Vrije Universiteit Brussel

Actions of Compact Quantum Groups I

Definition

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Course material

Material that will be treated:

- Actions and coactions of compact quantum groups.
- Actions on C^*-algebras and Hilbert modules.
- Crossed products.
- Free actions, ergodic actions, and their interrelationship.

Outline Lecture I

Compact quantum groups

Actions of compact quantum groups on compact quantum spaces

Actions on non-compact quantum spaces

Compact quantum groups

Definition (Woronowicz)

Compact quantum group (CQG) $\mathbb{G}$:

- *unital C^*-algebra $C(\mathbb{G})$,*
- *unital *-homomorphism, comultiplication*

$$\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$$

s.t.

- *coassociativity:*
$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta,$$
- *cancellation:*

$$[(C(\mathbb{G}) \otimes 1_{\mathbb{G}})\Delta(C(\mathbb{G}))] = [\Delta(C(\mathbb{G}))(1_{\mathbb{G}} \otimes C(\mathbb{G}))] = C(\mathbb{G}) \otimes C(\mathbb{G}).$$

Here:

$$[S] = \text{closed linear span of } S \text{ (in some Banach space).}$$

Classical CQG

Lemma

X, Y compact Hausdorff:

$$C(X) \otimes C(Y) \cong C(X \times Y), \quad (a \otimes b)(x, y) = a(x)b(y).$$

Example

G compact Hausdorff group $\Rightarrow$ CQG $(C(G), \Delta)$,

$$\Delta : C(G) \to C(G) \otimes C(G), \quad f \mapsto (\Delta(f) : (g, h) \mapsto f(gh)) .$$

Conversely:

CQG $\mathbb{G}$ with $C(\mathbb{G})$ commutative

$$\Downarrow$$

$G = \mathrm{Spec}(C(\mathbb{G}))$ compact Hausdorff group.

$C(\mathbb{G})$-corepresentations

Definition

Unitary $C(\mathbb{G})$-corepresentation:

- *finite dimensional Hilbert space $\mathcal{H}$,*
- $U \in B(\mathcal{H}) \otimes C(\mathbb{G})$

s.t.

- U *unitary,*
- $(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13}$, *where* $U_{12} = U \otimes 1$ *etc.*

$$U \in B(\mathcal{H}) \otimes C(\mathbb{G})$$

$$\updownarrow$$

$$\delta : \mathcal{H} \to \mathcal{H} \otimes C(\mathbb{G}), \quad \xi \mapsto U(\xi \otimes 1_{\mathbb{G}})$$

s.t. . . . ?

$\mathbb{G}$-representations

Definition

$\mathbb{G}$ compact quantum group.

(Continuous finite dimensional unitary left) $\mathbb{G}$-representation π:

- *finite dimensional Hilbert space $\mathcal{H}_\pi$,*
- *linear map*

$$\delta_\pi : \mathcal{H}_\pi \to \mathcal{H}_\pi \otimes C(\mathbb{G})$$

s.t.

- *right comodule: $(\mathrm{id} \otimes \Delta) \circ \delta_\pi = (\delta_\pi \otimes \mathrm{id}) \circ \delta_\pi$,*
- *isometric: $\delta_\pi(\xi)^* \delta_\pi(\eta) = \langle \xi, \eta \rangle 1_{C(\mathbb{G})}$,*
- *density: $[\delta_\pi(\mathcal{H})(1 \otimes C(\mathbb{G}))] = \mathcal{H} \otimes C(\mathbb{G})$.*

- Here $\mathcal{H}_\pi \cong B(\mathbb{C}, \mathcal{H}_\pi)$, so

$$(\xi \otimes a)^*(\eta \otimes b) = \xi^* \eta \otimes a^* b \cong \langle \xi, \eta \rangle a^* b.$$

- Density condition automatically satisfied.
- $C(\mathbb{G})$-corepresentations $\leftrightarrow$ $\mathbb{G}$-representations.

Classical representations

Example

Let G compact Hausdorff group. Then

$$G\text{-representations as compact quantum group}$$

$$\updownarrow$$

$$G\text{-representations as compact group}$$

by

$$\delta_\pi : \mathcal{H}_\pi \to \mathcal{H}_\pi \otimes C(G) \cong C(G, \mathcal{H}_\pi)$$

$$\updownarrow$$

$$\pi : G \times \mathcal{H}_\pi \to \mathcal{H}_\pi, \quad (g, \xi) \mapsto \pi(g)\xi = \delta_\pi(\xi)(g).$$

The canonical Hopf *-algebra

Theorem (Woronowicz)

