LYAPOUNOV EXPONENTS AND STABILITY IN PROJECTIVE HOLOMORPHIC DYNAMICS

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Abstract. Notes of the lectures given by François Berteloot at IMPAN, Warsaw, during the Simons Semester in Banach Center *Dynamical Systems*.

In these lectures we will discuss stability and bifurcation phenomena for holomorphic families of rational functions and, more generally, for holomorphic families of endomorphisms of $\mathbb{P}^k$. More precisely, we will discuss the classical results independently obtained by Lyubich and Mañé, Sad and Sullivan about the stability of rational function and its generalization to the setting of holomorphic endomorphisms of k-dimensional projectives spaces.

The four first lectures all deal with the case of $\mathbb{P}^1$. In the first lecture we briefly introduce the framework and present the tools which will be used. The second lecture is devoted to the proof of Lyubich-Mañé-Sad-Sullivan theorem and emphasizes the aspects which fail in higher dimension. In the third lecture we present an alternative approach based on pluripotential and ergodic theories, in particular we introduce the notion of bifurcation current by mean of Lyapounov exponents. The fourth lecture is a digression on some possibilities offered by the bifurcation currents techniques.

The remaining lectures deal with a generalization of Lyubich-Mañé-Sad-Sullivan theorem to $\mathbb{P}^k$ which has been obtained by C. Dupont and the authors of these notes. This generalization is based on new methods and ideas. In the fifth lecture, we first introduce some tools and, in particular, some concepts to deal with holomorphic motions of Julia sets, we then describe the main result and present a general strategy for its proof. In the two next lectures, admitting a fundamental lemma, we prove how to obtain stability (holomorphic motions) of Julia sets from stability of repelling cycles. The eighth lecture is devoted to the proof of the fundamental lemma. In the last lecture we cover some related aspects of our result.

For background on dynamics in several complex variables, we refer to the lecture notes [DS10] by T.C. Dinh and N. Sibony. For background on bifurcation currents, we refer to the survey [Du14] by R. Dujardin or the lecture notes [Be13] by the the first author. In particular, an extensive bibliography may be found in these three texts.

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1. Holomorphic families

1.1. Definitions and examples.

Definition 1.1. *A* holomorphic family *of endomorphisms of* $\mathbb{P}^k$ *is a holomorphic map* $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ *of the form* $f(\lambda, z) = (\lambda, f_\lambda(z))$ *such that:*

(1) M *is a complex manifold;*
(2) $\deg_a f_\lambda$ *(*the algebraic degree of f_λ*)* $\equiv d \geq 2$.

We shall say that d *is the degree of the holomorphic family* f*.*

Let us recall a few basic facts about holomorphic endomorphisms of projective spaces.

(1) Each f_λ can be lifted to $\mathbb{C}^{k+1}$:

$$\begin{array}{ccc} \mathbb{C}^{k+1} \setminus \{0\} & \xrightarrow{F_\lambda} & \mathbb{C}^{k+1} \setminus \{0\} \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{P}^k & \xrightarrow{f_\lambda} & \mathbb{P}^k \end{array}$$

Here F_λ is a non-degenerate d-homogeneous polynomial map, i.e., $F_\lambda^{-1}\{0\} = \{0\}$ and $F_\lambda(t, z) = t^d F_\lambda(z), \forall t \in \mathbb{C}, \forall z \in \mathbb{C}^{k+1}$.
(2) By definition, d is the algebraic degree of f_λ. Moreover, $f_\lambda : \mathbb{P}^k \to \mathbb{P}^k$ is a finite ramified cover whose topological degree $\deg_{\text{top}} f_\lambda$ is equal to d^k.
(3) For every $\lambda_0 \in M$, the family f can be lifted to a family

$$\begin{array}{rcl} F \colon \Omega \times \mathbb{C}^{k+1} & \to & \Omega \times \mathbb{C}^{k+1} \\ (\lambda, z) & \mapsto & (\lambda, F_\lambda(z)) \end{array}$$

for every sufficiently small ball $\Omega \subset M$ centered at λ_0.

The following are **basic examples** of holomorphic families.

(1) The space $\mathcal{H}_d(\mathbb{P}^k)$ of all holomorphic endomorphisms of algebraic degree d of $\mathbb{P}^k$. Actually $\mathcal{H}_d(\mathbb{P}^k) \approx \mathbb{P}^{N_{d,k}} \setminus Z$ where Z is an irreductible hypersurface in $\mathbb{P}^{N_{d,k}}$ and $N_{d,k} = \frac{(k+1)(d+k)!}{d!k!}$.
(2) M is any complex submanifold of $\mathcal{H}_d(\mathbb{P}^k)$. This is particularly interesting when M is dynamically defined.
(3) The polynomial quadratic family; $k = 1, M = \mathbb{C}$ and $f_\lambda(z) = z^2 + \lambda$.
(4) The degree d polynomial family; $k = 1, M = \mathbb{C}^{d-1}$ and f_λ is defined by the condition: f'_λ is unitary, the critical set $\mathsf{C}_{f_\lambda} = \{0, \lambda_1, \ldots, \lambda_{d-2}\}$ and $f_\lambda(0) = \lambda_d$.

1.2. The equilibrium measure of f_λ.

Our reference for this subsection and the next one is the lecture note [DS10]. Let ω denote the Fubini-Study form of $\mathbb{P}^k$. For each λ, one has $d^{-1} f_\lambda^* \omega = \omega + dd^c v_\lambda$, where v_λ is a smooth function on $\mathbb{P}^k$. One may for instance take $v_\lambda(z) = d^{-1} \log \frac{\|F_\lambda(\tilde{z})\|}{\|\tilde{z}\|}$, where $\|\cdot\|$ is the euclidean norm on $\mathbb{C}^{k+1}$ and $\pi(\tilde{z}) = z$.

By induction one gets $d^{-n}\left(f_\lambda^n\right)^*\omega = \omega + dd^c g_n(\lambda, z)$, where the function g_n is given by $g_n(\lambda, z) := v_\lambda + \cdots + d^{-n-1} v_\lambda \circ f_\lambda^{n-1}$.

It is clear that g_n is locally uniformly converging to some function $g(\lambda, z)$. One thus gets

$$\lim_n d^{-n}\left(f_\lambda^n\right)^*\omega = \omega + dd_z^c g(\lambda, z).$$

One sets $T_\lambda := \omega + dd^c g(\lambda, z)$. This is a positive closed $(1,1)$-current on $\mathbb{P}^k$, it is called the Green current of f_λ. The function $g(\lambda, z)$ is called "the" Green function of f_λ. By construction we have $f_\lambda^* T_\lambda = dT_\lambda$.

Let us stress that $g_n \to g$ locally uniformly and that the g_n's are smooth. It turns out that g is Hölder continuous, a fact which was first proved by M. Kosek[1].

The equilibrium measure μ_λ of f_λ is defined by $\mu_\lambda := \left(T_\lambda\right)^k = T_\lambda \wedge \cdots \wedge T_\lambda$. By construction μ_λ is a probability measure on $\mathbb{P}^k$ such that $f_\lambda^*\mu_\lambda = d^k\mu_\lambda$ and $\left(f_\lambda\right)_*\mu_\lambda = \mu_\lambda$. We shall use the facts that μ_λ does not give mass to proper analytic subsets of $\mathbb{P}^k$ and is ergodic.

1.3. Two equidistribution theorems (Briend-Duval[2,3], Dinh-Sibony[4]).

Theorem 1.2. Repelling periodic points equidistribute μ_λ : $d^{-kn}\sum_{z\in\mathcal{R}_n(\lambda)}\delta_z \to \mu_\lambda$, *where* $\mathcal{R}_n(\lambda) := \{$*n-periodic repelling points in* $J_\lambda\}$.

Theorem 1.3. Iterated preimages equidistribute μ_λ : *there exists a subset* $\mathcal{E}$ *of the post-critical set of* f_λ *such that* $d^{-kn}\sum_{f_\lambda^n(z)=a}\delta_z \to \mu_\lambda$ *for all* $a \in \mathbb{P}^k \setminus \mathcal{E}$.

Note that the exceptional set $\mathcal{E}$ is well understood when $k = 1$. For $k \geq 2$, one knows that $\mathcal{E}$ is a proper algebraic subset of $\mathbb{P}^k$ which is totally invariant by f_λ.

1.4. Goals.

In these lectures, we aim to

- study the stability of the ergodic dynamical systems $(J_\lambda, f_\lambda, \mu_\lambda)$;
- understand how J_λ depends on λ.

We will first review the classical results in dimension $k = 1$ and then discuss their generalization to higher dimension.

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[1] Marta Kosek, *Hölder continuity property of filled-in Julia sets in* $\mathbb{C}^n$. Proc. Amer. Math. Soc. **125** (1997), no. 7, 2029-2032

[2] Jean-Yves Briend and Julien Duval, *Exposants de Liapounoff et distribution des points périodiques d'un endomorphisme de* $\mathbb{CP}^k$, Acta Math. **182**.2 (1999), pp. 143–157.

[3] Jean-Yves Briend, Julien Duval, *Deux caractérisations de la mesure d'équilibre d'un endomorphisme de* $\mathbb{P}^k$, Publ. Math. Inst. Hautes Études Sci., **93** (2001), 145-159, and erratum in **109** (2009), 295-296.

[4] Tien-Cuong Dinh and Nessim Sibony, *Dynamique des applications d'allure polynomiale.* Journal de mathématiques pures et appliquées **82**.4 (2003): 367-423.

2. When $k = 1$: Lyubich-Mañé-Sad-Sullivan Theorem

Here we make the mild assumption that $f : M \times \mathbb{P}^1 \to M \times \mathbb{P}^1$ is a holomorphic family (of degree d) such that

$$\mathsf{C}_f = \mathrm{Crit} f = \{(\lambda, c_j(\lambda)) \colon \lambda \in M, 1 \leq j \leq 2d-2\},$$

where the maps $c_j : M \to \mathbb{P}^1$ are holomorphic.

The results obtained by Lyubich[5] and Mañé, Sad and Sullivan[6] in the early 80's are essentially summarized by the next theorem and its corollary.

Theorem 2.1. *For every $\lambda_0 \in M$ the following assertions are equivalent:*

(1) *J_λ moves holomorphically on some neighbourhood of λ_0;*
(1') *J_λ moves continuously on some neighbourhood of λ_0;*
(2) *the repelling cycles of f_λ move holomorphically on some neighbourhood of λ_0;*
(2') *f_λ has no unpersistent neutral cycle near λ_0;*
(3) *$(f_\lambda^n(c_j(\lambda)))$ is normal at λ_0 for all $1 \leq j \leq 2d-2$ (stability of the critical orbits).*

In view of that, one defines

$$\text{Stability locus of } f := \mathrm{Stab}(f) := \{\lambda_0 \colon (1)\ldots(3) \text{ occur}\}$$
$$\text{Bifurcation locus of } f := \mathrm{Bif}(f) := M \setminus \mathrm{Stab}(f).$$

The above theorem has the following fundamental consequence.

Corollary 2.2. $\mathrm{Stab}(f)$ *is dense in M.*

Example: the Mandelbrot set. *Consider the quadratic polynomial family $p : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{C} \times \mathbb{P}^1$, where $p(\lambda, z) = (\lambda, z^2 + \lambda)$. The Mandelbrot set is defined by:*

$$M_2 = \{\lambda \in \mathbb{C} \colon \ (p_\lambda^n(0)) \ \textit{ is bounded in } \mathbb{C}\}.$$

It follows from the above theorem that

$$\lambda_0 \in \mathrm{Bif}(p) \Leftrightarrow (p_\lambda^n(0))_n \ \textit{ is } \underline{\textit{not}} \textit{ normal at } \lambda_0 \Leftrightarrow \lambda_0 \in bM_2.$$

We now explain the concepts involved in the above Theorem.

