov, Binder, McMullen...

Dynamical setting: If $\partial\Omega$ is a regular fractal, e.g. a Julia set or a limit set of a Kleinian group, then

$$2\frac{d^2}{dp^2}\bigg|_{p=0}\beta_\varphi(p) = \sigma^2(\log\varphi') = \cdots .$$

Equality of Characteristics

Przytycki, Urbański, Zdunik, Makarov, Binder, McMullen...

Dynamical setting: If $\partial\Omega$ is a regular fractal, e.g. a Julia set or a limit set of a Kleinian group, then

$$2\frac{d^2}{dp^2}\bigg|_{p=0}\beta_\varphi(p) = \sigma^2(\log\varphi') = \cdots .$$

Theorem: True for universal bounds:

$$2\frac{d^2}{dp^2}\bigg|_{p=0}B_k(p) = \Sigma^2(k) := \sup_{\varphi\in\Sigma_k}\sigma^2(\log\varphi'),$$

$\Sigma^2(k)/k^2$ is a continuous increasing function for $k \in [0,1)$.

Asymptotic variance of the Beurling transform

Infinitesimal version. In view of $\|\log \varphi' - k\mathcal{S}\mu\|_{\mathcal{B}^*} = \mathcal{O}(k^2)$,

$$\Sigma^2 := \sup_{|\mu| \le \chi_{\mathbb{D}}} \sigma^2(\mathcal{S}\mu)$$

where $\mathcal{S}\mu$ is the Beurling transform

$$\mathcal{S}\mu(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\mu(w)}{(z-w)^2} dm(w), \quad |z| > 1.$$

It is easy to see that $\mathcal{S}\mu \in \mathcal{B}^*$.

Asymptotic variance of the Beurling transform

Infinitesimal version. In view of $\| \log \varphi' - k\mathcal{S}\mu\|_{\mathcal{B}^*} = \mathcal{O}(k^2)$,

$$\Sigma^2 := \sup_{|\mu|\leq\chi_{\mathbb{D}}} \sigma^2(\mathcal{S}\mu)$$

where $\mathcal{S}\mu$ is the Beurling transform

$$\mathcal{S}\mu(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\mu(w)}{(z-w)^2} dm(w), \quad |z| > 1.$$

It is easy to see that $\mathcal{S}\mu \in \mathcal{B}^*$.

Theorem: (AIPP) $0.879 \leq \Sigma^2 \leq 1$.

(Hedenmalm) $\Sigma^2 < 1$.

Equality of Characteristics II

Przytycki, Urbański, Zdunik, Makarov, Binder, McMullen...

Dynamical setting: If $\{\varphi_t\}$ is a holomorphic family of conformal maps, and $\varphi_t(\mathbb{S}^1)$ are invariant under hyperbolic dynamical systems,

$$\begin{aligned} 2\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{M.}\dim \varphi_t(\mathbb{S}^1) &= \sigma^2\left(\frac{d}{dt}\bigg|_{t=0} \log \varphi_t'\right), \\ &= \sigma^2(\mathcal{S}\mu), \\ &= \|\mu\|^2_{\mathrm{WP}}, \end{aligned}$$

where $\|\cdot\|^2_{\mathrm{WP}}$ is the Weil-Petersson metric.

Equality of Characteristics II

Przytycki, Urbański, Zdunik, Makarov, Binder, McMullen...

Dynamical setting: If $\{\varphi_t\}$ is a holomorphic family of conformal maps, and $\varphi_t(\mathbb{S}^1)$ are invariant under hyperbolic dynamical systems,

$$\begin{aligned} 2\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{M.}\dim \varphi_t(\mathbb{S}^1) &= \sigma^2\left(\frac{d}{dt}\bigg|_{t=0} \log \varphi_t'\right), \\ &= \sigma^2(\mathcal{S}\mu), \\ &= \|\mu\|^2_{\mathrm{WP}}, \end{aligned}$$

where $\|\cdot\|^2_{\mathrm{WP}}$ is the Weil-Petersson metric.

Theorem: True for universal bounds.

Dimensions of Quasicircles

Find $D(k)$, the maximal dimension of a k-quasicircle.

Theorem: (Smirnov) $D(k) \leq 1 + k^2$.

Is it sharp??

Dimensions of Quasicircles

Find $D(k)$, the maximal dimension of a k-quasicircle.

Theorem: (Smirnov) $D(k) \leq 1 + k^2$.

Is it sharp??

NO!

Theorem: $D(k) = 1 + \Sigma^2 k^2 + \mathcal{O}(k^{2.5})$.

(AIPP – Lower bound comes from polynomial perturbations of $z \to z^d$.)

Box Lemma

Infinitesimal version: For any $\varepsilon > 0$, if R is sufficiently large, then

$$\fint_B \left| \frac{2(\mathcal{S}\mu)'}{\rho_*}(z) \right|^2 \rho_* |dz|^2 < \Sigma^2 + \varepsilon,$$

for any μ with $|\mu| \le \chi_{\mathbb{D}}$ and every R-box B.

Box Lemma

Infinitesimal version: For any $\varepsilon > 0$, if R is sufficiently large, then

$$\fint_B \left| \frac{2(\mathcal{S}\mu)'}{\rho_*}(z) \right|^2 \rho_* |dz|^2 < \Sigma^2 + \varepsilon,$$

for any μ with $|\mu| \leq \chi_{\mathbb{D}}$ and every R-box B.