Let

$$\mathscr{O}(\mathbb{G}) = \{(\xi^* \otimes \mathrm{id})\delta_\pi(\eta) \mid \pi \ \mathbb{G}\text{-representation},\ \xi, \eta \in \mathcal{H}_\pi\}.$$

Then

- $(\mathscr{O}(\mathbb{G}), \Delta)$ *Hopf *-algebra*, $(\mathscr{O}(\mathbb{G}), \Delta, \epsilon, S)$,
- $\mathscr{O}(\mathbb{G})$ *dense in* $C(\mathbb{G})$,
- $(\mathscr{O}(\mathbb{G}), \Delta)$ *unique dense Hopf *-algebra*,
- $\delta_\pi : \mathcal{H} \to \mathcal{H} \otimes \mathscr{O}(\mathbb{G})$ *is* $\mathscr{O}(\mathbb{G})$*-comodule:*
  - $(\mathrm{id} \otimes \Delta) \circ \delta_\pi = (\delta_\pi \otimes \mathrm{id}) \circ \delta_\pi$,
  - $(\mathrm{id}_\mathcal{H} \otimes \epsilon)\delta_\pi = \mathrm{id}_\mathcal{H}$.

Notation (Sweedler-Heynemann notation)

$h \in \mathscr{O}(\mathbb{G})$:

$$\Delta(h) = h_{(1)} \otimes h_{(2)}, \ (\Delta \otimes \iota)\Delta(h) = \Delta^{(2)}(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}, \ \ldots$$

Example

Let $h \in \mathscr{O}(\mathbb{G})$. *Then*

$$\begin{aligned} \Delta(h_{(1)})(1 \otimes S(h_{(2)})) &= h_{(1)} \otimes h_{(2)} S(h_{(3)}) \\ &= h_{(1)} \otimes \epsilon(h_{(2)})1 \\ &= h \otimes 1. \end{aligned}$$

Hence

(Linear span) $\Delta(\mathscr{O}(\mathbb{G}))(1 \otimes \mathscr{O}(\mathbb{G})) = \mathscr{O}(\mathbb{G}) \underset{\mathrm{alg}}{\otimes} \mathscr{O}(\mathbb{G})$.

Universal C*-algebra

Lemma

$\mathbb{G}$ *CQG.*

- *Universal C*-envelope* $C(\mathbb{G}_u)$ *of* $\mathscr{O}(\mathbb{G})$ *exists.*
- *CQG* $\mathbb{G}_u$ *by*

$$\Delta_u : C(\mathbb{G}_u) \to C(\mathbb{G}_u) \otimes C(\mathbb{G}_u).$$

Definition

$\mathbb{G}_u$ *universal CQG (associated to* $\mathbb{G}$*).*

Right actions of compact quantum groups on C*-algebras

Definition (Podleś)

Right action $\mathbb{X} \curvearrowleft \mathbb{G}$:

- *Compact quantum group* $\mathbb{G}$,
- *C*-algebra* $C(\mathbb{X})$ *(with* $\mathbb{X}$ *'compact quantum space'),*
- *Unital *-homomorphism, right coaction*

$$\alpha : C(\mathbb{X}) \to C(\mathbb{X}) \otimes C(\mathbb{G})$$

s.t.

- *coaction property:*

$$(\alpha \otimes \mathrm{id}_{\mathbb{G}}) \circ \alpha = (\mathrm{id}_{\mathbb{X}} \otimes \Delta) \circ \alpha,$$

- *density (Podleś condition):*

$$[\alpha(C(\mathbb{X}))(1_{\mathbb{X}} \otimes C(\mathbb{G}))] = C(\mathbb{X}) \otimes C(\mathbb{G}).$$

Right translations

Example

Let $\mathbb{G}$ *compact quantum group. Then* $\mathbb{G} \overset{\Delta}{\curvearrowleft} \mathbb{G}$ *by*

$$\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G}).$$

Half-classical case

Lemma (All $C(\mathbb{G})$ commutative)

- *G compact Hausdorff group,*
- *C^*-algebra $C(\mathbb{X})$,*
- *$G \overset{\alpha}{\curvearrowright} C(\mathbb{X})$ continuous action:*
  - *$(g, a) \mapsto \alpha_g(a)$ continuous,*
  - *each α_g *-automorphism,*
  - *$\alpha_{gh} = \alpha_g \circ \alpha_h$,*
  - *$\alpha_e = \mathrm{id}_{\mathbb{X}}$, for $e \in G$ identity element.*

