From analysis to algebra and back, via representations

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29 June 2015

Compact quantum groups

Definition (Woronowicz, 1989)

An algebra of continuous functions on a compact quantum group is a unital C^*-algebra A with a unital $*$-algebra homomorphism $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ such that

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta \qquad \text{(coassociativity)}$$

and the quantum cancellation rules hold:

$$\overline{\mathrm{Lin}}\{\Delta(a)(b \otimes \mathbf{1}); a, b \in \mathsf{A}\} = \overline{\mathrm{Lin}}\{\Delta(a)(\mathbf{1} \otimes b); a, b \in \mathsf{A}\} = \mathsf{A} \otimes \mathsf{A}$$

The tensor products here are in the C^*-algebraic category. We will write $\mathsf{A} = C(\mathbb{G})$ and call $\mathbb{G}$ a compact quantum group.
Sometimes (A, Δ) is called a compact quantum group.

Convolution of probability measures on a compact group

Let G – compact group. Given two finite measures μ, ν on G their convolution $\mu \star \nu$ is defined by

$$\int_G f(g) d_{\mu\star\nu}(g) = \int_G \int_G f(g_1 g_2) d_\mu(g_1) d_\nu(g_2), \quad f \in C(G).$$

Here finite (signed) measures – continuous functionals on $C(G)$. The convolution of probability measures remains a probability measure.

The **Haar measure** on G is the unique bi-invariant measure $\mu_h \in \mathrm{Prob}(G)$: for any $g \in G$ and a Borel set $A \subset G$

$$\mu_h(gA) = \mu_h(Ag) = \mu_h(A).$$

In other words, it is a unique measure such that

$$\nu \star \mu_h = \mu_h = \mu_h \star \nu, \quad \nu \in \mathrm{Prob}(G).$$

Convolution of probability measures on a compact quantum group

Definition

Let $\mathbb{G}$ be a compact quantum group. Given two functionals $\varphi, \psi \in C(\mathbb{G})^*$ their convolution is defined by

$$\varphi \star \psi = (\varphi \otimes \psi) \circ \Delta.$$

Convolution of states (normalised positive functionals) is a state. We view states on $C(\mathbb{G})$ as probability measures on $\mathbb{G}$ (and may write simply $\mathrm{Prob}(\mathbb{G})$).

Haar state

Definition

A state $h \in \mathrm{Prob}(\mathbb{G})$ is called a Haar state if for all $a \in C(\mathbb{G})$)

$$(h \otimes \mathrm{id})(\Delta(a)) = (\mathrm{id} \otimes h)(\Delta(a)) = h(a)\mathbf{1};$$

equivalently for each $\mu \in C(\mathbb{G})^*$

$$h \star \mu = \mu \star h = \mu(1)h;$$

equivalently for each $\omega \in \mathrm{Prob}(\mathbb{G})$

$$h \star \omega = \omega \star h = h.$$

Haar state continued

Theorem

Every compact quantum group has a unique Haar state h.

This uses cancellation laws! (One) idea of the proof: take a faithful state $\omega \in \mathrm{Prob}(\mathbb{G})$ and show that

$$h = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \omega^{\star k}.$$

For $C(G)$ the Haar state is given by the integration with respect to the Haar measure. On $C_r^*(\Gamma)$ it is given by $h(\sum_{\gamma\in\Gamma} c_\gamma \lambda_\gamma) = c_e$.

Representations

A (finite-dimensional, unitary, continuous) representation of a compact group G is a continuous map $U : G \to \mathrm{U}(n)$ such that

$$U(gh) = U(g)U(h), \quad g, h \in G.$$

Looking at matrix entries we can view it as a single element $U \in M_n(C(G))$.

Definition

A finite-dimensional unitary, continuous representation of a compact quantum group $\mathbb{G}$ is a unitary $U = [u_{ij}]_{i,j=1}^{n} \in M_n(C(\mathbb{G}))$ such that

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \quad i, j = 1, \ldots, n.$$

Equivalently, identifying $M_n(C(\mathbb{G}))$ with $M_n \otimes C(\mathbb{G})$ we can write it as

$$(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13}.$$

Coefficients of U – linear combinations of u_{ij}.

Non-degenerate representation – invertible $U \in M_n(C(\mathbb{G}))$ + the formulas above.

Representations, II

$U \approx V$ if there is $y \in GL(n)$ such that

$$(y \otimes \mathrm{id})U = V(y \otimes \mathrm{id})$$

Fact

Any non-degenerate representation is equivalent to a unitary one.