2.1. **Holomorphic motions.**

Definition 2.3. *A holomorphic motion of $K \subset \mathbb{P}^1$ over a complex manifold Ω and centered at $\lambda_0 \in \Omega$ is a map $h : \Omega \times K \to \mathbb{P}^1$ such that*

(1) *$h(\lambda_0, \cdot) = Id_K$;*
(2) *$h(\lambda, \cdot)$ is one-to-one, for every $\lambda \in \Omega$;*
(3) *$h(\cdot, z)$ is holomorphic on Ω, for every $z \in K$.*

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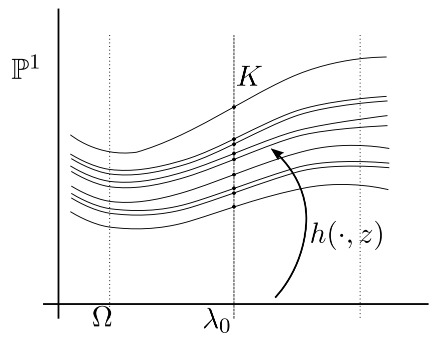
[5]Mikhail Lyubich. *Some typical properties of the dynamics of rational mappings.* Russian Math. Surveys, **38**(5):154-155, 1983.

[6]Ricardo Mañé, Paulo Sad, and Dennis Sullivan. *On the dynamics of rational maps. Ann. Sci. École Norm. Sup. (4),* **16***(2):193-217, 1983.*

Remark 2.4. *The λ-lemma says that such a motion may always be extended to $\Omega \times \overline{K}$ and is continuous on $\Omega \times \overline{K}$. We shall implicitely give the proof of this fact.*

It is useful to think about holomorphic motions in terms of laminations.



Definition 2.5. *Let $\mathcal{R}_n(\lambda) := \{n\text{-periodic repelling points of } f_\lambda\}$. We say that the repelling cycles of f_λ move holomorphically over $\Omega \subset M$ if for every $n \geq 1$ there exists a set of holomorphic maps $\rho_{j,n} : \Omega \to \mathbb{P}^1$ such that $\mathcal{R}_n(\lambda) = \{\rho_{j,n}(\lambda) \colon 1 \leq j \leq N_d(n)\}$, for every $\lambda \in \Omega$.*

Remark 2.6. *This is equivalent to say that there exists a holomorphic motion of $\mathcal{R}_n(\lambda_0)$ over Ω and centered at λ_0 for every $n \geq 1$. The important fact here is that Ω is the same for <u>all</u> n. Indeed, if n is fixed and Ω is a sufficiently small neighbourhood of λ_0, such a motion of $\mathcal{R}_n(\lambda_0)$ always exists by the implicit function theorem.*

Definition 2.7. *One says that J_λ moves holomorphically over Ω if there exists a holomorphic motion h of J_λ which is centered at $\lambda_0 \in \Omega$ and such that $h_\lambda := h(\lambda, \cdot)$ conjugates the dynamics:*

$$\begin{array}{ccc} J_{\lambda_0} & \xrightarrow{h_\lambda} & J_\lambda \\ {\scriptstyle f_{\lambda_0}}\downarrow & & \downarrow{\scriptstyle f_\lambda} \\ J_{\lambda_0} & \xrightarrow{h_\lambda} & J_\lambda \end{array}$$

2.2. **Continuity in the Hausdorff topology.**

$$\mathrm{Comp}^*(\mathbb{P}^k) := \{\text{non-empty compact subsets of } \mathbb{P}^k\}.$$

For $K \in \mathrm{Comp}^*(\mathbb{P}^k)$ we set $K_\varepsilon := \varepsilon$-nehighbourhood of K. For a map $E : M \to \mathrm{Comp}^*(\mathbb{P}^k)$ we have the following definitions:

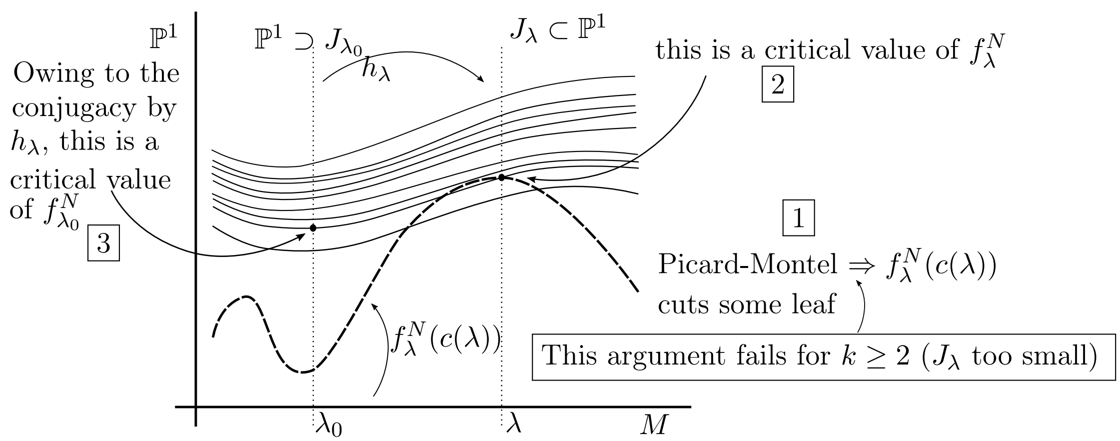
E is upper semicontinuous (*usc*) at $\lambda_0 \Leftrightarrow \forall \varepsilon > 0 : E(\lambda) \subset (E(\lambda_0))_\varepsilon$ for $\lambda \approx \lambda_0$
E is lower semicontinuous (*lsc*) at $\lambda_0 \Leftrightarrow \forall \varepsilon > 0 : E(\lambda_0) \subset (E(\lambda))_\varepsilon$ for $\lambda \approx \lambda_0$
E is continuous at $\lambda_0 \Leftrightarrow E$ is *ucs* and *lsc* at λ_0.

2.3. **Proof of Lyubich-Mañé-Sad-Sullivan Theorem.** The structure of the proof is as follows:

$$\begin{array}{ccccccc} (1) & \Longrightarrow & (3) & \Longrightarrow & (2) & \Longrightarrow & (1) \\ \Updownarrow & & & & \Updownarrow & & \\ (1') & & & & (2') & & \end{array}$$

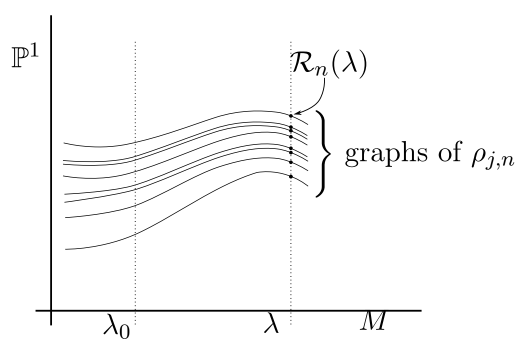
We focus on the proof of (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and the proof of the Corollary 2.2, stressing the arguments which cannot be adapted to $\mathbb{P}^k$, with $k \geq 2$.

(1) $\Rightarrow$ (3) Assume that $\left(f_\lambda^n(c(\lambda))\right)_n$ is <u>not</u> normal at λ_0 but J_λ moves holomorphically near λ_0. Then one has the following picture:



The picture shows that {critical values of $f_{\lambda_0}^N$} contains an open piece of J_{λ_0}: this is impossible.

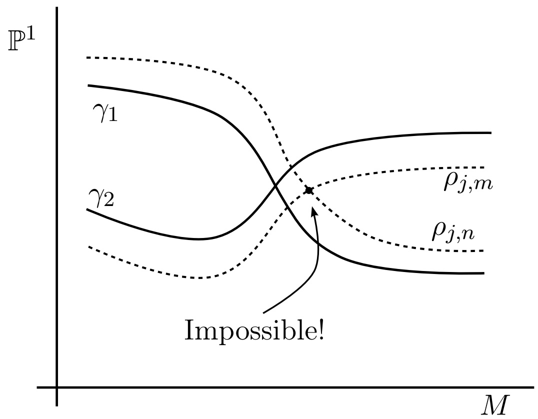
(2) $\Rightarrow$ (1) By assumption we have the following pictures for every n.



One easily sees (using lifts to $\mathbb{C}^2 \setminus \{0\}$) that the family $\{\rho_{j,n} \colon 1 \leq j \leq N_d(n), n \geq 1\}$ is normal. Since $\cup_n \mathcal{R}_n(\lambda)$ is dense in J_λ, the idea is to take limits to get the expected lamination. The only problem is that of uniqueness of limits.

Let us see on a picture why two distinct limits cannot meet.

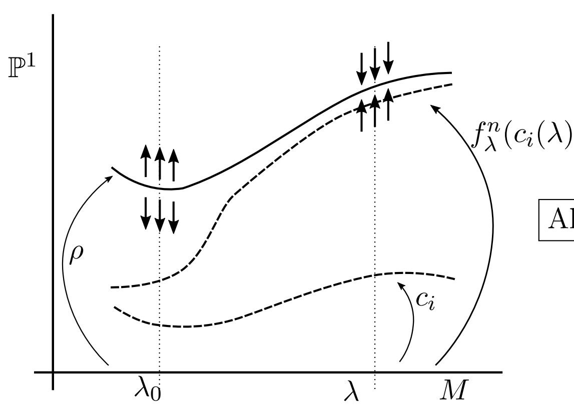
If the graphs of two limits γ_1, γ_2 meet, then for $\rho_{j,n}$ close to γ_1 and ρ_{j_n} close to γ_1...



... the graphs of $\rho_{j,n}$ and $\rho_{j,m}$ must meet by Hurwitz!

Fails when $k \geq 2$

(3) $\Rightarrow$ (2) The idea is as follows. Assume that a repelling cycle of f_{λ_0} does not move holomorphically over M. Then one finds a holomorphic map $\rho : U_0 \to \mathbb{P}^1$ such that $f_\lambda^{n_0}(\rho(\lambda)) = \rho(\lambda)$, $\lambda_0 \in U_0$ and $\rho(\lambda_0)$ is repelling for $f_{\lambda_0}^{n_0}$ but $\rho(\lambda_1)$ is attracting for $f_{\lambda_1}^{n_1}$. The orbit of a critical point $c_i(\lambda)$ is contained in the basin of attraction of $\rho(\lambda_1)$. In this situation, $\left(f_\lambda^n(c_i(\lambda))\right)_n$ cannot be normal on U_0 as the picture below shows.



Also fails for $k \geq 2$

2.4. **Proof of Corollary 2.2.** Let $\lambda_0 \in \mathrm{Bif}(f)$. After a small perturbation ($\lambda_0 \to \lambda_1$) one gets f_{λ_1}, which has an attracting cycle of period ≥ 3. Let $B(\lambda)$ be the basin of that cycle (exists for $\lambda \approx \lambda_1$). If $\left(f_\lambda^n(c_i(\lambda))\right)_n$ is <u>not</u> normal at λ_1, then, using Picard-Montel Theorem, one sees that after a small perturbation ($\lambda_1 \to \lambda_2$) one has $\left(f_{\lambda_2}^{n_0}(c_i(\lambda_2))\right) \in B(\lambda_2)$ and thus $\left(f_\lambda^n(c_i(\lambda))\right)_n$ is normal at λ_2.

One repeats this argument a finite ($\leq 2d-2$) number of times to get $\lambda_k \in \mathrm{Stab}(f)$ which is arbitrarily close to λ_0.

