Vrije Universiteit Brussel

Actions of Compact Quantum Groups II

Examples, spectral components and Podleś algebra

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Outline

Recall: actions on locally compact quantum spaces

Definition

Right action $\mathbb{X} \curvearrowleft \mathbb{G}$:

- *Compact quantum group* $\mathbb{G}$,
- *C*-algebra* $C_0(\mathbb{X})$,
- *non-degenerate* **-homomorphism, right coaction*

$$\alpha : C_0(\mathbb{X}) \to C_0(\mathbb{X}) \otimes C(\mathbb{G})$$

s.t.

- *coaction property,*
$$(\alpha \otimes \mathrm{id}_{\mathbb{G}}) \circ \alpha = (\mathrm{id}_{\mathbb{X}} \otimes \Delta) \circ \alpha,$$
- *density:*
$$[\alpha(C_0(\mathbb{X}))(1_{\mathbb{X}} \otimes C(\mathbb{G}))] = C_0(\mathbb{X}) \otimes C(\mathbb{G}).$$

Actions from representations

Example

Let $\mathbb{G}$ CQG, π $\mathbb{G}$-representation. Then $\mathbb{G} \curvearrowright B(\mathcal{H}_\pi)$,

$$\mathrm{Ad}_\pi : B(\mathcal{H}_\pi) \to B(\mathcal{H}_\pi) \otimes C(\mathbb{G}),$$

$$\xi\eta^* \mapsto \delta_\pi(\xi)\delta_\pi(\eta)^* = U_\pi(\xi\eta^* \otimes 1)U_\pi^*.$$

For G classical Hausdorff group, π representation:

$$(\mathrm{Ad}_\pi)_g(x) = \pi(g)x\pi(g)^*, \quad x \in B(\mathcal{H}_\pi).$$

Universal C*-envelopes

Definition

*Let $\mathscr{O}(\mathbb{X})$ *-algebra. Then $\mathscr{O}(\mathbb{X})$ admits universal C*-envelope if*

$$\|x\|_u = \sup\{\|\lambda(x)\| \mid \lambda \text{ non-degenerate } {}^*\text{-representation } \mathscr{O}(\mathbb{X}) \to B(\mathcal{H}_\lambda)\} < \infty.$$

Then

$$C_0(\mathbb{X}_u) \cong \left[\mathrm{Im}\left(\mathscr{O}(\mathbb{X}) \to \prod_\lambda B(\mathcal{H}_\lambda)\right)\right].$$

Remark:

- $\mathscr{O}(\mathbb{X}) \to C_0(\mathbb{X}_u)$ not necessarily injective...
- $C_0(\mathbb{X}_u)$ could be zero!

Examples

- *Examples:* $C_u(\mathbb{G}), \mathcal{O}_n, C(S_+^{N-1}), \ldots$
- *Non-example:* $\mathbb{C}[x]$, $x^* = x$.

C^*-algebraic actions from algebraic actions

Lemma

*Let $\mathscr{O}(\mathbb{X})$ *-algebra with Hopf *-algebraic coaction*

$$\alpha : \mathscr{O}(\mathbb{X}) \to \mathscr{O}(\mathbb{X}) \underset{\text{alg}}{\otimes} \mathscr{O}(\mathbb{G}).$$

Assume $\mathscr{O}(\mathbb{X})$ admits a universal C^*-envelope.

Then α extends to coaction

$$\alpha_u : C_0(\mathbb{X}_u) \to C_0(\mathbb{X}_u) \otimes C(\mathbb{G}_u).$$

Hopf *-algebraic coaction:

- $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$,
- $(\mathrm{id}_{\mathbb{X}} \otimes \epsilon)\alpha = \mathrm{id}_{\mathbb{X}}$.