Operations on representations ($U, V \in \mathrm{Rep}(\mathbb{G})$):

- direct sum: $U \oplus V \in M_{n+m}(C(\mathbb{G}))$;
- tensor product: $U \otimes V \in M_{nm}(C(\mathbb{G}))$:

$$(U \otimes V)_{(i,k),(j,l)} = u_{ij} v_{kl}$$

- adjoint:

$$\overline{U} \approx [u_{ij}^*]_{i,j=1}^n.$$

U is irreducible if it cannot be non-trivially decomposed as a direct sum; equivalently, there is no non-trivial projection $p \in M_n$ such that $(p \otimes \mathrm{id})U = U(p \otimes \mathrm{id})$. $\mathrm{Irr}(\mathbb{G})$ – the set of all (equivalence classes) of irreducible reps.

Representations, III

Theorem

Any unitary representation of $\mathbb{G}$ decomposes as a direct sum of irreducibles. The set of coefficients of all finite-dimensional unitary/non-degenerate representations of $\mathbb{G}$ forms a unital dense *-subalgebra of $C(\mathbb{G})$, denoted $\mathrm{Pol}(\mathbb{G})$ (or $\mathcal{A}$). The set

$$\{u_{ij}^{\alpha} : \alpha \in \mathrm{Irr}(\mathbb{G}), i, j = 1, \ldots, n_\alpha\}$$

is a linear basis of $\mathrm{Pol}(\mathbb{G})$. With

$$\epsilon(u_{ij}^{\alpha}) = \delta_{ij}, \quad S(u_{ij}^{\alpha}) = (u_{ji}^{\alpha})^*$$

$\mathrm{Pol}(\mathbb{G})$ becomes a Hopf*-algebra.

Neither ϵ nor S need to extend to $C(\mathbb{G})$!
$U \in \mathrm{Rep}(\mathbb{G})$ is called fundamental if its coefficients generate $C(\mathbb{G})$ as a C^*-algebra.

Orthogonality

Theorem

Haar state is faithful on $\mathrm{Pol}(\mathbb{G})$ (for $a \in \mathrm{Pol}(\mathbb{G})$ if $h(a^* a) = 0$ then $a = 0$). For each $\alpha \in \mathrm{Irr}(\mathbb{G})$ there exists a unique positive matrix $Q_\alpha \in GL(n_\alpha)$ such that $\mathrm{Tr}(Q_\alpha) = \mathrm{Tr}(Q_\alpha^{-1}) := d_\alpha \geq n_\alpha$ and we have for all $\alpha, \beta \in \mathrm{Irr}(\mathbb{G})$

- $h\left(u_{ij}^{\alpha} (u_{kl}^{\beta})^*\right) = \delta_{\alpha\beta}\delta_{ik}\frac{(Q_\alpha)_{l,j}}{d_\alpha}$;
- $h\left((u_{ij}^{\alpha})^* u_{kl}^{\beta}\right) = \delta_{\alpha\beta}\delta_{jl}\frac{(Q_\alpha^{-1})_{k,i}}{d_\alpha}$.

The matrices Q have various incarnations:

- as so-called Woronowicz characters on $\mathrm{Pol}(\mathbb{G})$
- generators of the 'scaling automorphism group' τ;
- witnesses of non-traciality of h;
- witnesses of unboundedness of S

Kac property

Definition

A compact quantum group $\mathbb{G}$ is of Kac type if all $Q_\alpha = I$; equivalently, $S^2 = \mathrm{id}_{\mathrm{Pol}(\mathbb{G})}$; equivalently h is a trace; equivalently the 'quantum dimensions' d_α are equal to n_α.

From $\mathrm{Pol}(\mathbb{G})$ to $C(\mathbb{G})$

We need 'good' C^*-norms on $\mathrm{Pol}(\mathbb{G})$.

- universal norm:

$$\|a\|_u := \sup\{\|\pi(a)\| : \pi : \mathrm{Pol}(\mathbb{G}) \to B(\mathrm{H}), \pi \text{ unital *-homomorphism}\}$$

  Completion of $\mathrm{Pol}(\mathbb{G})$ in this norm – $C_u(\mathbb{G})$ admits good Δ_u, h_u, etc..

- reduced norm:

$$\|a\|_r := \|\pi_h(a)\|,$$

  where π_h is the GNS representation of the Haar state on $\mathrm{Pol}(\mathbb{G})$.

  Completion of $\mathrm{Pol}(\mathbb{G})$ in this norm – $C_r(\mathbb{G})$ admits good Δ_r, h_r, etc..

Of course $\|\cdot\|_u \geq \|\cdot\|_r$.

Definition

A compact quantum group $\mathbb{G}$ is coamenable if $\|\cdot\|_u = \|\cdot\|_r$; equivalently, h_u is faithful on $C_u(\mathbb{G})$; equivalently, ϵ extends to a character on $C_r(\mathbb{G})$.