HOLOMORPHIC INTERPOLATION: FROM DYNAMICS TO ANALYSIS

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TOPICS

- variation of dimension of Julia sets
- quasiconvexity and Burkholder’s functional
- rotational estimates for bilipschitz maps
- dimension of quasicircles

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holomorphic interpolation

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VARIATION OF DIM OF JULIA SETS

Ransford

$(q_\lambda)_\lambda,\ \lambda \in D$ analytic family of rational maps, $d \geqslant 2$

D simply connected, q_λ hyperbolic

Theorem: $1/\dim J_\lambda = \inf_{u \in \mathcal{H}} u(\lambda)$ $\quad$ $\mathcal{H}$ a collection of harmonic functions

Corollary:
$$\frac{1/\dim(J_{\lambda_1}) - \frac{1}{2}}{1/\dim(J_{\lambda_2}) - \frac{1}{2}} \leqslant \exp \rho_D(\lambda_1, \lambda_2)$$

Pf: $\dim \leqslant 2,\quad u - \dfrac{1}{2} \geqslant 0 \quad \xrightarrow{\text{Harnack}} \quad \dfrac{u(\lambda_1) - \frac{1}{2}}{u(\lambda_2) - \frac{1}{2}} \leqslant \exp \rho_D(\lambda_1, \lambda_2)$

Example: $q_\lambda = z^2 + \lambda z$, $\lambda \in \mathbb{D}$ $\qquad \dim J_\lambda \leqslant 1 + |\lambda|$

THERMODYNAMICS

variational principle (Ruelle, Bowen)

$$\dim(J_\lambda) = \sup_{\mu \in \mathcal{M}_\lambda} \frac{h_\mu(q_\lambda)}{\int_{J_\lambda} \log|q'_\lambda| d\mu}$$

$$\varphi_\lambda : J_{\lambda_0} \to J_\lambda$$

$$q_\lambda = \varphi_\lambda \circ q_{\lambda_0} \circ \varphi_\lambda^{-1}$$

Mañé-Sad-Sullivan

holomorphic motion

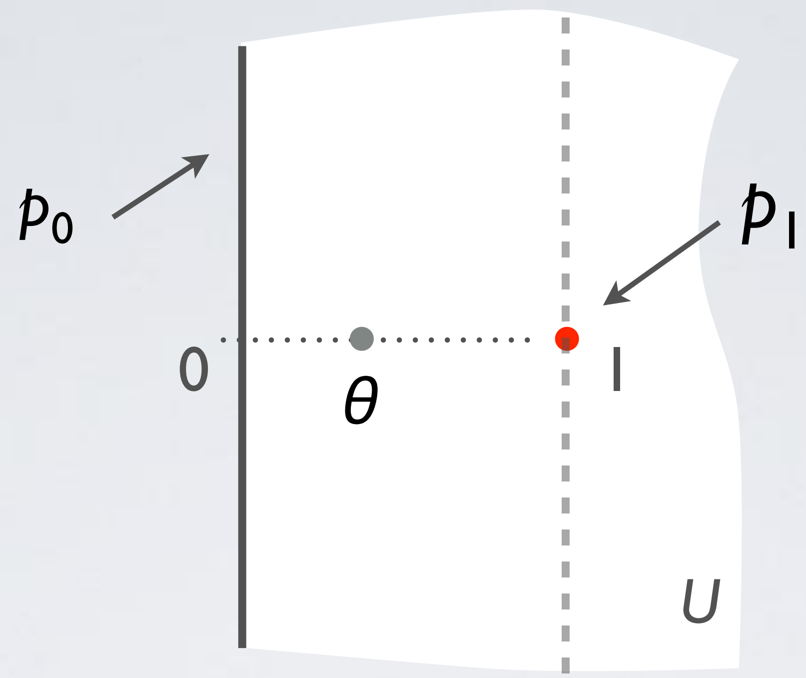
$$\frac{1}{\dim J_\lambda} = \inf_{\mu \in \mathcal{M}_{\lambda_0}} \frac{\int_{J_{\lambda_0}} \log|q'_\lambda \circ \varphi_\lambda| d\mu}{h_\mu(q_{\lambda_0})} \quad \longleftarrow \quad \text{harmonic}$$

HOLOMORPHIC INTERPOLATION

variation of dimension	Riesz-Thorin
holomorphic motions	holomorphic exponent
dimension	norm
variational principle	duality
apriori bounds	endpoint estimates
Harnack's inequality	Hadamard's three lines theorem

INTERPOLATION LEMMA

Astala-Iwaniec-Prause-Saksman



$$0 < p_0, p_1 \leqslant \infty, \qquad \theta \in (0,1)$$

$\phi_\lambda(z)$ analytic family, $\lambda \in U = \{\mathrm{Re}\,\lambda > 0\}$