The fact that this argument cannot be extended to $\mathbb{P}^k$, $k \geq 2$, is essentially due to the non-finiteness of C_{f_λ}.

3. When $k=1$: a potential theoretic approach to bifurcations

3.1. **The main idea explained for polynomials families.** We consider here a holomorphic polynomial family

$$\begin{array}{rcl} p\colon M \times \mathbb{C} & \to & M \times \mathbb{C} \\ (\lambda, z) & \mapsto & (\lambda, p_\lambda(z)) \end{array}$$

where p_λ is a unitary polynomial of degree $d \geq 2$. As in the former lecture, we assume that the critical set $\mathsf{C}_p = \{(\lambda, c_j(\lambda))\colon 1 \geq j \geq d-1\}$, where $c_j : M \to \mathbb{C}$ are holomorphic.

The Green function G of p on $M \times \mathbb{C}$ is defined by

$$G(\lambda, z) := \lim_n d^{-n} \log^+ |p_\lambda^n(z)| \,.$$

As the convergence is locally uniform, the function G is *psh* and continuous. Moreover, one easily sees that

- $G(\lambda, z) = 0 \Leftrightarrow (p_\lambda^n(z))$ is bounded
- $J_\lambda = b\{G(\lambda, z) = 0\}$
- $\mu_\lambda = \Delta_z G(\lambda, z)$.

The Lyapounov exponent $L(\lambda)$ of p_λ with respect to its equilibrium measure μ_λ is defined by

$$L(\lambda) := \int \log \left|p'_\lambda(z)\right| \mu_\lambda(z).$$

Note that $\log |p'_\lambda| \in L^1(\mu_\lambda)$, since μ_λ has a continuous potential.

Proposition 3.1. *$L(\lambda)$ is the exponential rate of growth of $(f_\lambda^n)'(z)$ for μ_λ-almost every point z.*

Proof. Apply Birkhoff Theorem (μ_λ is ergodic). For μ_λ-almost every z,

$$\int \log \left|p'_\lambda(z)\right| \mu_\lambda(z) = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \log \left|p'_\lambda(p_\lambda^k(z))\right| = \lim_n \frac{1}{n} \log \left|(p_\lambda^n)'(z)\right| .$$

□

Przytycki Formula[7] $L(\lambda) = \log d + \sum_{j=1}^{d-1} G(\lambda, c_j(\lambda))$.

Corollary 3.2. *The function $L(\lambda)$ is psh, continuous and bigger than* $\log d$.

[7] Feliks Przytycki, *Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map,* Invent. Math. **80**, 161-179, (1985)

Proof. Integrate by parts.

$$L(\lambda) = \int \log |p'_\lambda| \, \mu_\lambda = \int \log \left| d \prod_j (z - c_j(\lambda)) \right| \mu_\lambda$$

$$= \log d + \sum_j \int \log |z - c_j(\lambda)| \, \Delta_z G(\lambda, z) = \log d + \sum_j \left\langle \delta_{c_j(\lambda)}, G(\lambda, \cdot) \right\rangle .$$

□

Applying dd^c_λ to Przytycki Formula one gets the following fundamental formula:

$$\boxed{dd^c_\lambda L(\lambda) = \sum_{j=1}^{d-1} dd^c_\lambda G(\lambda, c_j(\lambda))} \tag{1}$$

deals with holomorphic motions of repelling cycles ($dd^c_\lambda L(\lambda)$) — detects the instability of the orbit of $c_j(\lambda)$ ($dd^c_\lambda G(\lambda, c_j(\lambda))$)

Proposition 3.3. *The following hold:*

1. $G(\lambda, c_i(\lambda))$ *is pluriharmonic on a ball* $B \subset M \Leftrightarrow (p^n_\lambda(c_i(\lambda)))_n$ *is normal on* B*;*
2. *if the repelling cycles of* p *move holomorphically over* $U \subset M$*, then* L *is pluriharmonic on* U*.*

Remark 3.4. *Combining* (1) *and the above Proposition, one gets a proof of* (2) $\Rightarrow$ (3) *in Lyubich-Mañé-Sad-Sullivan Theorem which avoids Picard-Montel arguments. Formula (1) thus provides another way to look at stability questions in dimension one.*

Proof. (1) Set $u_n(\lambda) := p^n_\lambda(c_i(\lambda))$, and recall that $d^{-n} \log^+ |u_n(\lambda)| \to G(\lambda, c_i(\lambda))$ locally uniformly.

- $(u_n)_n$ normal on $B \xRightarrow[B]{\text{shrink}} \begin{cases} u_n \xrightarrow{\text{unif}} \infty \Rightarrow & G(\lambda, c_i(\lambda)) \text{ is a limit of pluriharmonic functions on B} \\ u_{n_k} \xrightarrow{\text{unif}} h \text{ holomorphic} \Rightarrow G(\lambda, c_i(\lambda) = 0 \text{ on } B \end{cases}$
- Assume $G(\lambda, c_i(\lambda))$ is pluriharmonic on B.
  First case: $G(\lambda, c_i(\lambda_0)) = 0$, $\lambda_0 \in B$. Then $G(\lambda, c_i(\lambda)) = 0$ on B (maximum principle) and thus $(u_n(\lambda))_n$ is uniformly bounded on B.
  Second case: $G(\lambda, c_i(\lambda)) > 0$ on B. Then $p^n_\lambda(c_i(\lambda)) \to \infty$ locally uniformly on B.

(2) This is a direct consequence of the approximation formula which will be discussed in the next lecture.

□

3.2. **The bifurcation current.** It follows immediately from formula (1) and Lyubich-Mañé-Sad-Sullivan Theorem that the support of $dd^c L(\lambda)$ is precisely the bifurcation locus

$\mathrm{Bif}(p)$. This can be extended to families of rational maps of $\mathbb{P}^1$. Let us say a few words about that, precise references and proofs can be found in [Be13].

One first has to generalize Przytycki formula to rational maps. This has been done by De Marco and a different proof, more in the spirit of integration by part, has been given by Bassanelli and Berteloot. Applying dd^c_λ, this formula yields:

$$dd^c_\lambda L(\lambda) = (\pi_M)_* \left((\omega + dd^c_{\lambda,z} g(\lambda, z)) \wedge [\mathsf{C}_f] \right). \tag{2}$$

Recall that $g(\lambda, z)$ is the Green function of f, ω the Fubini-Study form on $\mathbb{P}^1$ and $[\mathsf{C}_f]$ the current of integration on C_f, the critical set of f in $M \times \mathbb{P}^1$.

Exactly as in the polynomial case, the above formula (2) allows to see that $\mathrm{Supp}\, dd^c_\lambda L(\lambda)$ coincides with the bifurcation locus $\mathrm{Bif}(f)$. Here, $L(\lambda) := \int \log |f'_\lambda| \, \mu_\lambda$ is the Lyapounov exponent of the system $(J_\lambda, f_\lambda, \mu_\lambda)$. One may easily check that it does not depend on the choice of the metric $|\cdot|$ on $\mathbb{P}^1$. This has lead De Marco to define the bifurcation current T_{bif} of the family f by

$$T_{\mathrm{bif}} := dd^c_\lambda L(\lambda).$$

As shown by the formula (2), T_{bif} is a $(1, 1)$-current on M, which is positive and closed.

4. When $k = 1$: an application of the bifurcation currents techniques

We consider here a holomorphic family $f : M \times \mathbb{P}^1 \to M \times \mathbb{P}^1$, $(\lambda, z) \mapsto (\lambda, f_\lambda(z))$ of degree d rational maps on $\mathbb{P}^1$. Our aim is to use bifurcation current techniques to prove that certain parameters in M (for a sufficently "big" parameter space M) can be accumulated by hyperbolic parameters or by either parameters for which f_λ has the maximal number of distinct neutral cycles (i.e. $2d - 2$ by the Fatou-Shishikura inequality). Let us recall that a parameter λ is hyperbolic if and only if f_λ has $2d - 2$ distinct attracting basins. A parameter for which f_λ has $2d-2$ distinct neutral cycles will be called a Shishikura parameter.

Let us recall that the equilibrium measure of f_λ is denoted by μ_λ. The Lyapounov exponent $L(\lambda)$ of $(J_\lambda, f_\lambda, \mu_\lambda)$ is given by

$$L(\lambda) = \int_{\mathbb{P}^1} \log |f'_\lambda| \, \mu_\lambda.$$

It follows from the Manning formula $\dim_H \; \mu_\lambda = \frac{h_{top}(f_\lambda)}{L(\lambda)}$ and the Misiurewicz-Przytycki inequality $h_{top}(f_\lambda) \geq \log d$ that

$$L(\lambda) \geq \frac{\log d}{2}.$$

We shall use here the following

Approximation Formula: $L(\lambda) = \lim_n d^{-n} \sum_{z \in \mathcal{R}_n(\lambda)} \frac{1}{n} \log |(f^n_\lambda)'(z)|$.

Recall that $\mathcal{R}_n(\lambda)$ is the set of n-periodic repelling points of f_λ. The formula says that the Lyapounov exponent of f_λ (with respect to μ_λ) is the asymptotic limit of the averages of Lyapounov exponents of repelling n-cycles. This approximation formula is proved in

[Be13]. Okuyama[8] has given another proof.

Let us now consider the following subsets of the parameter space M:

$$\mathsf{Per}_n(w) := \{\lambda \in M \colon f_\lambda \text{ has a } n\text{-cycle of multiplier } w\}\,.$$

One may show that $\mathsf{Per}_n(w)$ is a (singular) hypersurface in M. More precisely, the exists a collection of functions $p_n(\lambda, w)$ which are unitary polynomials of degree $N_d(n) \sim \frac{d^n}{n}$ in w and whose coefficients are holomorphic function in λ, such that $\mathsf{Per}_n(w) = \{p_n(\cdot, w) = 0\}$ for $w \neq 1$ (the case $w = 1$ is more delicate).

According to the Poincaré-Lelong formula, the integration current $[\mathsf{Per}_n(w)]$ on the hypersurface $\mathsf{Per}_n(w)$ is thus given by

$$[\mathsf{Per}_n(w)] = dd^c_\lambda \log |p_n(\lambda, w)|\,.$$

We are interested in comparing the limits of $d^{-n}\,[\mathsf{Per}_n(w)]$ and the bifurcation current $T_{bif} = dd^c_\lambda L(\lambda)$.

4.1. **A computation.** Set $L^r_n(\lambda) := \frac{d^{-n}}{2\pi}\int_0^{2\pi} \log\left|p_n(\lambda, re^{i\theta})\right| d\theta$, for $r \geq 0$. Denoting by $w_{n,j}(\lambda), 1 \leq j \leq N_d(n)$ the roots of $p_n(\lambda, \cdot)$ (taken with mutiplicity) we get

$$\begin{aligned} L^r_n(\lambda) &= \frac{d^{-n}}{2\pi}\int_0^{2\pi} \log\left|\prod_{j=1}^{N_d(n)} \left(re^{i\theta} - w_{n,j}(\lambda)\right)\right| d\theta \\ &= \frac{d^{-n}}{2\pi}\sum_{j=1}^{N_d(n)}\int_0^{2\pi} \log\left|\left(re^{i\theta} - w_{n,j}(\lambda)\right)\right| d\theta \\ &= d^{-n}\sum_{j=1}^{N_d(n)} \log\max\left(r, |w_{n,j}(\lambda)|\right) d\theta. \end{aligned}$$

By Fatou Theorem, there exists $n(\lambda) \in \mathbb{N}$ such that all n-cycles of f_λ are repelling for $n \geq n(\lambda)$. Thus, for $0 \leq r \leq 1$ and $n \geq n(\lambda)$, we obtain

$$L^r_n(\lambda) = d^{-n}\sum_{j=1}^{N_d(n)} \log|w_{n,j}(\lambda)| = d^{-n}\sum_{\mathcal{R}_n(\lambda)} \frac{1}{n}\log\left|(f^n_\lambda)'(z)\right|$$

and by the approximation formula $\lim_n L^r_n(\lambda) = L(\lambda)$.