Proof

- Existence α_u as *-homomorphism: clear by universality.
- α_u coaction:
  - Coaction property: clear by continuity.
  - Density:
    - Write

$$\alpha(a) = a_{(0)} \otimes a_{(1)}, (\mathrm{id} \otimes \Delta)\alpha(a) = a_{(0)} \otimes a_{(1)} \otimes a_{(2)}, \ldots$$

    - Then

$$\begin{aligned} \alpha(a_{(0)})(1 \otimes S(a_{(1)})) &= a_{(0)} \otimes a_{(1)} S(a_{(2)}) \\ &= a_{(0)} \otimes \epsilon(a_{(1)})1 \\ &= a \otimes 1. \end{aligned}$$

    - Hence $\alpha(\mathscr{O}(\mathbb{X}))(1_{\mathbb{X}} \otimes \mathscr{O}(\mathbb{G})) = \mathscr{O}(\mathbb{X}) \underset{\mathrm{alg}}{\otimes} \mathscr{O}(\mathbb{G})$.
    - Hence $[\alpha_u(C_0(\mathbb{X}_u))(1 \otimes C(\mathbb{G}_u))] = C_0(\mathbb{X}_u) \otimes C(\mathbb{G}_u)$.

Actions from representations II

Definition

Let $\mathcal{H}$ finite dimensional Hilbert space. Cuntz C^*-algebra $\mathcal{O}(\mathcal{H})$,

- $\mathcal{H} \subseteq \mathcal{O}(\mathcal{H})$ *linearly,*
- $\xi^*\eta = \langle \xi, \eta \rangle$,
- $\sum_i \xi_i\xi_i^* = 1$ *for* $\{\xi_i\}$ *o.n. basis.*

Example

$\mathcal{O}_n = \mathcal{O}(\mathbb{C}^n)$.

Example

Let $\mathbb{G}$ CQG, π $\mathbb{G}$-representation. Then action $\mathbb{G} \curvearrowright \mathcal{O}(\mathcal{H}_\pi)$,

$$\alpha_\pi : \mathcal{O}(\mathcal{H}_\pi) \to \mathcal{O}(\mathcal{H}_\pi) \otimes C(\mathbb{G}), \quad \xi \mapsto \delta_\pi(\xi).$$

Liberated and free

Definition (Wang)

Universal C^*-algebra $C(O_N^+)$,

$$C^*(u_{ij} \mid 1 \leq i,j \leq N, u_{ij}^* = u_{ij} \text{ and } U = (u_{ij})_{i,j} \text{ unitary})$$

is CQG by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

Example

$S_+^{N-1} \curvearrowleft O_N^+$ *by*

$$\alpha(V_i) = \sum_j V_j \otimes u_{ji}.$$

Half-classical revisited

Definition (Wang)

Universal C^*-algebra

$$C(\mathrm{Sym}_n^+) = C(O^+(n))/ < u_{ij} - u_{ij}^2 >$$

is CQG by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

Example

Let $X_n = \{1, 2, \dots, n\}$. *Then* $X_n \curvearrowleft \mathrm{Sym}_n^+$,

$$\alpha : C(X_n) \to C(X_n) \otimes C(\mathrm{Sym}_n^+), \quad \delta_i \mapsto \sum_j \delta_j \otimes u_{ji}.$$

Actions of discrete group duals

Example (C*-algebraic bundles and Γ-graded C*-algebras)

- Γ *discrete group.*
- *Banach spaces* A_g *with 'associative' contractive multiplication*

$$A_g \times A_h \to A_{gh}.$$

- $* : A_g \to A_{g^{-1}}$ *antilinear, 'involutive', isometric.*
- $\|b^*b\| = \|b\|^2$ *for* $b \in A_g$*,*
- $b^*b \geq 0$ *in (the C*-algebra)* A_e *for* $b \in A_g$*.*

Then $\widehat{\Gamma} \curvearrowright A =$ *universal C*-envelope* $\oplus_g A_g$*,*

$$\alpha : A \to A \otimes C^*(\Gamma), \quad a_g \mapsto a_g \otimes \lambda_g.$$

The C*-algebra of coinvariants

Definition

Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$*. Space of orbits:* $\mathbb{Y} = \mathbb{X}/\mathbb{G}$*, given by C*-algebra*

$$C_0(\mathbb{Y}) = \{a \in C_0(\mathbb{X}) \mid \alpha(a) = a \otimes 1_{\mathbb{G}}\}.$$