Global version: For any $\varepsilon > 0$, if R is sufficiently large, then

$$\fint_B \left| \frac{2(\log \varphi')'}{\rho_*}(z) \right|^2 \rho_* |dz|^2 < \Sigma^2(k) + \varepsilon,$$

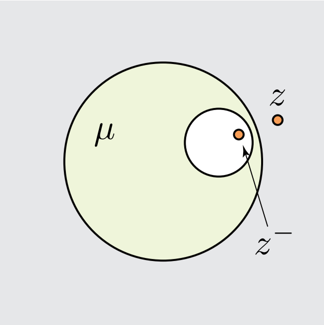
for any conformal map $\varphi \in \Sigma_k$ and every R-box B.

(Asymptotic) locality of non-linearity

Bishop, Jones, McMullen...

Infinitesimal version: Suppose $\mu \in M(\mathbb{D})$ with $\|\mu\|_\infty \leq 1$. Then,

- For $z \in \mathbb{D}^*$, $\left|((\mathcal{S}\mu)'/\rho_*)(z)\right| \lesssim 1$.
- If $d_{\mathbb{D}}^{\square}(z^-, \operatorname{supp}\mu) \geq R$ then $\left|((\mathcal{S}\mu)'/\rho_*)(z)\right| \lesssim e^{-R}$.

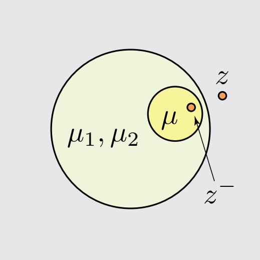


(Asymptotic) locality of non-linearity

Global version: Suppose $\mu \in M(\mathbb{D})$ with $\|\mu\|_\infty \leq k$. Then,

- For $z \in \mathbb{D}^*$, $\left|(N_\varphi/\rho_*)(z)\right| \lesssim 1$.
- If $d_{\mathbb{D}}(z^-, \operatorname{supp}(\mu_1 - \mu_2)) \geq R$ then

$$\left|\frac{N_{\varphi_1} - N_{\varphi_2}}{\rho_*}(z)\right| \lesssim e^{-C(k)\,R} + o(|z| - 1).$$



Sketch of proof

Want to estimate

$$\int_{|z|=r} |\varphi'|^p, \qquad \text{with} \quad 0 \leq k < 1 \quad \text{and} \quad pk \approx 0.$$

Sketch of proof

Want to estimate

$$\int_{|z|=r} |\varphi'|^p, \qquad \text{with} \quad 0 \le k < 1 \quad \text{and} \quad pk \approx 0.$$

Trick! We instead estimate

$$u(r) := \int_{A(r)} |\varphi'|^p \, \rho_* |dz|^2$$

where $A(r) = \left\{ z : \; r < |z| < 1 + \frac{r-1}{R} \right\}$, $R > 0$ large.

Sketch of proof II

Hardy's identity $\implies$

$$u''(r) \leq \frac{1}{(r-1)^2} \cdot \frac{p^2}{4} \int_{A(r)} |\varphi'|^p \cdot \left| \frac{2(\log \varphi')'}{\rho_*}(z) \right|^2 \rho_* |dz|^2$$

Sketch of proof II

Hardy's identity $\Longrightarrow$

$$u''(r) \leq \frac{1}{(r-1)^2} \cdot \frac{p^2}{4} \int_{A(r)} |\varphi'|^p \cdot \left| \frac{2(\log \varphi')'}{\rho_*}(z) \right|^2 \rho_* |dz|^2$$

Box lemma $\Longrightarrow$

$$u''(r) \leq \frac{1}{(r-1)^2} \cdot \frac{p^2}{4} \int_{A(r)} |\varphi'|^p \cdot (\Sigma^2(k) + \varepsilon)\, \rho_* |dz|^2$$

Sketch of proof II

Hardy's identity $\implies$

$$u''(r) \leq \frac{1}{(r-1)^2} \cdot \frac{p^2}{4} \int_{A(r)} |\varphi'|^p \cdot \left| \frac{2(\log \varphi')'}{\rho_*}(z) \right|^2 \rho_* |dz|^2$$

Box lemma $\implies$

$$u''(r) \leq \frac{1}{(r-1)^2} \cdot \frac{p^2}{4} \int_{A(r)} |\varphi'|^p \cdot (\Sigma^2(k) + \varepsilon)\, \rho_* |dz|^2$$

Since $u''(r) \leq \dfrac{cu}{(r-1)^2}$, u is bounded above by solution of the differential equation. (cf. Becker-Pommerenke, $1 + 37k^2$).

Additional applications

Proof is very simple and **robust**:

- Provides an alternative proof of "dynamical equalities" – including parabolic cases
- Applies to coefficients invariant under Fuchsian groups, with $\Sigma_F^2 < 2/3$ replacing Σ^2
- Improved estimates for sparse Beltrami coefficients
- Can use any equivalent norm on the Bloch space (we use the L^∞ norm, classically people in '80s used the Bloch norm)
- Works in higher dimensions (if one wants to forgo connections to Hausdorff dimension)

Additional applications

Proof is very simple and **robust**:

- Provides an alternative proof of "dynamical equalities" – including parabolic cases
- Applies to coefficients invariant under Fuchsian groups, with $\Sigma_F^2 < 2/3$ replacing Σ^2
- Improved estimates for sparse Beltrami coefficients
- Can use any equivalent norm on the Bloch space (we use the L^∞ norm, classically people in '80s used the Bloch norm)
- Works in higher dimensions (if one wants to forgo connections to Hausdorff dimension)

A similar argument gives connections to Makarov's constant in the law of iterated logarithm. (Joint with I. Kayumov.)

Thank you for your attention!