$\Rightarrow \mathbb{X} \curvearrowleft G$,

$$\alpha : C(\mathbb{X}) \to C(\mathbb{X}) \otimes C(G) \cong C(G, C(\mathbb{X})),$$

$$a \mapsto (\alpha(a) : g \mapsto \alpha_g(a)) \,.$$

Proof, Part I

- Forgetting group structure:
  - Using partitions of unity on G:
    - $C(\mathbb{X}) \otimes C(G) \overset{\cong}{\to} C(G, C(\mathbb{X}))$ by $a \otimes f \mapsto (g \mapsto f(g)a)$.
    - $C(\mathbb{X}) \otimes C(G) \otimes C(G) \cong C(G \times G, C(\mathbb{X}))$, etc.
  - continuous $G \overset{\alpha}{\curvearrowright} C(\mathbb{X})$ by unital *-endomorphisms $\Leftrightarrow \alpha : C(\mathbb{X}) \to C(G, C(\mathbb{X}))$ unital *-homomorphism.
- $((\mathrm{id} \otimes \Delta)\alpha)(a)(g, h) = ((\alpha \otimes \mathrm{id})\alpha)(a)(g, h) \Leftrightarrow \alpha_{gh}(a) = \alpha_g(\alpha_h(a))$.

Conclusion: one-to-one correspondence between

- α with coaction property, and
- actions of a group on a C*-algebra by endomorphisms.

To do: Density $\Leftrightarrow \alpha_e = \mathrm{id}_{C(\mathbb{X})}$ for e unit G.

Proof, Part II

- *-homomorphism

$$\widetilde{\alpha} : C(\mathbb{X}) \otimes C(G) \to C(\mathbb{X}) \otimes C(G), \quad a \otimes f \mapsto \alpha(a)(1 \otimes f).$$

- Density $\Leftrightarrow$ $\widetilde{\alpha}$ surjective.
- On level of $C(G, C(\mathbb{X})) \cong C(\mathbb{X}) \otimes C(G)$:

$$\forall F \in C(G, C(\mathbb{X})), \quad \widetilde{\alpha}(F)(g) = \alpha_g(F(g)).$$

- Assume $\alpha_e = \mathrm{id}_{C(\mathbb{X})}$. Then $\widetilde{\alpha}$ has inverse $\widetilde{\beta}$,

$$\widetilde{\beta}(F)(g) = \alpha_{g^{-1}}(F(g)).$$

  Hence range $\widetilde{\alpha}$ dense.
- If $\alpha_e \neq \mathrm{id}_{C(\mathbb{X})} \Rightarrow \alpha_e$ non-trivial idempotent *-endomorphism.
- Put $C(\mathbb{X}_e) = \alpha_e(C(\mathbb{X})) \neq C(\mathbb{X})$.
- $\forall g \in G$: $\alpha_g(C(\mathbb{X})) = \alpha_e(\alpha_g(C(\mathbb{X}))) \subseteq C(\mathbb{X}_e)$.
- $\Rightarrow$ If $a \notin C(\mathbb{X}_e)$, then $g \mapsto a$ not in range $\widetilde{\alpha}$.

Classical

Example (All $C(\mathbb{G})$ and $C(\mathbb{X})$ commutative)

G compact Hausdorff group, X compact Hausdorff space,

$$X \curvearrowleft G \text{ continuous} \quad \Rightarrow \quad G \curvearrowright C(X), \quad \alpha_g(f)(x) = f(x \cdot g).$$

Example

Consider sphere

$$S^{N-1} = \{z = (z_1, \ldots, z_N) \in \mathbb{R}^N \mid \sum_i z_i^2 = 1\}.$$

Then $S^{N-1} \curvearrowleft O(N)$ by

$$(z, g) \mapsto zg.$$

Example: Half-classical I

Example

Cuntz algebras,

$$\mathcal{O}_n = C^*(V_1, \ldots, V_n \mid V_i^* V_j = \delta_{ij}, \sum_i V_i V_i^* = 1).$$

Then $U(n) \curvearrowright \mathcal{O}_n$ *by*

$$\alpha_u(V_i) = \sum_j u_{ji} V_j.$$

In particular, $S^1 \curvearrowright \mathcal{O}_n$ *by*

$$\alpha_z(V_i) = zV_i.$$

Example: Half-classical II

Example (Banica)

Free spheres,

$$C(S_+^{N-1}) =< V_1, \ldots, V_N \mid V_i = V_i^*, \sum_i V_i^2 = 1\}.$$

Then $O(N) \curvearrowright C(S_+^{N-1})$ *by*

$$\alpha_g(V_i) = \sum_j g_{ji} V_j.$$

Left actions of compact quantum groups on C*-algebras

Definition (Podleś)

Left action $\mathbb{G} \curvearrowright \mathbb{H}$*:*

- *Compact quantum group* $\mathbb{G}$*,*
- *C*-algebra* $C(\mathbb{X})$*,*
- *Unital *-homomorphism, left coaction*

$$\alpha : C(\mathbb{X}) \to C(\mathbb{G}) \otimes C(\mathbb{X})$$

s.t.