When $r > 1$ we have (always for $n \geq n(\lambda)$)

$$L^r_n(\lambda) = d^{-n}\sum_{j=1}^{N_d(n)} \log|w_{j,n}(\lambda)| + d^{-n}\sum_{1\leq|w_{j,n}(\lambda)|<r} \log\frac{r}{|w_{j,n}(\lambda)|}$$

[8]Yûzuke Okuyama. *Repelling periodic points and logarithmic equidistribution in non-archimedean dynamics.* Acta Arith., **153**(3):267-277, 2012.

and therefore $0 \leq L_n^r(\lambda) - L_n^0(\lambda) \leq d^{-n} N_d(n) \log r$. As $d^{-n} N_d(n) \sim \frac{1}{n}$, we get again $\lim_n L_n^r(\lambda) = L(\lambda)$. Since the sequence $\left(L_n^r(\lambda)\right)_n$ is a sequence of *psh* functions which is locally uniformly bounded from above (easy to check), the convergence actually occurs in $L^1_{loc}(M)$. Taking dd^c we thus have

$$T_{bif} = dd^c L = \lim_n dd^c L_n^r = \lim_n \frac{d^{-n}}{2\pi} \int_0^{2\pi} \left[\mathsf{Per}_n(re^{i\theta})\right] d\theta. \tag{3}$$

4.2. **Perturbation of Lattès examples.** We aim here to use the above formula and simple potential theoretic arguments to show that rigid Lattès examples are accumulated by hyperbolic parameters or by Shishikura parameters. Let us recall that a Lattès map is a rational function f which is induced on the Riemann sphere from a dilation on a complex torus $D : T \to T$ by mean of some elliptic function $p : T \to \mathbb{P}^1$.

$$\begin{array}{ccc} T & \xrightarrow{D} & T \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1 \end{array}$$

Such maps are of course very rare. They have been characterized by A. Zdunik[9] by the minimality of their Lyapounov exponent.

Theorem 4.1. *f_λ is a Lattès example* $\Leftrightarrow L(\lambda) = \frac{\log d}{2}$.

For simplicity, we consider the moduli space of quadratic rational maps which (according to Milnor[10]) can be considered as a family $f : M \times \mathbb{P}^1 \to M \times \mathbb{P}^1$ parametrized by $M = \mathbb{C}^2$.

We shall use the Monge-Ampère measure associated to the *psh* function L. By definition this positive measure is given by $dd^c L \wedge dd^c L$. The fact that this product of two closed positive current is well defined follows from the continuity of L. Indeed, for any closed positive current S, the function L is integrable with respect to the trace measure of S and therefore the current LS is well defined. One then set $dd^c L \wedge S := dd^c(LS)$ (recall that S closed means that $dd^c S = 0$).

The *bifurcation measure* of the family f has been introduced by Bassanelli and Berteloot who gave its first properties. It is defined by

$$\mu_{\text{bif}} := \frac{1}{2} dd^c L \wedge dd^c L.$$

It is an elementary property that the strict minima of L belong to the support of $dd^c L \wedge dd^c L$. Since degree two Lattès example are rigid, they correspond to isolated parameters in

[9]Anna Zdunik. *Parabolic orbifolds and the dimension of the maximal measure for rational maps.* Invent. Math., **99**(3):627-649, 1990.

[10]John Milnor. *Geometry and dynamics of quadratic rational maps.* Experiment. Math., **2**(1):37-83, 1993. With an appendix by the author and Lei Tan.

M. The theorem of Zdunik thus tells us that they belong to the support of the bifurcation measure. This is also true for all Lattès example as it has been proved by X. Buff and T. Gauthier[11]. To achieve our goal it remains to show that any parameter λ_0 in the support of the bifurcation measure μ_{bif} can be accumulated either by hyperbolic parameters or by Shishikura parameters. This is actually a direct consequence of the following approximation formula which is obtained from (3) by mean of elementary potential theoretic arguments:

$$\mu_{\text{bif}} = \lim_n \frac{2^{-(n+k(n))}}{2(2\pi)^2} \int_{[0,2\pi]^2} [Per_n(re^{i\theta_1})] \wedge [Per_{k(n)}(re^{i\theta_2})]\, d\theta_1 d\theta_2.$$

In the above formula, $k(n)$ is a suitable increasing sequence of integers. The choice of $0 < r < 1$ shows that the support of μ_{bif} is accumulated by hyperbolic parameters while the choice $r = 1$ shows that it can be accumulated by Shishikura parameters.

Full details of the contents of this section are given in [Be13].

5. Extension of Lyubich-Mañé-Sad-Sullivan Theorem to higher dimension

We consider here a holomorphic family

$$\begin{aligned} f\colon M \times \mathbb{P}^k &\to M \times \mathbb{P}^k \\ (\lambda, z) &\mapsto (\lambda, f_\lambda(z)) \end{aligned}$$

whose degree is $d \geq 2$. Recall that we have an ergodic dynamical system $(J_\lambda, f_\lambda, \mu_\lambda)$, where μ_λ is the equilibrium measure of f_λ (see Lecture 1). The sum of the Lyapounov exponents of $(J_\lambda, f_\lambda, \mu_\lambda)$ is given by the following expression:

$$L(\lambda) := \int_{\mathbb{P}^k} \log \left|\det f'_\lambda(z)\right| \mu_\lambda(z).$$

Our aim in this lecture is to describe a recent result[12] due to Berteloot, Bianchi and Dupont which extends Lyubich-Mañé-Sad-Sullivan theorem in higher dimension.

5.1. **A substitute to the notion of holomorphic motions of Julia sets.** We endow $\mathcal{O}(M, \mathbb{P}^k) := \left\{\gamma : M \to \mathbb{P}^k \colon \gamma \text{ holomorphic}\right\}$ with the topology of local uniform convergence; this is a metric space. The space of interest here is the (possibly empty) subspace

$$\mathcal{J} := \left\{\gamma \in \mathcal{O}(M, \mathbb{P}^k) \colon \gamma(\lambda) \in J_\lambda, \forall \lambda \in M\right\}.$$

We have two natural maps. The first one is a self-map on $\mathcal{J}$

$$\begin{aligned} \mathcal{F} : \mathcal{J} &\to \mathcal{J} \\ \gamma &\mapsto \mathcal{F}(\gamma) \end{aligned}$$

[11] Xavier Buff and Thomas Gauthier. *Perturbations of flexible Lattes maps.* Bulletin de la société mathématique de France **141**.4 (2013): 603-614.

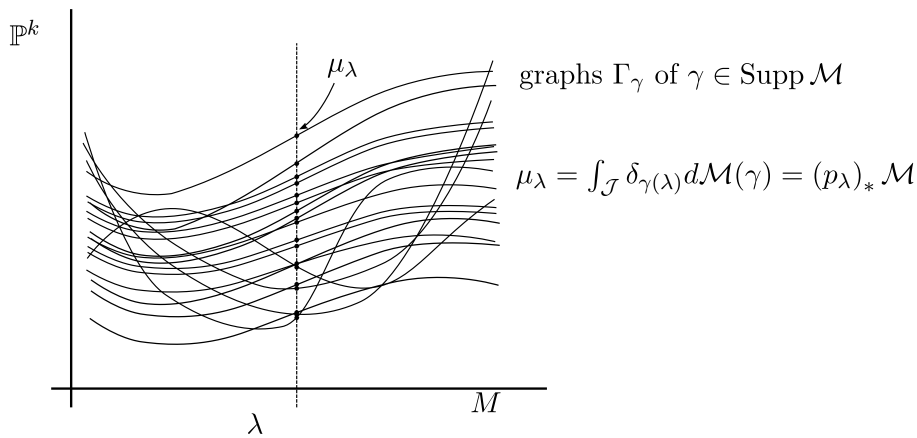
[12] François Berteloot, Fabrizio Bianchi and Christophe Dupont, *Dynamical stability and Lyapunov exponents for holomorphic endomorphisms of* $\mathbb{P}^k$, arXiv preprint arXiv:1403.7603

defined by $\mathcal{F}(\gamma)(\lambda) = f_\lambda(\gamma(\lambda))$ and the second one is a projection

$$\begin{aligned} p_\lambda : \mathcal{J} &\to \mathbb{P}^k \\ \gamma &\mapsto \gamma(\lambda). \end{aligned}$$

Definition 5.1. *An equilibrium web for f is a compactly supported probability measure $\mathcal{M}$ on $\mathcal{J}$ such that $\mathcal{F}_*\mathcal{M} = \mathcal{M}$ and $(p_\lambda)_* \mathcal{M} = \mu_\lambda$ for all $\lambda \in M$.*

Interpretation:



The existence of such $\mathcal{M}$ somehow means that the equilibrium measures μ_λ's are "holomorphically glued" over M. It is also possible to associate to M a web current in the sense of Dinh: $W_{\mathcal{M}} := \int_{\mathcal{J}} [\Gamma_\gamma]\ d\mathcal{M}(\gamma)$. This allows the use of "calculus".

The support of an equilibrium web $\mathcal{M}$ is a quite wild object; we seek for a cleaner one:

Definition 5.2. *An equilibrium lamination for f is an $\mathcal{F}$-invariant subset $\mathcal{L}$ of $\mathcal{J}$ such that:*

(1) $\Gamma_\gamma \cap \Gamma_{\gamma'} = \emptyset, \forall \gamma \neq \gamma' \in \mathcal{L}$;
(2) $\mu_\lambda(\{\gamma(\lambda) \colon \gamma \in \mathcal{L}\}) = 1, \forall \lambda \in M$;
(3) $\Gamma_\gamma \cap GO(C_f) = \emptyset, \forall \lambda \in \mathcal{L}$;
(4) $\mathcal{F} : \mathcal{L} \to \mathcal{L}$ *is* d^k*-to-1.*

Our main result is as follows.

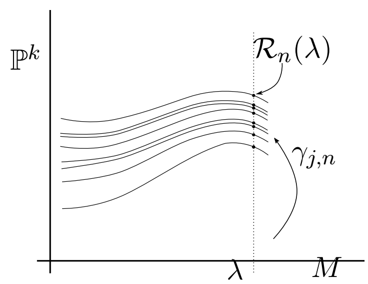
Theorem 5.3. *Let M be an open and simply connected subset of $\mathcal{H}_d(\mathbb{P}^k)$. Then the following assertions are equivalent:*

(1) *the repelling J-cycles of f_λ move holomorphically over M;*
(2) $dd^c L \equiv 0$;
(3) *f admits an equilibrium lamination.*

Let us precise a couple of definitions. The repelling J-cycles of f_λ are the repelling cycles of f_λ which belong to J_λ. This is not automatic when $k \geq 2$, note however that in the Briend-Duval equidistribution therem (see Lecture 1), the repelling J-cycles of f_λ equidistribute μ_λ. $GO(\mathsf{C}_f)$ denotes the grand orbit (by f) of the critical set C_f of f.

5.2. **Strategy of the proof and tools.** We shall only prove (1)⇒{(2) and (3)} and we shall briefly discuss (3) ⇒ (1).

Recall that (1) means that



where $\gamma_{j,n} \in \mathcal{J}$ and $\{\gamma_{j,n}(\lambda), j\} = \mathcal{R}_n(\lambda) := \{n-\text{periodic periodic points of } f_\lambda \text{ in } J_\lambda\}$.

• The implication (1) ⇒ (2) can be proved by using a generalization of the approximation formula for L seen in Lecture 4.