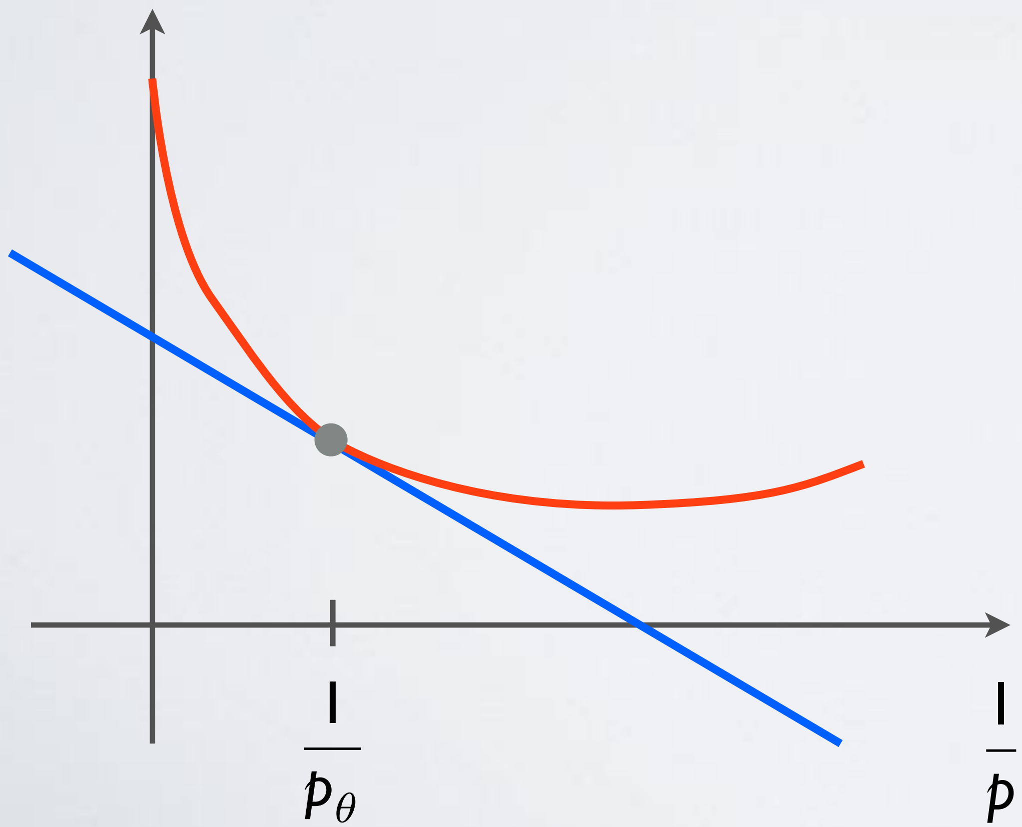
non-vanishing $\phi_\lambda(z) \neq 0$

$$\|\phi_\lambda\|_{p_0} \leqslant M_0 \qquad \|\phi_1\|_{p_1} \leqslant M_1 \quad \Rightarrow \quad \|\phi_\theta\|_{p_\theta} \leqslant M_0^{1-\theta} \cdot M_1^{\theta}, \qquad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

cf. RIESZ-THORIN

duality ⤳ *log-convexity*

change p	freeze p
subharmonic	harmonic
Hadamard	Harnack



$$\log \|\phi_\theta\|_p \geqslant A \cdot \frac{1}{p} + B$$

MARTINGALE INEQUALITY

Burkholder

$X_n \prec Y_n$ subordinated martingales

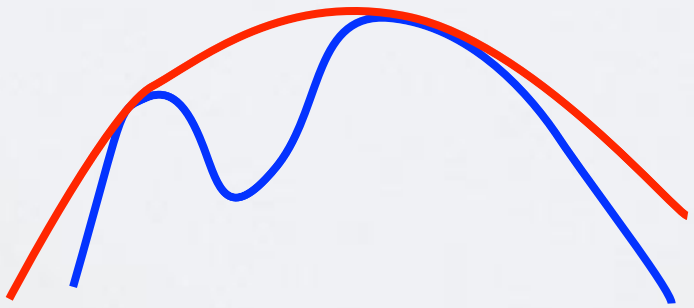
$$|X_n - X_{n-1}| \leq |Y_n - Y_{n-1}| \text{ a.s.}$$

$$\Rightarrow \quad \|X_n\|_p \leqslant (p-1)\,\|Y_n\|_p.$$

$$B_p(z,w) = \big(|z| - (p-1)\,|w|\big)\cdot\big(|z| + |w|\big)^{p-1}$$

$$|z|^p - (p-1)^p\,|w|^p \leqslant c_p\,B_p(z,w)$$

$$\mathbb{E}\,B_p(X_n,Y_n) \leqslant \mathbb{E}\,B_p(X_{n-1},Y_{n-1}) \leqslant \ldots \leqslant 0$$



Rank-one convexity vs Quasiconvexity

local Morrey *global*

$$\mathbf{E}: \mathbb{R}^{n\times n} \to \mathbb{R}$$

rank $X = 1$

$t \mapsto \mathbf{E}(A + tX)$ *convex*

$$\Leftarrow \quad \int_\Omega \mathbf{E}(Df) \geqslant \int_\Omega \mathbf{E}(A) = \mathbf{E}(A)|\Omega|$$

$$f \in A + C_0^\infty(\Omega, \mathbb{R}^n)$$

(ellipticity of Euler-Lagrange)