- *coaction property:*

$$(\mathrm{id}_{\mathbb{G}} \otimes \alpha) \circ \alpha = (\Delta \otimes \mathrm{id}_{\mathbb{X}}) \circ \alpha,$$

- *density:*

$$[(C(\mathbb{G}) \otimes 1_{\mathbb{X}})\alpha(C(\mathbb{X}))] = C(\mathbb{X}) \otimes C(\mathbb{G}).$$

From left to right

Definition

Let $\mathbb{G}$ *CQG. Then* $\mathbb{G}^{\mathrm{op}}$ *CQG by*

$$C(\mathbb{G}^{\mathrm{op}}) = C(\mathbb{G}), \quad \Delta_{\mathbb{G}^{\mathrm{op}}} = \Delta_{\mathbb{G}}^{\mathrm{op}} = \varsigma \circ \Delta,$$

where

$$\varsigma : C(\mathbb{G}) \otimes C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G}), \quad g \otimes h \mapsto h \otimes g.$$

Lemma

$$\mathbb{G} \overset{\alpha}{\curvearrowright} \mathbb{X} \quad \leftrightarrow \quad \mathbb{X} \overset{\alpha^{\mathrm{op}}}{\curvearrowleft} \mathbb{G}^{\mathrm{op}}.$$

Non-unital C*-algebras

Definition (Multiplier C*-algebras)

$C_0(\mathbb{X})$ *non-unital C*-algebra ('locally compact quantum space').*

Multiplier C-algebra* $M(C_0(\mathbb{X})) = C_b(\mathbb{X})$:

- *Concrete: For* $C_0(\mathbb{X}) \subseteq B(\mathcal{H})$ *with* $[C_0(\mathbb{X})\,\mathcal{H}] = \mathcal{H}$*:*

$$C_b(\mathbb{X}) = \{T \in B(\mathcal{H}) \mid \forall a \in C_0(\mathbb{X}), Ta, aT \in C_0(\mathbb{X})\}.$$

- *Abstract:* $C_b(\mathbb{X})$ *collection maps* $T : C_0(\mathbb{X}) \to C_0(\mathbb{X})$ *s.t.*

$$\exists T^*, \forall a, b \in C_0(\mathbb{X}), \quad a^*(Tb) = (T^*b)^*a.$$

If $T \in C_b(\mathbb{X})$: $T(ab) = T(a)b$, and $C_0(\mathbb{X}) \subseteq C_b(\mathbb{X})$.

Example

If X *locally compact Hausdorff space,*

$$M(C_0(X)) = C_b(X).$$

Morphisms between locally compact quantum spaces

Definition

*-homomorphism $\pi : C_0(\mathbb{Y}) \to M(C_0(\mathbb{X}))$ non-degenerate:

$$[\pi(C_0(\mathbb{Y}))C_0(\mathbb{X})] = C_0(\mathbb{X}).$$

Example

Let X, Y locally compact Hausdorff spaces.

- *Non-degenerate maps $C_0(Y) \to C_b(X) \Leftrightarrow$ continuous maps $X \to Y$.*
- *Non-degenerate maps $C_0(Y) \to C_0(X) \Leftrightarrow$ continuous proper maps $X \to Y$.*
- *Degenerate map $C_0(Y) \to C_b(X)$: points of X to infinity.*

Lemma

$\pi : C_0(\mathbb{Y}) \to C_b(\mathbb{X})$ *non-degenerate* $\Rightarrow \exists! \pi : C_b(\mathbb{Y}) \to C_b(\mathbb{X})$.

Actions on locally compact quantum spaces

Definition

Right action $\mathbb{X} \curvearrowleft \mathbb{G}$:

- *Compact quantum group* $\mathbb{G}$,
- C^**-algebra* $C_0(\mathbb{X})$,
- *non-degenerate* **-homomorphism, right coaction*

$$\alpha : C_0(\mathbb{X}) \to C_b(\mathbb{X} \times \mathbb{G})$$

s.t.

- *coaction property:*
$$(\alpha \otimes \mathrm{id}_{\mathbb{G}}) \circ \alpha = (\mathrm{id}_{\mathbb{X}} \otimes \Delta) \circ \alpha,$$
- *density:*
$$[\alpha(C_0(\mathbb{X}))(1_{\mathbb{X}} \otimes C(\mathbb{G}))] = C_0(\mathbb{X}) \otimes C(\mathbb{G}).$$

In particular...

$$\alpha : C_0(\mathbb{X}) \to C_0(\mathbb{X}) \otimes C(\mathbb{G}) \quad \text{(proper action)}.$$