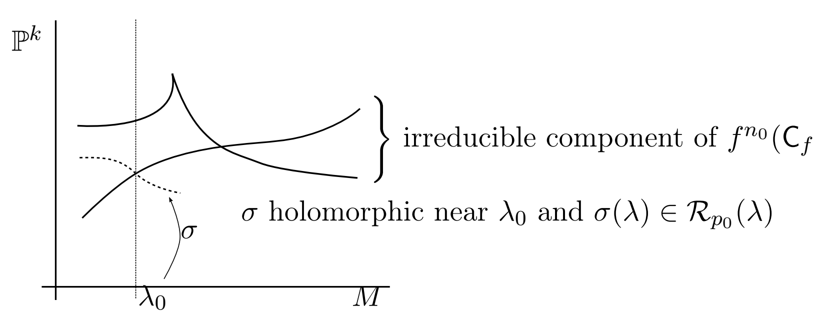
• The implication (1) ⇒ *existence of an equilibrium web* $\mathcal{M}$ can be proved by applying Banach-Alaoglu Theorem to $\mathcal{M}_k := d^{-kn} \sum_{\delta_{\gamma_{j,n}}}$ and using the equidistribution theorem for repelling orbits seen in Lecture 1.

• To deduce the *existence of an equilibrium lamination* from the existence of $\mathcal{M}$ is much harder. Its proof exploits the dynamical properties of the system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$. In particular, we need first prove that $\mathcal{M}$ is acritical (in the sense that $\mathcal{M}(\{\gamma \in \Gamma_\gamma : \Gamma_\gamma \cap GO(\mathsf{C}_f) \neq \emptyset\}) = 0$) and then replace it with a new web which is both acrtitical and ergodic.

The fact that $\mathcal{M}$ is acritical is proved by using the

Fundamental Lemma: $dd^cL \equiv 0 \Rightarrow$ No Misiurewicz parameters in M

Saying that λ_0 is a Misiurewicz parameter means that the following picture occurs (note that $\Gamma_\sigma \not\subseteq f^{n_0}(\mathsf{C}_f)$):



An argument of extremality based on Choquet's decomposition theorem then allows to replace $\mathcal{M}$ by an acritical and ergodic equilibrium web.

• (3) $\Rightarrow$ (1). The proof here follows an idea which works well in dimension one: if a repelling cycle does not move holomorphically then a Siegel disc must appear and this obviously creates a discontinuity of $\lambda \to J_\lambda$ in the Hausdorff topology, in particular this is not compatible with the holomorphic motion of J_λ.

In $\mathbb{P}^{k\geq 2}$, a Siegel $k-$polydisc would also create a discontinuity but when a repelling cycle does not move holomorphically one simply obtains a Siegel disc, that is a one dimensional object. Is is however possible to show that this is an obstruction to the existence of an holomorphic motion of Julia sets. The question of the discontinuity of $\lambda \to J_\lambda$ in the Hausdorff topology remains open.

Although this strategy seems quite natural, one has to face several technical difficulties to implement it in higher dimension. In particular we must deal with possible persistent resonances in the holomorphic family which is considered. This is exactly why, at list when $k \geq 3$, our result is given for the full family $\mathcal{H}_d(\mathbb{P}^k)$.

To end this description, let us stress that the technical (and potential theoretic) part of the proof is concentrated in the Fundamental Lemma and relies on the following generalization of the Przytycki-De Marco formula (see Lecture 3) due to Bassanelli and Berteloot:

$$dd^c L = (p_M)_* \left(\left(dd^c_{\lambda,z} g + \omega \right)^k \wedge [\mathsf{C}_f] \right).$$

As a by-product of our approach we also obtain:

Theorem 5.4. *Misiurewicz parameters are dense in* $\operatorname{Supp} dd^c L$.

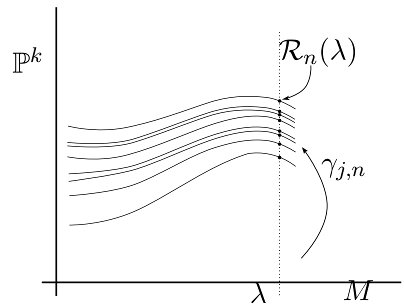
The remaining lectures will be devoted to the proof of the above results. In Lecture 6 we will construct an acritical and ergodic equilibrium web from holomorphic motions of repelling $\mathcal{J}$-cycles (assuming the Fundamental Lemma). In Lecture 7, we will explain how to extract an equilibrium lamination from an acritical and ergodic equilibrium web. In Lectures 8 and 9 we will respectively prove the Fundamental Lemma and the density of Misiurewicz parameters in $\operatorname{Supp} dd^c L$.

6. Construction of acritical ergodic equilibrium webs

We assume here that a family $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ enjoys the property that its repelling J-cycles move holomorphically over M. Our aim is to show that f admits an ergodic acritical equilibrium web.

6.1. f admits an equilibrium web.

Proof. The holomorphic motion of repelling J-cycles means that for every $n \in \mathbb{N}^*$ we have a subset $\{\gamma_{j,n} \colon 1 \leq j \leq N_d(n)\}$ of $\mathcal{J}$ such that $\mathcal{R}_n(\lambda) = \{\gamma_{j,n}(\lambda) \colon 1 \leq j \leq N_d(n)\}$ for all $\lambda \in M$. Recall that $\mathcal{R}_n(\lambda) := \{n\text{-periodic repelling points of } f_\lambda \text{ in } J_\lambda\}$.



Set $\mathcal{M}_n := d^{-kn} \sum_{j=1}^{N_d(n)} \delta_{\gamma_{j,n}}$. This is a sequence of discrete probability measures on $\mathcal{J}$ (actually $|\mathcal{M}_n| \sim 1$). Moreover, $\cup_n \operatorname{Supp} \mathcal{M}_n$ is relatively compact (a simple Montel normality argument using local lifts of f to $M \times \mathbb{C}^{k+1}$ explains that). Then, by Banach-Alaoglu Theorem we get:

$$\mathcal{M}_n \to \mathcal{M}.$$

$\mathcal{M}$ is clearly a compactly supported measure on $\mathcal{J}$. Moreover, we have:

$$\mathcal{F}_* \mathcal{M} = \lim_i \mathcal{F}_* \mathcal{M}_{n_i} = \lim_i \mathcal{M}_{n_i} = \mathcal{M}$$

and

$$(p_\lambda)_* \mathcal{M} = \lim_i (p_\lambda)_* \mathcal{M}_{n_i} = \lim_i d^{-kn_i} \sum_j \delta_{\gamma_{j,n_i}(\lambda)} = \lim_i d^{-kn_i} \sum_{z \in \mathcal{R}_n(\lambda)} \delta_z = \mu_\lambda.$$

μ_λ equidistributes the repelling cycles

$\mathcal{M}$ is thus an equilibrium web of f. □

6.2. The current $dd^c L$ vanishes on M.

Proof. We use the following approximation formula by Berteloot-Dupont-Molino[13]

$$L(\lambda) = \lim_n d^{-kn} \sum_{\mathcal{R}_n(\lambda)} \frac{1}{n} \log \left|\det \left(f_\lambda^n\right)'(z)\right|,$$

which in our case yields $L(\lambda) = \lim_n d^{-kn} \sum_{j=1}^{N_d(n)} \frac{1}{n} \log |\det (df_\lambda^n)(\gamma_{j,n}(\lambda)|$ and shows that L is a pointwise limit of pluriharmonic functions. Since these function are locally uniformly bounded from above, the convergence occurs in L^1_{loc} and thus L is pluriharmonic. □

We shall see in Lecture 9 another proof of this fact.

[13]François Berteloot, Christophe Dupont, and Laura Molino. *Normalization of bundle holomorphic contractions and applications to dynamics.*Annales de l'institut Fourier. **58.** No. 6. 2008.

6.3. $\mathcal{M}$ is acritical.

Proof. First step: $\mathcal{M}$ satifies the following key property

$$\Gamma_\gamma \cap \mathsf{C}_f^+ \neq \emptyset \Rightarrow \Gamma_\gamma \subset \mathsf{C}_f^+, \forall \gamma \in \operatorname{Supp} \mathcal{M}. \tag{4}$$

Recall that $\mathsf{C}_f^+ = \cup_{m \geq 0} f^m(\mathsf{C}_f)$. According to the former subsection and the Fundamental Lemma, there are no Misiurewicz parameter in M. In other words: $\mathcal{M}_{n_k}$ satisfy (4) for all k. By Hurwitz: $\mathcal{M} = \lim_k \mathcal{M}_{n_k}$ also satisfies (4).

Second step: $\mathcal{M}(\mathcal{J}_s) = 0$, where $\mathcal{J}_s := \{\gamma \in \Gamma \colon \Gamma_\gamma \cap GO(\mathsf{C}_f) \neq \emptyset\}$.
First of all,

$$\begin{aligned} \mathcal{M}\left(\left\{\gamma \in \mathcal{J} \colon \Gamma_\gamma \cap \mathsf{C}_f^+ \neq \emptyset\right\}\right) &\overset{\boxed{\text{by (4)}}}{\leq} \mathcal{M}\left(\left\{\gamma \in \mathcal{J} \colon \Gamma_\gamma \subset \mathsf{C}_f^+\right\}\right) \\ &\underset{\boxed{\forall \lambda_0 \in M}}{=} \mathcal{M}\left(\left\{\gamma \in \mathcal{J} \colon \gamma(\lambda_0) \in \mathsf{C}_{f_{\lambda_0}}^+\right\}\right) \underset{\boxed{(p_{\lambda_0})_\star \mathcal{M} = \mu_{\lambda_0}}}{=} \mu_{\lambda_0}\left(\mathsf{C}_{f_{\lambda_0}}^+\right) = 0 \end{aligned}$$

where the last equality is due to the fact that μ_{λ_0} does not give mass to pluripolar sets. Then $\mathcal{M}(\mathcal{J}_s) = 0$ follows from the $\mathcal{F}$-invariance of $\mathcal{M}$. □

6.4. Existence of acritical ergodic webs.

Proof. Let $\mathcal{M}_0$ be an acritical equilibrium web for f (constructed in the above step) and let $\mathcal{K} := \operatorname{Supp} \mathcal{M}_0$. Recall that $\mathcal{K}$ is compact in the metric space $\mathcal{J}$. We consider the spaces

$$\begin{aligned} \mathcal{P}_{web}(\mathcal{K}) &:= \{\text{equilibrium webs of } f \text{ supported in } \mathcal{K}\} \\ &\cap \\ \mathcal{P}_{inv}(\mathcal{K}) &:= \{\mathcal{F} - \text{invariant probability measures supported in } \mathcal{K}\}. \end{aligned}$$

Note that these are compact metric spaces for the weak$\star$-topology.

The ergodic $\mathcal{F}$-invariant probability measures on $\mathcal{K}$ are precisely the extremal points of $\mathcal{P}_{inv}(\mathcal{K})$. It thus suffices to check that:

(1) extremality in $\mathcal{P}_{web}(\mathcal{K}) \Rightarrow$ extremality in $\mathcal{P}_{inv}(\mathcal{K})$;
(2) there exists an acritical web in $\mathcal{P}_{web}(\mathcal{K})$ which is extremal.

(1) follows easily from the fact that $(p_\lambda)_\star \mathcal{M} = \mu_\lambda$ for every $\mathcal{M} \in \mathcal{P}_{web}(\mathcal{K})$ and that the μ_λ's are ergodic.

(2) follows from Choquet Decomposition Theorem applied to $\mathcal{M}_0$:

$$\mathcal{M}_0 = \int_{\operatorname{Ext}(\mathcal{P}_{web})} \mathcal{E} d\nu_0(\mathcal{E})$$

where ν_0 is a probability measure for which $\operatorname{Ext}(\mathcal{P}_{web})$ has full measure.