(lower semicontinuity)

$n \geqslant 3$ Šverák $\not\Rightarrow$

$n = 2$? Faraco-Székelyhidi: "localization"

Burkholder: $B_p(Df) = B_p(f_z, f_{\bar z})$ rank-one concave

$$B_p(A) = \left(\frac{p}{2} \det A + (1 - \frac{p}{2}) |A|^2 \right) \cdot |A|^{p-2}$$

Quasiconvexity result

Astala-Iwaniec-Prause-Saksman

$$B_p(z,w) = \big(|z| - (p-1)\,|w|\big)\cdot\big(|z| + |w|\big)^{p-1} \qquad p \geqslant 2$$

$$B_p(Df) = B_p(f_z, f_{\bar{z}})$$

Theorem: $f(z) \in z + C_0^\infty(\Omega), \quad B_p(Df) \geqslant 0, \; z \in \Omega$

$$\int_\Omega B_p\,(Df) \leqslant \int_\Omega B_p\,(Id) = |\Omega|$$

full quasiconvexity →

$$\|S\|_{L^p(\mathbb{C})} = p - 1$$

$$Sf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{(\zeta - z)^2}\,dm(\zeta)$$

Proof of main thm: $\quad p \geq 2$

$$f_{\bar z} = \mu f_z \qquad |\mu(z)| \leq \frac{1}{p-1} \chi_{\mathbb{D}}(z) \qquad f(z) = z + \mathcal{O}(1/z)$$

want: $$\frac{1}{\pi} \int_{\mathbb{D}} \left(1 - p \frac{|\mu(z)|}{1+|\mu(z)|}\right) |Df(z)|^p \, dm(z) \leq 1$$

$p = 2$ $$\frac{|Df(z)|^2}{K(z,f)} = J(z,f) \quad \text{null-Lagrangian} \checkmark$$

$p = \infty$ $$K = 1 \qquad f(z) = z \quad \checkmark$$

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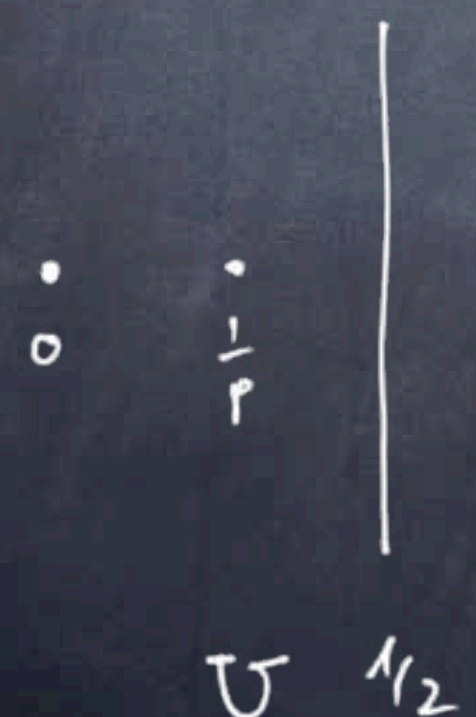

$$\lambda \in U = \{\operatorname{Re} \lambda < 1/2\} \qquad f \leadsto f^\lambda$$

$$f^\lambda_{\bar z} = \mu_\lambda f^\lambda_z$$

$$f_{\frac{1}{p}} = f$$

$$f_0 = id$$

$$\mu_\lambda(z) = \alpha_\lambda(z) \cdot \frac{\mu(z)}{|\mu(z)|} \qquad \text{"} \frac{0}{0} = 0 \text{"}$$

$$\frac{\alpha_\lambda(z)}{1 + \alpha_\lambda(z)} = \lambda \cdot \rho \underbrace{\frac{|\mu(z)|}{1 + |\mu(z)|}}_{\leq 1} \Rightarrow \|\mu_\lambda\|_\infty = \|\alpha_\lambda\|_\infty < 1$$

$$\mu_0 \equiv 0$$

$$\mu_{\frac{1}{\rho}} = \mu$$

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$$\mu_{\frac{1}{\rho}} = \mu$$

$$\varphi_\lambda(z) = f_z^\lambda(z) + \frac{|\mu(z)|}{\mu(z)} f_{\bar{z}}^\lambda(z) \neq 0$$

$$\frac{J(z, f^\lambda)}{|\varphi_\lambda(z)|^2} = \frac{1 - |\mu_\lambda(z)|^2}{|1 + \alpha_\lambda(z)|^2} = 1 - 2\,\mathrm{Re} \frac{\alpha_\lambda}{1+\alpha_\lambda} \geq 1 - \rho \frac{|\mu|}{1+|\mu|} = w$$

$$\mu_\lambda(z) = \alpha_\lambda(z) \cdot \frac{\mu(z)}{|\mu(z)|} \qquad " \frac{0}{0} = 0 "$$

$$\frac{\alpha_\lambda(z)}{1+\alpha_\lambda(z)} = \lambda \cdot \underbrace{\rho \frac{|\mu(z)|}{1+|\mu(z)|}}_{\leq 1} \quad \Rightarrow \quad \|\mu_\lambda\|_\infty = \|\alpha_\lambda\|_\infty < 1$$

