Differentiable dynamics near and far from homoclinic bifurcations

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Abstract

In this mini-course, we will give a panorama of various results on the global behavior of differentiable dynamical systems, far and near from homoclinic bifurcations. We will focus on examples, and we will indicate some ideas of the proofs involved.

1 Introduction to uniformly Hyperbolic dynamical systems (1h)

First we will recall the definition of uniformly hyperbolic compact set. The folklore theory will be recall (stable manifold theorem, expansiveness...). We will present all the known examples. At this occasion, we will give the definition of dynamics which are Morse-Smale, axiom A, and those which satisfy the strong transversality condition.

2 Structural stability (3h)

We will first recall the definition of structural stability for dynamical systems. Then we will present the main conjectures involved (Smale conjecture, Fatou conjecture).

2.1 Far from bifurcation, dynamical systems are structurally stable

We will present the known results in this direction:

- C^1-structurally stable diffeomorphisms are Axiom A [Mañ88].
- Structural stability Theorem for holomorphic 1d-dynamics far from bifurcations and its genericity [MSS83].
- Structural stability Theorems for holomorphic dynamics of $\mathbb{C}^2$ far from bifurcations [DL, BD].

The proofs based on the lambda-Lemma will be briefly explained.

2.2 Uniformly hyperbolic dynamical systems are structurally stable

Then, we will present the following results:

- Structural stability of diffeomorphisms which are axiom A which satisfy the strong transversality condition [Rob71, Rob76].
- Structural stability of the inverse limit of endomorphisms which are axiom A and which satisfy the strong transversality condition [BK13].
- Structural stability of endomorphisms with singularity [Ber12].

For the two first results, we will present the idea of the proof based on the implicit function Theorem.

For the last Theorem, we will present the analogy in the state of the art between singularity theory and structural stability.

3 Abundance of infinitely many attractors (3h)

We will study the Newhouse phenomena, and how it is abundant. First we will describe how a perturbation of a homoclinic tangency produces a sink. Then we will study two mechanisms to create robust homoclinic tangencies.

3.1 Wild Horseshoes

The first mechanism holds for horseshoe of a surface, so that the transverse space to the unstable and stable laminations are sufficiently "large". Several criteria will be given [New74, MY01, MY10]. Then we will explain how such a mechanism produces generic surface diffeomorphisms with infinitely many sinks [New79]. Moreover we will explain that this set has co-dimension at least 1/2 in the dissipative case [BDS15].

3.2 Blenders and parablender

The second mechanism holds for local diffeomorphisms of surface or diffeomorphisms of higher dimensional manifolds. This is constructed thank to a special class of horseshoes called blenders [BD99, DNP06]. We will generalize this construction to obtain a counter example to the following conjecture [Ber14]:

Conjecture 3.1. *Diffeomorphisms with finitely many attractors are typical in the sense of Kolmogorov.*

4 Abundance of non-uniformly hyperbolic dynamics (2h)

We will finish this mini-course by giving the known examples of abundant non-uniformly hyperbolic sets for 1-dimensional maps and surface diffeomorphisms [Jak81, Ree86, Yoc, PY09, H7́6, BC91, Ber]. We will define the SRB measure and give sufficient conditions to obtain them. We will end by giving a toy model for the parameter selection of non-uniformly hyperbolic maps [BM13].

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