Then $0 = \mathcal{M}_0(\mathcal{J}_s) = \int_{\mathrm{Ext}(\mathcal{P}_{web})} \mathcal{E}(\mathcal{J}_s) d\nu_0(\mathcal{E})$ implies that ν_0-almost all $\mathcal{E}$ in $\mathrm{Ext}(\mathcal{P}_{web})$ are actually acritical. □

7. From ergodic acritical equilibrium webs to equilibrium laminations

We assume here that a family $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ admits an ergodic and acritical web $\mathcal{M}$. Our goal is to explain how the support of $\mathcal{M}$ can be "cleaned" in order to get an equilibrium lamination $\mathcal{L}$ for f. We refer to the lecture 5 for definitions. The manifold M is supposed to be simply connected.

7.1. **A dynamical system associated to $\mathcal{M}$.** Let us recall that $\mathcal{M}$ is a compactly supported probability measure on $\mathcal{J}$ such that $\mu_\lambda = \int_{\mathcal{J}} \delta_{\gamma(\lambda)} d\mathcal{M}(\gamma)$ and which is invariant by the map

$$\begin{aligned} \mathcal{F} : \mathcal{J} &\to \mathcal{J} \qquad\qquad \mathcal{F}(\gamma)(\lambda) = f_\lambda(\gamma(\lambda)). \\ \gamma &\mapsto \mathcal{F}(\gamma) \end{aligned}$$

Let $\mathcal{K}$ be the support of $\mathcal{M}$ and $\mathcal{X} := \mathcal{K} \setminus \mathcal{J}_s$ where, as before, $\mathcal{J}_s$ is the subset of elements γ in $\mathcal{J}$ whose graphs Γ_γ do not meet the grand orbit $GO(\mathsf{C}_f)$ of the critical set C_f of f. Saying that $\mathcal{M}$ is acritical means that $\mathcal{M}(\mathcal{J}_s) = 0$ and thus that $\mathcal{X}$ has full measure. Using the fact that M is simply connected, one easily sees that $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is onto. To summarize:

$$(\mathcal{X}, \mathcal{F}, \mathcal{M}) \text{ is an ergodic dynamical system and the map } \mathcal{F} : \mathcal{X} \to \mathcal{X} \text{ is onto.}$$

We will now transform the dynamical system $(\mathcal{X}, \mathcal{F}, \mathcal{M})$ into an invertible one by means of the classical construction of natural extension. The natural extension $\widehat{\mathcal{X}}$ of $\mathcal{X}$ is the set of all possible "histories" for points $\gamma \in \mathcal{X}$:

$$\widehat{\mathcal{X}} := \{\widehat{\gamma} := (\ldots, \gamma_{-j}, \gamma_{-j+1}, \ldots, \gamma_{-1}, \gamma_0, \gamma_1, \ldots, \gamma_j) \text{ such that } \gamma_j \in \mathcal{X} \text{ and } \mathcal{F}(\gamma_j) = \gamma_{j+1}\}.$$

We have projections $\pi_j : \widehat{\mathcal{X}} \to \mathcal{X}$, $\widehat{\gamma} \mapsto \gamma_j$ which are onto for all $j \in \mathbb{Z}$.

The shift $\widehat{\mathcal{F}} : \mathcal{X} \to \mathcal{X}$ is clearly invertible and satisfies $\pi_0 \circ \widehat{\mathcal{F}} = \mathcal{F} \circ \pi_0$. It is a classical result that there exists a probability measure $\widehat{\mathcal{M}}$ on $\mathcal{X}$ which is $\widehat{\mathcal{F}}$-invariant and such that $(\pi_j)_* \widehat{\mathcal{M}} = \mathcal{M}$ for all $j \in \mathbb{Z}$. Moreover, $\widehat{\mathcal{M}}$ is ergodic when $\mathcal{M}$ is ergodic. We have thus transfered our problem to the

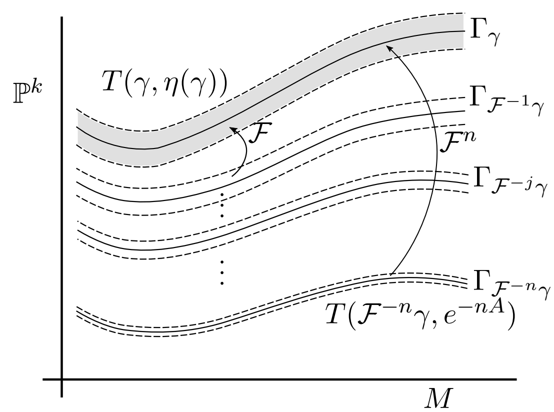
$$\text{invertible and ergodic dynamical system } \left(\widehat{\mathcal{X}}, \widehat{\mathcal{F}}, \widehat{\mathcal{M}}\right).$$

In the following, in order to keep the notation as light as possible, we shall continue to use the system $(\mathcal{X}, \mathcal{F}, \mathcal{M})$. The arguments above show that we can actually *think* that our system is invertible.

7.2. **Contraction and conclusion.** The following lemma gives the key estimate that will allow us to recover an equilibrium lamination out of the support of our ergodic and acritical equilibrium web $\mathcal{M}$. As explained in the preceding subsection, we can think that our system $(\mathcal{X}, \mathcal{F}, \mathcal{M})$ is invertible.

Lemma 7.1. *There exist a measurable function $\eta : \mathcal{X} \to]0,1]$ and a positive constant A such that for $\mathcal{M}$-almost every $\gamma \in \mathcal{X}$ the following hold:*

(1) *f^{-n} is defined on a tubular neighbourhood $T(\gamma, \eta(\gamma))$ of Γ_γ; and*
(2) *$f^{-n}(T(\gamma, \eta(\gamma))) \subset T(\mathcal{F}^{-n}\gamma, e^{-nA})$.*



We can now show how to cut out a set of zero measure from the support of $\mathcal{M}$ in order to recover a set of non-intersecting graphs. Let us fix a small ball $B \subset M$. We define the *ramification* $R_B(\gamma)$ over B of an element $\gamma \in \mathcal{X}$ as

$$R_B(\gamma) = \sup_{\gamma' \in \operatorname{Supp} \mathcal{M}, \Gamma_\gamma \cap \Gamma_{\gamma'} \neq \emptyset} \sup_B d(\gamma(\lambda), \gamma'(\lambda)).$$

Fix now a $\alpha > 0$ and consider the set of elements $\{\gamma\colon R_B(\gamma) > \alpha\}$. Notice that, by the contraction Lemma 7.1, we have $R_B(\mathcal{F}^{-n}\gamma) \to 0$ as $n \to +\infty$. Then, Poincaré recurrence Theorem (applied to the inverse system $\big(X, \mathcal{F}^{-1}, \mathcal{M}\big)$) implies that $\{\gamma\colon R_B(\gamma) > \alpha\}$ has zero $\mathcal{M}$-measure for every $\alpha > 0$. So, $\mathcal{M}\left(\{\gamma\colon R_B(\gamma) > 0\}\right) = 0$ and we can thus consider the full-measure subset $\mathcal{L}^+$ of graphs with zero ramification. It is then not difficult to build the desired lamination $\mathcal{L}$ starting with the set $\mathcal{L}^+$.

We are thus left with proving Lemma 7.1. This is the content of the next subsections.

7.3. **Proof of the contraction Lemma 7.1: reduction to some estimate.** Let us start describing the general philosophy of the proof. We exploit a method, developed by Briend-Duval, which proves the analogous statement at a fixed parameter. Namely, given an endomorphism g of $\mathbb{P}^k$ of algebraic degree d, almost every point x (with respect to the equilibrium measure μ) is contained in a ball $B(x, \eta(x))$ where the inverse g^{-n} is defined for every n and satisfies the contraction property $g^{-n}(B(x, \eta(x))) \subset B(g^{-n}(x), e^{-n\chi_1})$. Here χ_1 denotes the smallest Lyapounov exponent of the system $\big(\mathbb{P}^k, g, \mu\big)$, which is known[14] to be greater that $\frac{\log d}{2}$.

[14]Jean-Yves Briend and Julien Duval, *Exposants de Liapounoff et distribution des points périodiques d'un endomorphisme de* $\mathbb{CP}^k$, Acta Mathematica 182.2 (1999), pp. 143–157.

The main steps of the method are as follows.

- An asymptotic estimate of $\left\|dg^{-1}(\cdot)\right\|$ over the inverse orbit $\{g^{-j}(x)\}$ of a point x yields an estimate of the radius of a ball centered at x where g^{-n} is defined, for every n, and of the asymptotic rate of contraction of g^{-n} on this ball.
- The asymptotic estimate of $\left\|dg^{-1}\right\|$ is obtained from

$$\text{for } \mu - \text{a.e. } x \in \mathbb{P}^k\colon \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log\left\|dg^{-1}(g^{-j}(x))\right\| = \int_{\mathbb{P}^k} \log\left\|dg^{-1}(x)\right\|\mu(x) = -\chi_1 < 0$$

where the first equality comes from the ergodicity of μ and Birkhoff Theorem.

In our setting, the same method reduces the problem to prove that

$$\text{for } \mathcal{M} - \text{a.e. } \gamma \in \mathcal{X}\colon \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log\max_{\lambda}\left\|df_\lambda^{-1}(\mathcal{F}^{-j}\gamma(\lambda))\right\| < 0.$$

As $\mathcal{M}$ is ergodic, we still know that the limit equals $\int_{\mathcal{X}} \log\max_\lambda \left\|df_\lambda^{-1}(\gamma(\lambda))\right\|\mathcal{M}(\gamma)$. So, we only need to prove that this integral is negative. In order to get this, we show that

$$\lim_{n\to\infty} \frac{1}{n}\int_{\mathcal{X}} \log\max_{\lambda}\left\|df_\lambda^{-n}(\gamma(\lambda))\right\|\mathcal{M}(\gamma) < 0 \tag{5}$$

and then get the desired estimate after replacing our system by a suitable high iterate f^N.

7.4. **Proof of the contraction Lemma 7.1: the estimate (5).** Let us introduce some notations.

Notation 7.2. *We set*

- $u_n(\gamma,\lambda) = \log\left\|df_\lambda^{-n}(\gamma(\lambda))\right\|$*; and*
- $\widehat{u}_n(\gamma,\lambda) = \log\max_\lambda\left\|df_\lambda^{-n}(\gamma(\lambda))\right\|$.