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- $\frac{1}{\pi}\int_{\mathbb{D}} |\varphi_\lambda|^2 w \leq \frac{1}{\pi}\int_{\mathbb{D}} J(z, f^\lambda) = \frac{|f^\lambda(\mathbb{D})|}{\pi} \leq 1 \qquad (p=2)$
- $\varphi_0 \equiv 1 \qquad (p=\infty)$

$$\mu_\lambda(z) = \alpha_\lambda(z) \cdot \frac{\mu(z)}{|\mu(z)|} \qquad \text{"}\frac{0}{0} = 0\text{"}$$

$$\frac{\alpha_\lambda(z)}{1+\alpha_\lambda(z)} = \lambda \cdot \underbrace{p\,\frac{|\mu(z)|}{1+|\mu(z)|}}_{\leq 1} \quad \Rightarrow \quad \|\mu_\lambda\|_\infty = \|\alpha_\lambda\|_\infty < 1$$

$$\mu_0 \equiv 0$$

$$\mu_{\frac{1}{p}} = \mu$$

$$\varphi_\lambda(z) = f_z^\lambda(z) + \frac{|\mu(z)|}{\mu(z)} f_{\bar z}^\lambda(z) \neq 0$$

$$\frac{J(z, f^\lambda)}{|\varphi_\lambda(z)|^2} = \frac{1 - |\mu_\lambda(z)|^2}{|1 + \alpha_\lambda(z)|^2} = 1 - 2\,\mathrm{Re}\,\frac{\alpha_\lambda}{1+\alpha_\lambda} \geq 1 - p\,\frac{|\mu|}{1+|\mu|} = w$$

- $\frac{1}{\pi}\int_{\mathbb{D}} |\varphi_\lambda|^2 w \leq \frac{1}{\pi}\int_{\mathbb{D}} J(z, f^\lambda) = \frac{|f^\lambda(\mathbb{D})|}{\pi} \leq 1 \qquad (p=2)$
- $\varphi_0 \equiv 1 \qquad (p = \infty)$

Interpolation lemma gives: $\frac{1}{\pi}\int_{\mathbb{D}} |\varphi_{\frac{1}{p}}|^p w \leq 1$

$$|\varphi_{\frac{1}{p}}| = |Df| \qquad w = 1 - p\,\frac{|\mu|}{1+|\mu|} \quad \square$$

STRETCHING vs ROTATION

harmonic dependence *"conjugate harmonic"*

stretching	rotation
quasiconformal	**bilipschitz**
Grötzsch problem	John's problem
Hölder exponent	rate of spiralling
$\log J(z,f) \in BMO$	$\arg f_z \in BMO$
higher integrability	exponential integrability
multifractal spectrum	

