Dynamics of the scenery flow and conical density theorems
Simons Semester minicourse, October 2015
Exercises

1. Show that for each $0 < \beta < \pi$ there is $q \in \mathbb{N}$ such that in any set of q points in $\mathbb{R}^2$ there always exist three points which determine an angle between β and π.

2. Show that for each $1 < s \le 2$ and $0 < \alpha \le 1$ there is a constant $\varepsilon = \varepsilon(s, \alpha) > 0$ satisfying the following: For every $E \subset \mathbb{R}^2$ with $\mathcal{H}^s(E) < \infty$ it holds that

$$\limsup_{r\downarrow 0} \inf_{\theta_1,\theta_2 \in S^1} \frac{\mathcal{H}^s(E \cap X(x, r, \theta_1, \alpha) \setminus H(x, \theta_2, \alpha))}{(2r)^s} \ge \varepsilon$$

for $\mathcal{H}^s$-almost all $x \in A$. Here $H(x, \theta, \alpha) = \{y \in \mathbb{R}^2 : (y - x) \cdot \theta > \alpha|y - x|\}$.

We say that $E \subset \mathbb{R}^2$ is (α, k)-*porous*, $0 < \alpha < 1/2$ and $k \in \{1, 2\}$, if for each $x \in E$ and $0 < r < \operatorname{diam}(E)$ there are $y_1, \ldots, y_k \in \mathbb{R}^2$ such that

$$B(y_i, \alpha r) \subset B(x, r) \setminus E$$

for all $i \in \{1, \ldots, k\}$ and $(y_i - x) \cdot (y_j - x) = 0$ whenever $i \ne j$.

3. Show that

$$\sup\{s > 0 : E \text{ is } (\alpha, 2)\text{-porous and } \dim_{\mathrm{H}}(E) > s \text{ for some } E \subset \mathbb{R}^2\} \to 0$$

as $\alpha \to 1/2$.

4. Show that the standard Cantor 4-corner set with dimension strictly less than one can be covered by a single Lipschitz curve.

5. Show that a distribution Q on Ω is adapted if and only if there is $\overline{Q}$ such that

$$\int f(\mu, x)\, \mathrm{d}Q(\mu, x) = \iint f(\mu, x)\, \mathrm{d}\mu(x)\, \mathrm{d}\overline{Q}(\mu)$$

for all essentially bounded and measurable f.

6. Show that if a distribution Q on Ω is adapted, then MQ is adapted.

A set $A' \subset [0, 1]^2$ is a *miniset* of $A \subset [0, 1]^2$ if $A' \subset (\lambda A + t) \cap [0, 1]^2$ for some $\lambda \ge 1$ and $t \in \mathbb{R}^2$. A set $A' \subset [0, 1]^2$ is a *microset* of a compact set $A \subset [0, 1]^2$ if there exists a sequence $(A'_n)_n$ of minisets of A such that $A'_n \to A'$ in the Hausdorff metric. A compact set $A \subset [0, 1]^2$ is *homogeneous* is every microset of A is a miniset of A.

7. Show that a self-homothetic set $E \subset [0, 1]^2$ satisfying the strong separation condition if homogeneous. Does the claim remain true if the strong separation is replaced by the open set condition?

8. Let μ be a Radon measure on $\mathbb{R}^2$ and define $\underline{\dim}_{\mathrm{H}}(\mu) = \operatorname{ess\,inf}_{x\sim\mu} \underline{\dim}_{\mathrm{loc}}(\mu, x)$. Show that

$$\underline{\dim}_{\mathrm{H}}(\mu) = \inf\{\dim_{\mathrm{H}}(A) : A \subset \mathbb{R}^2 \text{ is a Borel set with } \mu(A) > 0\}.$$

9. Show that the set $\mathcal{A}_\varepsilon = \{v \in \mathcal{M}_1 : \nu(B(y, \alpha)) \leq \varepsilon \text{ for some } y \in \overline{B}(0, 1-\alpha)\}$ is closed in $\mathcal{M}_1$ for all $\varepsilon \geq 0$.

A Radon measure μ is $(\alpha, 1)$-*porous at a point* x *and scale* r *with threshold* ε if there exists $y \in \mathbb{R}^2$ with

$$B(y, \alpha r) \subset \overline{B}(x, r) \quad \text{and} \quad \mu(B(y, \alpha r)) \leq \varepsilon\mu(\overline{B}(x, r)).$$

If for all small enough $\varepsilon > 0$ this happens for all enough small $r > 0$ (depending on ε), then μ is $(\alpha, 1)$-*porous at* x and if μ is $(\alpha, 1)$-porous at μ-almost every x, then μ is $(\alpha, 1)$-*porous*.

10. Show that

$$\begin{aligned} &\sup\{\underline{\dim}_{\mathrm{H}}(\mu) : \mu \text{ is an } (\alpha, 1)\text{-porous Radon measure on } \mathbb{R}^2\} \\ =&\sup\{\dim_{\mathrm{H}}(E) : E \subset \mathbb{R}^2 \text{ is a } (\alpha, 1)\text{-porous set}\}. \end{aligned}$$