Notice that the functions $u_n(\gamma,\cdot)$'s are *psh* (this will be useful in the sequel). With these notations, (5) becomes

$$\lim_{n\to\infty} \frac{1}{n}\int_{\mathcal{X}} \widehat{u}_n(\gamma)\mathcal{M}(\gamma) < 0. \tag{6}$$

In order to establish (6), we shall need the following two facts:

Fact 7.3. *For $\mathcal{M}$-almost every $\gamma \in \mathcal{X}$ the following holds:*

$$\tfrac{1}{n}u_n(\gamma,\lambda) \to -\chi_1(\lambda), \textit{ for almost every } \lambda \in \mathcal{M}.$$

Proof. First of all recall that, by Oseledec Theorem, for every $\lambda \in M$ the set

$$J_{\lambda,1} := \left\{x \in J_\lambda\colon \frac{1}{n}\log\left\|df_\lambda^{-n}\right\| \to -\chi_1(\lambda)\right\}$$

has full μ_λ-measure. So, since $(p_\lambda)_*\mathcal{M} = \mu_\lambda$, we have that

$$\forall\lambda \in M\colon \tfrac{1}{n}u_n(\gamma,\lambda) \to -\chi_1(\lambda) \text{ for } \mathcal{M}\text{-almost every } \gamma \in \mathcal{X}.$$

We can then consider the subset E of the product space $\mathcal{X} \times M$ given by

$$E := \left\{ (\gamma, \lambda) \in \mathcal{X} \times M \colon \frac{1}{n} u_n(\gamma(\lambda)) \to -\chi_1(\lambda) \right\}.$$

By Tonelli Theorem, we have (denoting by Leb the Lebesgue measure on M):

$$(\mathcal{M} \otimes \mathrm{Leb})(E)$$

$$\int_M \underbrace{\mathcal{M}(\{\gamma \in \mathcal{X} \colon (\gamma, \lambda) \in E\})}_{=1 \forall \lambda \in M} \mathrm{Leb}(\lambda) \quad \Rightarrow \quad \int_{\mathcal{X}} \underbrace{\mathrm{Leb}(\{\lambda \in M \colon (\gamma, \lambda) \in E\})}_{=\mathrm{Leb}(M) \text{ for } \mathcal{M}-\text{a.e.}\gamma \in \mathcal{X}} \mathcal{M}(\gamma)$$

□

Fact 7.4. *Let $V_0 \Subset W_0 \Subset M$. Then*

(1) $\frac{u_n}{n}(\gamma, \cdot)$ *are locally uniformly bounded on V_0, for $\mathcal{M}$-a.e. $\gamma \in \mathcal{X}$;*
(2) $\widehat{u}_n \in L^1(\mathcal{M})$.

This fact is crucial for the proof of (6). Its proof is elementary but quite technical and we shall therefore only sketch it. We shall make use of the following elementary fact about holomorphic functions from the unit disk $\mathbb{D}$ to $\mathbb{D}^*$:

Compactness statement: *there exists* $0 < \alpha \leq 1$ *such that* $\sup_{V_0} |\phi| \leq |\phi(\lambda)|^\alpha$ *for every* $\lambda \in \mathbb{D}(0, 1/2)$ *and every holomorphic function* $\phi : \mathbb{D} \to \mathbb{D}^*$.

Idea of proof of Fact 7.4. Notice that $u_n(\gamma, \lambda) = \log\left(\frac{1}{\delta(df_\lambda^n(\gamma(\lambda)))}\right)$, where we denote by $\delta(A)$ the smallest singular value of a matrix A. Moreover, notice that all the matrices that we are considering satisfy $\delta(A) \neq 0$. If the function δ were holomorphic, the above statement and the compactness of V_0 and $\operatorname{Supp} \mathcal{M}$ would imply that for every n the function $u_n(\gamma, n)$ satisfy the estimate

$$\delta\left(df_\lambda^n\left(\gamma(\lambda')\right)\right) \leq c^{n(1-\alpha)} \delta\left(df_\lambda^n\left(\gamma(\lambda)\right)\right)^\alpha, \tag{7}$$

holding for any two points $\lambda, \lambda' \in V_0$, where $c = \max_{\gamma, \lambda} \delta(df_\lambda(\gamma(\lambda)))$. The difficulty is overcome as follows. For every matrix A involved, $c_1 |\det A|^k \leq \delta(A)^k \leq |\det A|$ for some constant c_1. So, even if δ is not a holomorphic function, it is bounded above and below by two such functions, for which we can find estimates as above.

Once we have established (7), we can transfer all the estimates we have to do with a variable λ to a single suitable one (given by the Fact 7.3), and thus apply the analogous (known) results in the fixed-parameter setting in order to conclude. □

We can now explain why the limit in (6) is negative. Since

(1) the sequence $\widehat{u}_n$ is subadditive (i.e., $\widehat{u}_{n+m} \leq \widehat{u}_n + \widehat{u}_m \circ \mathcal{F}^n$),
(2) $\widehat{u}_n \in L^1(\mathcal{M})$, and
(3) $\mathcal{M}$ is ergodic

we can apply the ergodic version of Kingman Subadditive Theorem and thus get the existence of an L such that

(a) $\frac{1}{n}\int_{\mathcal{X}} \widehat{u}_n(\gamma)\mathcal{M}(\gamma) \to L$; and
(b) for $\mathcal{M}$-a.e. $\gamma \in \mathcal{X}$: $\frac{1}{n}\widehat{u}_n(\gamma) \to L$.

In particular, this proves that the limit in (6) exists and that we can compute it on almost every element $\gamma \in \mathcal{X}$. We are thus left with proving that $L < 0$. In order to do so, we consider a $\gamma \in \mathcal{X}$ with the following properties:

(1) $\frac{1}{n}\widehat{u}_n(\gamma) \to L$ (from (b));
(2) $\frac{1}{n}u_n(\gamma,\cdot)$ is uniformly bounded on $V_0 \Subset M$ (from Fact 7.4); and
(3) $\frac{1}{n}u_n(\gamma,\lambda) \to -\chi_1(\lambda)$ for Leb-a.e. $\lambda \in V_0$ (from Fact 7.3).

We then claim that $L \le -\frac{\log d}{2}$. Suppose to the contrary that $L > -\frac{\log d}{2}$. Pick $U_0 \Subset V_0$. Recall that $\widehat{u}_n(\gamma) = \max_\lambda u_n(\gamma,\lambda)$. Up to extracting a subsequence, one finds points $\lambda_n \in U_0$ and a positive ε such that $\frac{u_n(\gamma,\lambda_n)}{n} \ge -\frac{\log d}{2} + \varepsilon$. Up to extracting another subsequence, we can assume that $\lambda_n \to \lambda_0 \in \overline{U_0}$. Now, there exists an r such that $B(\lambda_n, r) \subset V_0$ for every n, and, since every $u_n(\gamma,\cdot)$ is *psh*, the submean inequality for each of them at λ_n yields

$$-\frac{\log d}{2} + \varepsilon \le \frac{u_n(\gamma,\lambda_n)}{n} \le \frac{1}{|B(\lambda_n,r)|}\int_{B(\lambda_n,r)} \frac{u_n(\gamma,\lambda)}{n}.$$

Since by (3) the sequence $\frac{u_n(\gamma,\lambda)}{n}$ converges to $\chi_1(\lambda)$ for almost every λ and is uniformly bounded (by (2)), the Lebesgue dominated convergence Theorem gives:

$$-\frac{\log d}{2} + \varepsilon \le \frac{1}{|B(\lambda_0,r)|}\int_{B(\lambda_0,r)} -\chi_1(\lambda).$$

Since $\chi_1(\lambda) \ge \frac{\log d}{2}$, this gives the desired contradiction.

8. Proof of the fundamental lemma

Let $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ be a holomorphic family of degree $d \ge 2$. Let $L(\lambda) := \int_{\mathbb{P}^k} \log|\det f'_\lambda(z)|\,\mu_\lambda(z)$ be the sum of the Lyapounov exponents of the system $(J_\lambda, f_\lambda, \mu_\lambda)$. Our aim is to prove the following

Lemma 8.1. *Misiurewicz parameters belong to* $\mathrm{Supp}\, dd^c L$.

Recall that $\mu_\lambda = (dd^c_z g(\lambda,z) + \omega)^k$ and $g = \lim_n g_n$ (locally uniformly on $M \times \mathbb{P}^k$) with g smooth (see Lecture 1).

We shall use the formula (see Lecture 5)

$$dd^c L = (p_M)_* \left(\left(dd^c_{\lambda,z} g + \omega\right)^k \wedge [\mathsf{C}_f] \right). \tag{8}$$

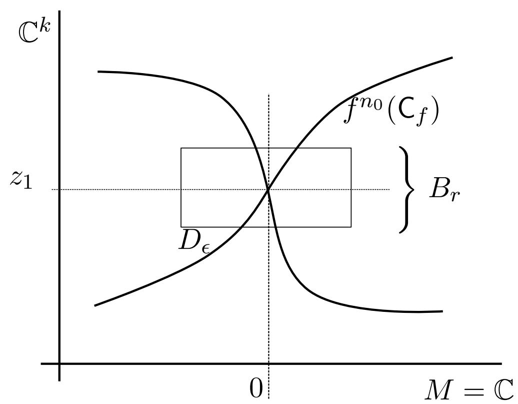
The function L is *psh* and continuous on M.

8.1. **Step 1: Simplifications.** In our computations, we will think that g is smooth. This is possible since

$$f^*\left(dd^c g+\omega\right)=d\left(dd^c g+\omega\right)$$
$$f^*\left(dd^c g_{n+1}+\omega\right)=d\left(dd^c g_n+\omega\right)$$

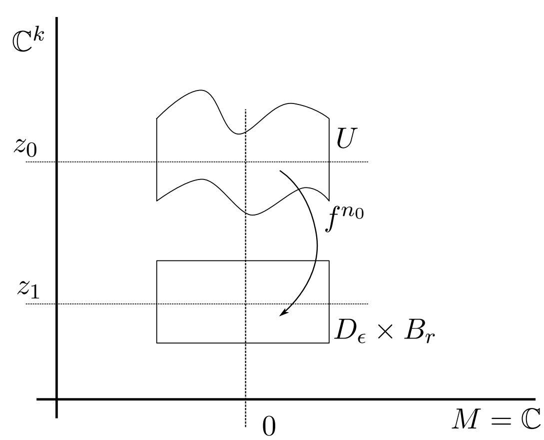
where the g_n are smooth and locally uniformly converging to g.

We assume that the parameter space M is a disc centered at the origin 0 in $\mathbb{C}$ and replace $\mathbb{P}^k$ by $\mathbb{C}^k$. The assumption that 0 is a Misiurewicz parameter is then summarized by the following picture.



- $f_\lambda(z_1)=z_1, \forall\lambda\in D_\varepsilon$ and $z_1\in J_\lambda$ and repelling;
- $\exists A>1$ such that $\|f_\lambda(z)-f_\lambda(z')\| \geq A\|z-z'\|$, $\forall\lambda\in D_\varepsilon, z,z'\in B_r$;
- $(\lambda,z_1)\in f^{n_0}(\mathsf{C}_f)\cap(D_\varepsilon\times B_r)\Rightarrow \lambda=0$.

8.2. **Step 2: Lower bound for $\langle dd^c L, 1_{D_\varepsilon}\rangle$ using** (8)**.** Pick $(0,z_0)\in\mathsf{C}_f$ such that $f^{n_0}(z_0)=z_1$. After shrinking, we have $f^{n_0}: U\to D_\varepsilon\times B_r$ proper.