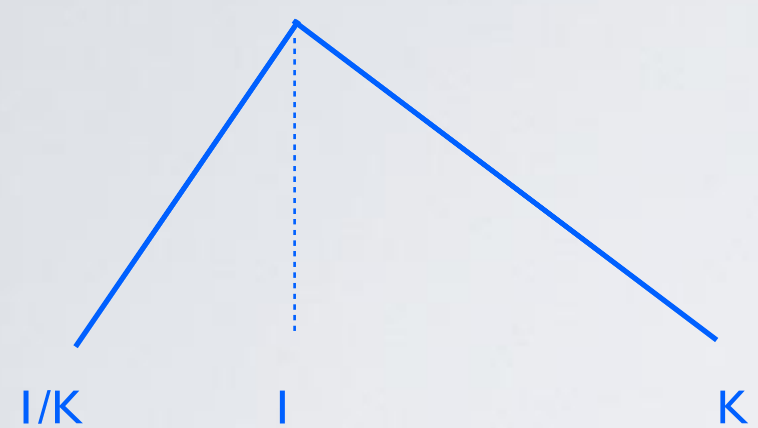
MULTIFRACTAL SPECTRA

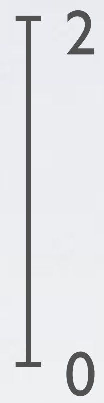
Astala-Iwaniec-Prause-Saksman

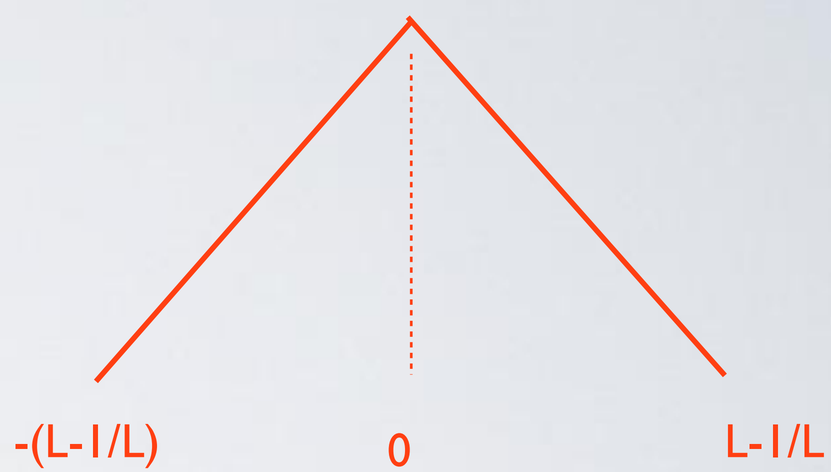
K-quasiconformal

$f\colon \mathbb{C} \to \mathbb{C}$

L-bilipschitz







$$\dim_H\{z \in \mathbb{C} : \alpha(z) = \alpha\} \leq 1 + \alpha - \frac{|1-\alpha|}{k}$$

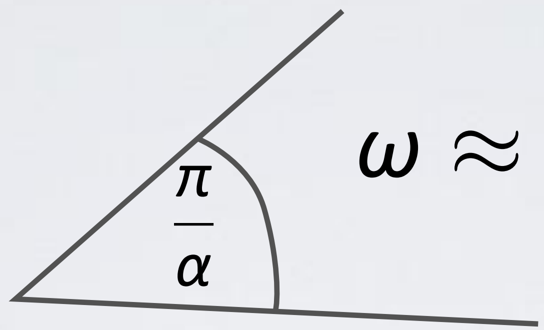
$$\dim_H\{z : \gamma(z) = \gamma\} \leq 2 - \frac{2L}{L^2 - 1}|\gamma|$$

$$\alpha(z_0) = \lim_{|z-z_0|=r_n \to 0} \frac{\log|f(z) - f(z_0)|}{\log|z - z_0|}$$

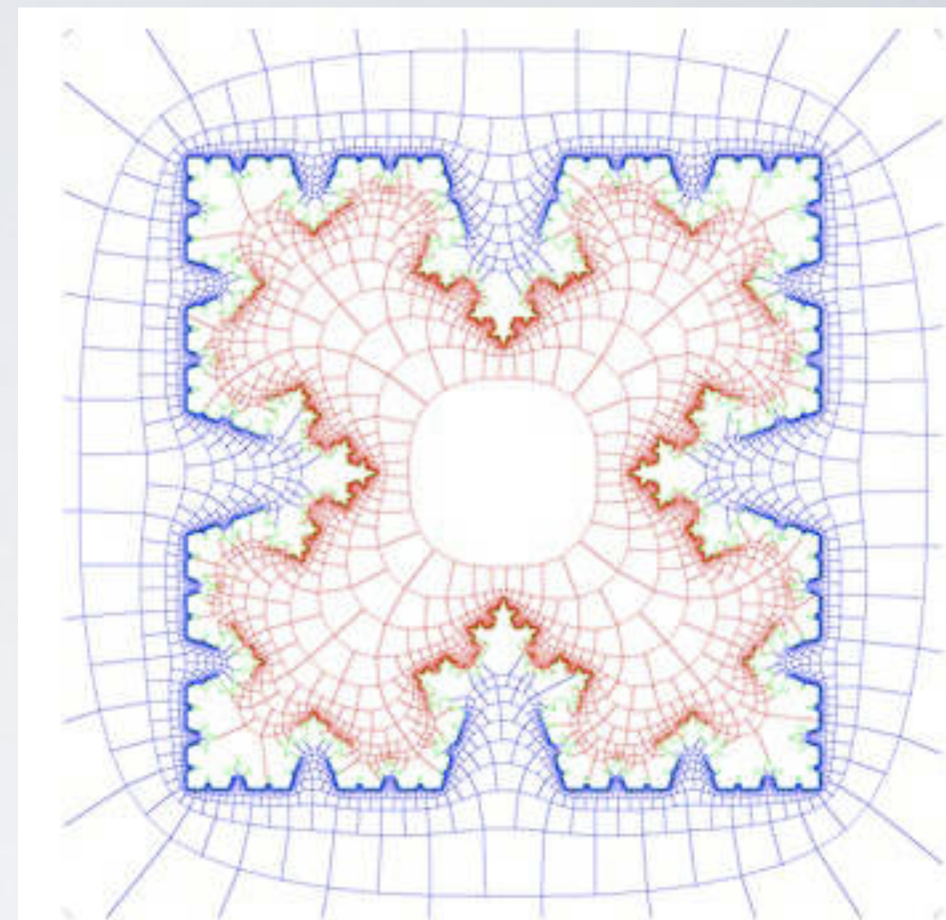
$$\gamma(z_0) = \lim_{r_n \to 0} \frac{\arg(f(z_0 + r_n) - f(z_0))}{\log|f(z_0 + r_n) - f(z_0)|}$$

What about conformal maps?
multifractality of ω

$\mathcal{F}_\alpha$ scaling: $\omega B(z,r) \approx r^\alpha$



$\omega \approx r^\alpha$



Courtesy of D. Marshall

multifractal spectrum: $f(\alpha) = \dim \mathcal{F}_\alpha$

Makarov: $\dim \omega = 1$

$$f(\alpha_-, \alpha_+, \gamma) \leq \frac{2 - (1+\gamma^2)\left(\frac{1}{\alpha_-} + \frac{1}{\alpha_+}\right)}{1 - \frac{1+\gamma^2}{\alpha_- \alpha_+}} ?$$

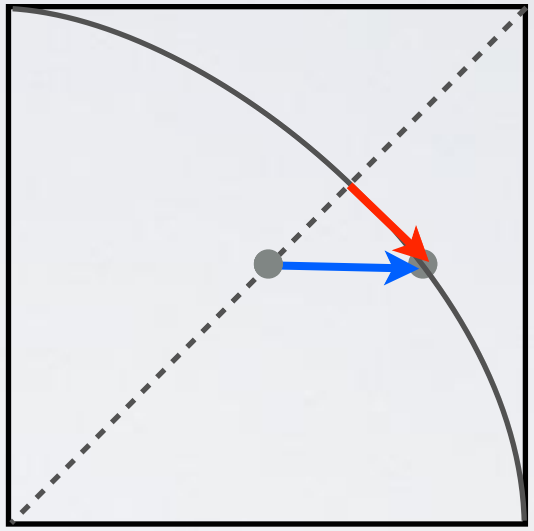
DIMENSION OF QUASICIRCLES

Example: $f_\lambda = z^2 + \lambda z$

Ruelle: $\dim J(f_\lambda) = 1 + \frac{1}{4\log 2}\left(\frac{|\lambda|}{2}\right)^2 + \cdots$

$1 + |\lambda|$

$\rightsquigarrow$ $1 + \left(\frac{|\lambda|}{2}\right)^2 + \ldots$



$F_{\lambda,\bar\eta} = \frac{z^2 + \lambda z}{1 + \bar\eta z}$

$\mu(z) = -\overline{\mu\left(\frac{1}{\bar z}\right)\frac{\bar z^2}{z^2}}$

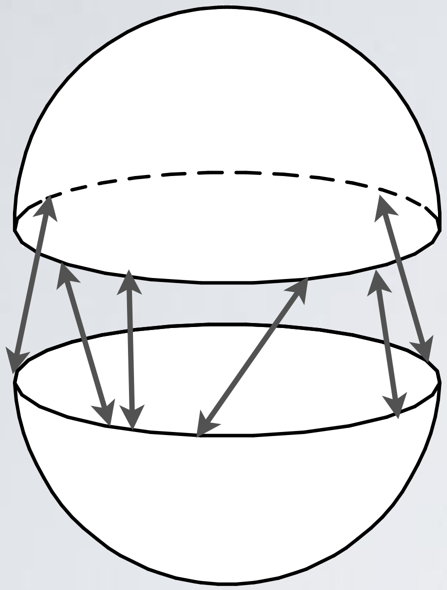
$\psi_{\lambda^2} = \sqrt{\phi_\lambda \cdot \phi_{-\lambda}}$

Smirnov: $\dim \varphi\left(\mathbb{S}^1\right) \leqslant 1 + k^2$

$\varphi : \mathbb{C} \to \mathbb{C}$ k-quasiconformal, $\|\mu\|_\infty \leqslant k$

SHARPNESS ?

existence of quasicircle with dim=1+k² → lower bound for multifractal spectrum $f(\alpha) \geqslant 2 - \frac{1}{\alpha}$



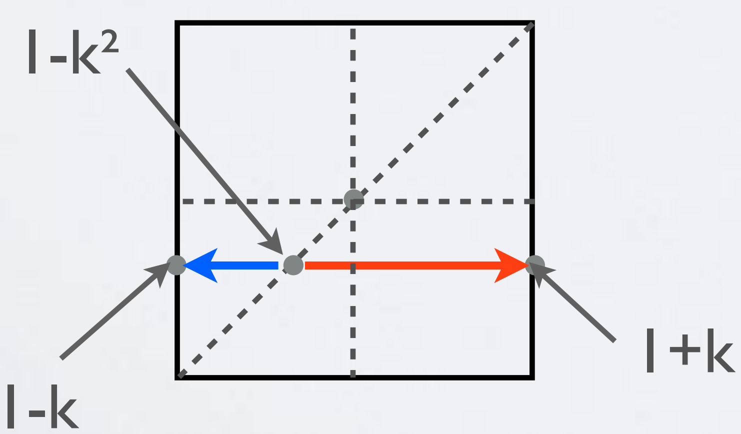
$$\Phi(\lambda, \eta) = 1 - \frac{h_m}{\Lambda_m(\lambda, \eta)}$$

$$\Phi(\cdot, 0) = \Phi(0, \cdot) = 0$$

$$\Phi : \mathbb{D}^2 \to \mathbb{D}$$

$$\dim \omega = 1$$

$$\partial_\lambda \partial_{\bar{\lambda}} \Phi(\lambda, \bar{\lambda})|_{\lambda=0} = 1 \quad \Rightarrow \quad \Phi(\lambda, \eta) = \lambda\eta$$



compressing/
expanding
conformal map

BLASCHKE PRODUCTS

$$B^d = \begin{matrix}\text{Blaschke products of degree } d \\ \text{with an attracting fixed point}\end{matrix} \Big/ \mathrm{Aut}\,\mathbb{D}$$