Note that $1_U \leq 1_{D_\varepsilon\circ p_M}$, where

$$\begin{aligned} p_M\colon M\times\mathbb{C}^k &\to M\\ (\lambda,z)&\mapsto\lambda. \end{aligned}$$

We now compute:

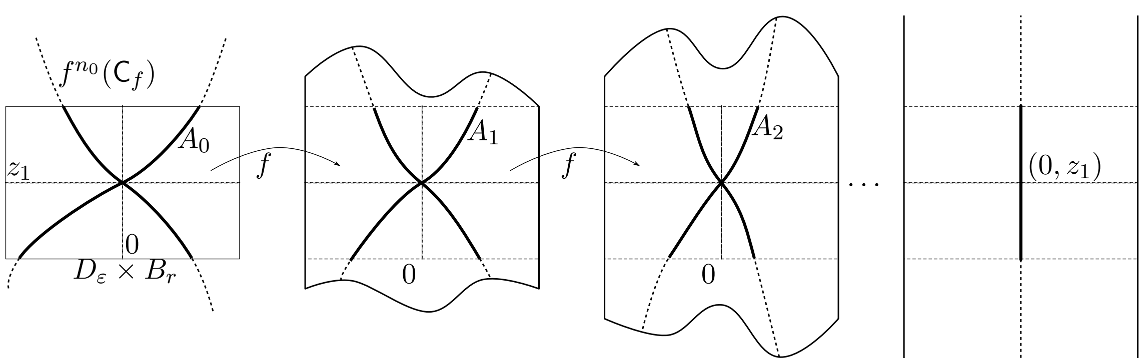
$$
\begin{aligned}
\langle dd^c L, 1_{D_\varepsilon}\rangle &\overset{\text{by (8)}}{=} \Big\langle \overbrace{(dd^c g+\omega)^k \wedge [\mathsf{C}_f]}^{\text{positive measure}}, 1_{D_\varepsilon}\circ p_M \Big\rangle \\
&\underset{1_U \le 1_{D_\varepsilon}\circ p_m}{\ge} \Big\langle (dd^c g+\omega)^k \wedge [\mathsf{C}_f], 1_U \Big\rangle \underset{\text{think } g \text{ smooth}}{=} \Big\langle 1_U\,[\mathsf{C}_f], (dd^c g+\omega)^k \Big\rangle \\
&= \Big\langle 1_U\,[\mathsf{C}_f], d^{-n_0 k}\,(f^{n_0})^*\,(dd^c g+\omega)^k \Big\rangle \\
&= \Big\langle d^{-n_0 k} f^{n_0}_*\,(1_U\,[\mathsf{C}_f]), (dd^c g+\omega)^k \Big\rangle \\
&\underset{f^{n_0}: U\to D_\varepsilon\times B_r \text{ proper}}{\ge} d^{-n_0 k}\Big\langle [f^{n_0}(\mathsf{C}_f)]\,1_{D_\varepsilon\times B_r}, (dd^c g+\omega)^k \Big\rangle .
\end{aligned}
$$

Setting $A_0 := 1_{D_\varepsilon\times B_r}\,[f^{n_0}(\mathsf{C}_f)]$ we have proved that

$$\langle dd^c L, 1_{D_\varepsilon}\rangle \ge d^{-n_0 k}\left\| A_0 \wedge (dd^c g+\omega)^k \right\| .$$

8.3. **Step 3: Transfer from the dynamical space $\{0\}\times\mathbb{P}^k$ to the parameter space via a dynamical rescaling.** Define inductively A_p by $A_{p+1} := 1_{D_\varepsilon\times B_r} f_*(A_p)$, one may check that

$$A_p \to m\,[\{0\}\times B_r] \text{ for some } m \ge 1.$$



Claim: $\left\| A_p \wedge (dd^c g+\omega)^k \right\| \le d^{pk}\left\| A_0 \wedge (dd^c g+\omega)^k \right\|$

Conclusion:

$$\langle dd^c L, 1_{D_\varepsilon} \rangle \underset{\boxed{\text{Step 1}}}{\geq} d^{-n_0 k} \left\| A_0 \wedge (dd^c g + \omega)^k \right\| \underset{\boxed{\text{Claim}}}{\geq} d^{-(n_0+p)k} \left\| A_p \wedge (dd^c g + \omega)^k \right\|.$$

This implies that $\langle dd^c L, 1_{D_\varepsilon} \rangle > 0$ since

$$\left\| A_p \wedge (dd^c g + \omega)^k \right\| \to \left\langle m\,[\{0\} \times B_r]\,, (dd^c g + \omega)^k \right\rangle \underset{\boxed{\mu_{\lambda_0} = (dd^c_z g(0,z) + \omega)^k}}{=} m\mu_{\lambda_0}(B_r) \underset{\boxed{z_1 \in J_{\lambda_0}}}{>} 0.$$

Proof of the Claim: (again think g smooth)

$$\begin{aligned}
\left\| A_{p+1} \wedge (dd^c g + \omega)^k \right\| &= \left\langle 1_{D_\varepsilon \times B_r} f_*(A_p), (dd^c g + \omega)^k \right\rangle \\
&= \left\langle A_p, f^* \left(1_{D_\varepsilon \times B_r} \, (dd^c g + \omega)^k \right) \right\rangle \\
&= \left\langle A_p, (1_{D_\varepsilon \times B_r} \circ f)\, d^k \, (dd^c g + \omega)^k \right\rangle \\
&\leq d^k \left\langle A_p, (dd^c g + \omega)^k \right\rangle = d^k \left\| A_p \wedge (dd^c g + \omega)^k \right\|.
\end{aligned}$$

□

9. Density of Misiurewicz parameters in the support of $dd^c L$

We consider a family $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ like in the last lecture.

9.1. A general condition for the vanishing of $dd^c L$.

Theorem 9.1. *If f admits an equilibrium web $\mathcal{M}$ such that*

$$\forall \gamma \in \operatorname{Supp} \mathcal{M} \colon \Gamma_\gamma \cap C_f \neq \emptyset \Rightarrow \Gamma_\gamma \subset C_f$$

then $dd^c L \equiv 0$ on M.

Proof. 1) To simplify we consider a lifted family $F : B \times \mathbb{C}^{k+1} \to B \times \mathbb{C}^{k+1}$ where B is a (small) disc in $\mathbb{C}$. Note that in this case $\operatorname{Supp} \mathcal{M} \subset \mathcal{O}(B, \mathbb{C}^{k+1})$. Set $\operatorname{Jac}(\lambda, z) := \det df_\lambda(z)$. We associate to $\mathcal{M}$ the web current $W_\mathcal{M} := \int [\Gamma_\gamma] \, d\mathcal{M}(\gamma)$. It is a result due to Pham[15] that the current $\log |\operatorname{Jac}| \, W_\mathcal{M}$ is well defined. Moreover, Pham has given a generalized version of the $dd^c L$ formula (8) of the last lecture which yields:

$$0 \leq dd^c L \leq (p_B)_* \left(dd^c \left(\log |\operatorname{Jac}| \, W_\mathcal{M} \right) \right).$$

It thus suffices to show that $\log |\operatorname{Jac}| \, W_\mathcal{M}$ is dd^c-closed.

[15] Ngoc-mai Pham. *Lyapunov exponents and bifurcation current for polynomial-like maps,* 2005. Preprint arXiv:math.DS/0512557v1.

2) A formal computation.

test form

$$\begin{aligned}
\Big\langle \log|\mathrm{Jac}|\, W_{\mathcal{M}}, dd^c\,\phi \Big\rangle &= \langle W_{\mathcal{M}}, \log|\mathrm{Jac}|\, dd^c\phi\rangle \\
&= \int \langle [\Gamma_\gamma, \log|\mathrm{Jac}|\, dd^c\phi]\rangle\, d\mathcal{M}(\gamma) \\
&= \int d\mathcal{M}(\gamma) \underbrace{\left(\int_{\mathbb{C}} \log|\mathrm{Jac}\,(\lambda,\gamma(\lambda))|\, dd^c(\phi\circ\gamma)\right)}_{=0 \text{ if } \Gamma_\gamma\cap\{\mathrm{Jac}=0\}=\emptyset} \\
&= 0.
\end{aligned}$$

3) To make the above computation rigorous, one uses the following estimate:

$$\mathcal{M}\left(\{\gamma\colon \Gamma_\gamma\cap\{|\mathrm{Jac}\,|<\varepsilon\}\neq\emptyset\}\right) \lesssim \varepsilon^\tau.$$

Let us set $\mathcal{S}_\varepsilon := \{\gamma\in \mathrm{Supp}\,\mathcal{M}\colon \Gamma_\gamma\cap\{|\mathrm{Jac}\,|<\varepsilon\}\neq\emptyset\}$. Let $\lambda_0\in B$ fixed.

Claim: $\mathcal{S}_\varepsilon \subset \left\{\gamma\colon \gamma(\lambda_0)\in \left(\mathsf{C}_{F_{\lambda_0}}\right)_{A\varepsilon^a}\right\}$, for some $A, a>0$.

The estimate above follows from the Claim:

$$\mathcal{M}(\mathcal{S}_\varepsilon) \leq \mu_{\lambda_0}\left(\left(\mathsf{C}_{F_{\lambda_0}}\right)_{A\varepsilon^a}\right) \lesssim \varepsilon^\tau$$

$(p_{\lambda_0})_*\,\mathcal{M}=\mu_{\lambda_0}$ $\qquad$ μ_{λ_0} has Hölder potentials

To prove that claim:

i) $\gamma\in\mathcal{S}_\varepsilon$ and $\Gamma_\gamma\not\subseteq\{\mathrm{Jac}\,=0\} \Rightarrow \gamma\in\mathcal{S}_\varepsilon$ and $\Gamma_\gamma\cap\{\mathrm{Jac}\,=0\}=\emptyset$. Then, as $B\ni\lambda\mapsto \mathrm{Jac}\,(\lambda,\gamma(\lambda))\in\Delta_R\setminus\{0\}$ contracts the Kobayashi distances we get $|\mathrm{Jac}\,(\lambda_0,\gamma(\lambda_0))|<\varepsilon^\alpha$, for some $0<\alpha\leq 1$. By compactness of $\mathrm{Supp}\,\mathcal{M}$, the estimate is uniform in γ.

ii) The claim then follows from a Łojasiewicz inequality. □

Remark 9.2. *The above theorem provides another proof of the fact that the holomorphic motion of repelling J-cycles implies the vanishing of dd^cL ((1) $\Rightarrow$ (2) in Lecture 5).*

9.2. **Density of Misiurewicz parameters.**

Lemma 9.3. *Let $\lambda_0\in M$ and B be a ball centered at λ_0 with no Misiurewicz parameters inside. Then λ_0 does not belong to the support of dd^cL.*

Proof. After shrinking B, we find $\gamma_0 : B\to\mathbb{P}^k$ holomorphic such that $\gamma(\lambda_0)\in J_\lambda$ and $\gamma_0(\lambda)$ is repelling and n_0-periodic for all λ. It is not clear at all that the graph of such a γ_0 is not included in C_f^+. We claim that one can actually find γ_0 in such a way that $\Gamma_{\gamma_0}\not\subseteq\mathsf{C}_f^+$. The proof of this fact is not elementary and will not be discussed here.

Since there are no Misiurewicz parameters in B and $\Gamma_{\gamma_0}\not\subseteq\mathsf{C}_f^+$ we have $\Gamma_{\gamma_0}\cap\mathsf{C}_f^+=\emptyset$. We may thus lift γ_0 by f^n:

$$\begin{array}{ccc} & & B \times \mathbb{P}^k \\ & \overset{(\lambda, \gamma_{n,j}(\lambda))}{\nearrow} & \big\downarrow f^n \\ B & \longrightarrow & B \times \mathbb{P}^k \\ \lambda & \longmapsto & (\lambda, \gamma_0(\lambda)) \end{array}$$

Let us set $\mathcal{M}_n := d^{-kn} \sum_j \delta_{\gamma_{n,j}}$. Using Banach-Alaoglu Theorem and the theorem of equidistribution of iterated preimages we find an equilibrium web $\mathcal{M}$ for f such that

$$\mathcal{M}_{n_i} \to \mathcal{M}.$$

Since by construction $\Gamma_\gamma \cap \mathsf{C}_f = \emptyset$ for all $\gamma \in \operatorname{Supp} \mathcal{M}_{n_i}$, Hurwitz Theorem shows that $\Gamma_\gamma \cap \mathsf{C}_f = \emptyset$ or $\Gamma_\gamma \subset \mathsf{C}_f$ for all $\gamma \in \operatorname{Supp} \mathcal{M}$. By the above theorem we thus have $dd^c L \equiv 0$ on B. □

Combining the above Lemma with the Fundamental Lemma proved in Lecture 8, we get:

Theorem 9.4. $\operatorname{Supp} dd^c L = \overline{\{\textit{Misiurewicz parameters}\}}$.

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