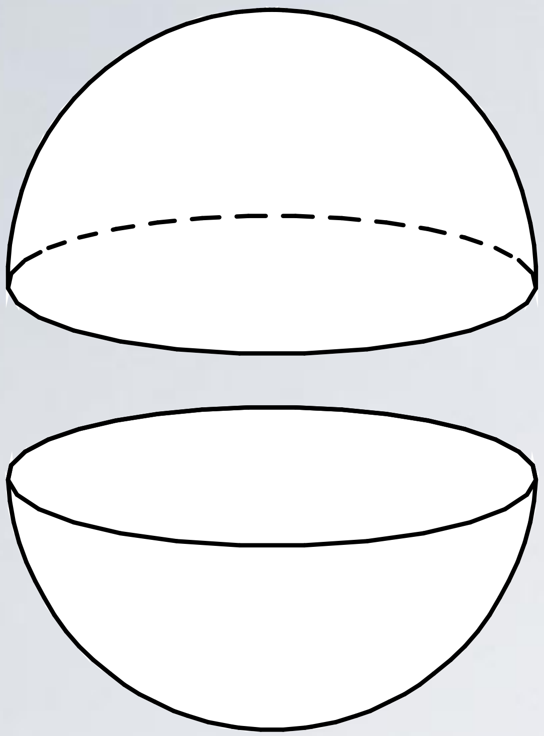
Example: $B^2 \cong \mathbb{D}$

$$\lambda \in \mathbb{D} \qquad f_\lambda(z) = \frac{z^2 + \lambda z}{1 + \bar{\lambda} z}$$

Julia set = S^1

quasisymmetrically conjugate to each other

MATING



f_λ

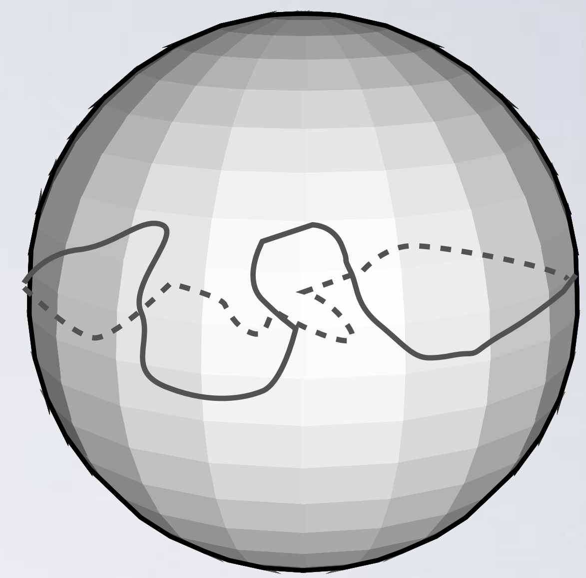
$\bar{f}_\eta$

F rational map

$$F|\Omega_+ \cong f_\lambda$$

$$F|\Omega_- \cong \bar{f}_\eta$$

$$F = [f_\lambda, f_\eta]$$



γ

Jordan curve

Example: d=2

$$F = \frac{z^2 + \lambda z}{1 + \bar{\eta} z}$$

cf. Bers' simultaneous uniformization

WEIL-PETERSSON METRIC

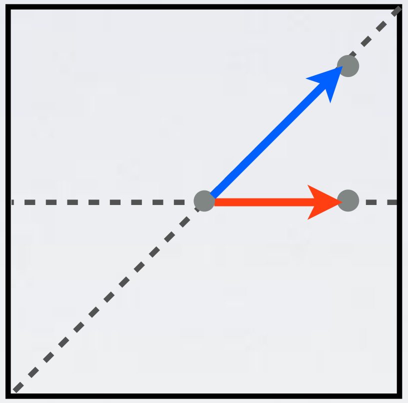
McMullen

$$f_t \in B^d$$

$$F_t = [f_0, f_t]$$

$$J(F_0) = S^1$$

$$m_{0,0} = \text{Lebesgue m.}$$



$$\varphi_t : \mathbb{D}^* \to \mathbb{C}$$

conformal conjugacy

$$-\frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} \dim(m_{t,t}) = 2\frac{d^2}{dt^2}\Big|_{t=0} \dim(J(F_t)) = \sigma^2(v')$$

$$v = \frac{d\varphi_t}{dt}\Big|_{t=0}$$

ASYMPTOTIC VARIANCE

$$g \in \mathcal{B}$$

$$\|g\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|g'(z)| < \infty$$

$$\sigma^2(g) = \frac{1}{2\pi} \limsup_{r \to 1} \frac{1}{|\log(1-r)|} \int_{|z|=r} |g|^2 d\theta \quad \left(\leqslant \|g\|_{\mathcal{B}}^2 \right)$$

Example: $g(z) = \sum_{n=1}^{\infty} z^{d^n}$ $\qquad \sigma^2 = \frac{1}{\log d}$

WP METRIC at z^d

Ruelle

$$F_t = z^d + tz \qquad t \in \mathbb{D}$$

$$\varphi_t(z^d) = \varphi_t^d(z) + t\varphi_t(z) \qquad \varphi_0(z) = z$$

$$z \in \mathbb{D}^*$$

$$v = \dot{\varphi}_0$$

$$v(z) = v_0(z) + \frac{1}{dz^{d-1}} v(z^d), \quad v_0(z) = -\frac{1}{d} z^{2-d}$$

$$v(z) = \sum_{k=0}^{\infty} v_k(z), \quad v_{k+1}(z) = \frac{1}{dz^{d-1}} v_k(z^d)$$

$$v(z) = -\frac{z}{d} \sum_{n=0}^{\infty} \frac{z^{-(d-1)d^n}}{d^n}, \qquad |z| > 1.$$

WP METRIC at z^d

$$v'(z) = \frac{(d-1)}{d} \cdot \sum_{n\geq 0} z^{-(d-1)d^n} + b_0, \quad b_0 \in \mathcal{B}_0{}^*$$

$$\sigma^2(v') = \frac{(d-1)^2}{d^2 \log d}$$

$$\text{H.dim}\, J(F_t) = 1 + \sigma^2(v')|t|^2 + \mathcal{O}(|t|^3)$$

$J(F_t)$ is a quasicircle (φ_t has |t|-qc extension)

Degree	λ-lemma	Explicit repr.
$d = 2$	$0.3606\ldots$	$0.3606\ldots$
$d = 3$	$0.4045\ldots$	$0.5394\ldots$
$d = 4$	$0.4057\ldots$	$0.6441\ldots$
$d = 20$	$0.3012\ldots$	$0.8791\ldots$

quadratic terms in dimension expansion for *k*-quasicircles

EXPLICIT REPRESENTATION

Astala-Ivrii-Perälä-Prause

$$\mathcal{C}\mu(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\mu(w)}{z-w}\,dm(w)$$

Find $\quad \mu, \ \operatorname{spt}\mu \subset \mathbb{D}, \quad v = \mathcal{C}\mu \quad \& \quad \min \|\mu\|_\infty$

pull-back $\quad \mu^*(z) = \left((z^d)^*\mu)\right)(z) = \mu(z^d)\dfrac{\bar{z}^{d-1}}{z^{d-1}}$

Lemma: $\quad \mathcal{C}\Big((z^d)^*\mu\Big)(z) = \dfrac{1}{dz^{d-1}}\Big\{\mathcal{C}\mu(z^d) - \mathcal{C}\mu(0)\Big\}, \quad z \in \mathbb{C}$

Proof: $\quad$ take $\bar{\partial}$ and $z \to \infty \quad \square$

EXPLICIT REPRESENTATION

Building block: $\mu_0(z) := \left(\bar{z}/|z|\right)^{d-3} \chi_{A(r,\rho)}$ $\rho^d = r$

$$\mathcal{C}\mu_0(z) = \frac{2}{d-1}\left(\rho^{d-1} - r^{d-1}\right) z^{2-d}, \quad |z| > 1$$

$$\mathcal{C}\mu_0 = \alpha_d\, v_0$$

$$\mu_{k+1} := \mu_k^*, \quad \mathcal{C}\mu_{k+1}(z) = \alpha_d \frac{1}{dz^{d-1}} v_k(z^d) = \alpha_d\, v_{k+1}(z)$$

disjoint spt

$$\mu := \sum_{k=0}^{\infty} \mu_k, \quad \mathcal{C}\mu = \alpha_d\, v$$

optimize over r $\quad -\alpha_d^{-1} = \frac{d^{1/(d-1)}}{2} \leqslant 1$

optimize over d (d=20) → $\sigma^2(S\mu) = 0.879\ldots$

QUASICONFORMAL EXTENSION

Astala-Ivrii-Perälä-Prause

$$F_t = z^d + t\,z \qquad t \in \mathbb{D}$$

$\varphi_t : \mathbb{D}^* \to \mathbb{C}$ conformal conjugacy

λ-lemma: H_t $|t|$-qc extension

Thm: $\frac{d^{1/(d-1)}}{2}\,|t| + \mathcal{O}(|t|^2)$ qc extension

Corollary: $J(F_t)$ is a *k*-quasicircle with dimension > *1+0.879 k²*
for *k* small *(d=20)*

QUASICONFORMAL EXTENSION

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$$F_t = z^d + t\,z \qquad t \in \mathbb{D}$$

$\varphi_t : \mathbb{D}^* \to \mathbb{C}$ conformal conjugacy

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Thm: $\frac{d^{1/(d-1)}}{2}|t| + \mathcal{O}(|t|^2)$ qc extension

Corollary: $J(F_t)$ is a k-quasicircle with dimension > $1+0.879\,k^2$ for k small (d=20)

Oleg Ivrii's talk: No k-quasicircles with dimension $1+k^2$