Examples

- $G \overset{\alpha}{\curvearrowright} C_0(\mathbb{X})$

$$C_0(\mathbb{Y}) = C_0(\mathbb{X})^G = \{a \in C_0(\mathbb{X}) \mid \alpha_g(a) = a \textit{ for all } g \in G\}.$$

- $X \overset{\alpha}{\curvearrowleft} G$:

$$\begin{aligned} C_0(X)^G &= \{G\textit{-constant continuous functions on } X\} \\ &\cong \{\textit{continuous functions on } X/G\}. \end{aligned}$$

Intertwiners and fixed point subalgebras

Definition (Space of intertwiners)

Let π_1 *and* π_2 $\mathbb{G}$*-representations. Then*

$$\mathrm{Mor}(\pi_1, \pi_2) = \{T : \mathcal{H}_1 \to \mathcal{H}_2 \mid \delta_2 \circ T = (T \otimes \mathrm{id}) \circ \delta_1\} \subseteq B(\mathcal{H}_1, \mathcal{H}_2).$$

Lemma

- $T \in \mathrm{Mor}(\pi_1, \pi_2), T' \in \mathrm{Mor}(\pi_2, \pi_3) \Rightarrow T' \circ T \in \mathrm{Mor}(\pi_1, \pi_3).$
- $T \in \mathrm{Mor}(\pi_1, \pi_2) \Rightarrow T^* \in \mathrm{Mor}(\pi_2, \pi_1).$

Example

$$B(\mathcal{H}_\pi)^{\alpha_\pi} = \mathrm{Mor}(\pi, \pi),$$

$$\alpha_\pi(\xi\eta^*) = \delta_\pi(\xi)\delta_\pi(\eta)^* = U_\pi(\xi\eta^* \otimes 1)U_\pi^*.$$

Commuting actions

Definition

Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$ *and* $\mathbb{H} \overset{\beta}{\curvearrowright} \mathbb{X}$. *Commutation of* α *and* β*:*

$$(\beta \otimes \mathrm{id}_{\mathbb{G}})\alpha = (\mathrm{id}_{\mathbb{G}} \otimes \alpha)\beta.$$

Example

Assume $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$ *and* $\mathbb{H} \overset{\beta}{\curvearrowright} \mathbb{X}$ *commute. Then* $\mathbb{H} \curvearrowright \mathbb{X}/\mathbb{G}$*,*

$$\beta_{|C(\mathbb{X}/\mathbb{G})} : C(\mathbb{X}/\mathbb{G}) \to C(\mathbb{H}) \otimes C(\mathbb{X}/\mathbb{G}).$$

Invariant functionals on CQG

Theorem (Woronowicz)

$\mathbb{G}$ *compact quantum group.*

$\exists!$ *state* φ *on* $C(\mathbb{G})$, *Haar state, s.t.*

$$(\mathrm{id} \otimes \varphi)\Delta(f) = (\varphi \otimes \mathrm{id})\Delta(f) = \varphi(f)1_{\mathbb{G}}, \quad \forall f \in C(\mathbb{G}).$$

For G compact Hausdorff group, μ *Haar (probability) measure*,

$$\varphi(f) = \int_G f(g)\mathrm{d}\mu(g).$$

Lemma

φ *faithful on* $\mathscr{O}(\mathbb{G})$*:*

$$\forall h \in \mathscr{O}(\mathbb{G}), \quad \varphi(h^*h) = 0 \quad \Rightarrow \quad h = 0.$$

In fact: if $h \in \mathscr{O}(\mathbb{G})$ *positive in* $C(\mathbb{G})$ *and* $\varphi(h) = 0$*, then* $h = 0$*.*

Conditional expectation

Lemma (Integration over fibers)

$\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$, $\mathbb{Y} = \mathbb{X}/\mathbb{G}$. *Conditional expectation onto* $C_0(\mathbb{Y})$

$$E_{\mathbb{Y}} : C_0(\mathbb{X}) \to C_0(\mathbb{X}), \quad a \mapsto (\mathrm{id} \otimes \varphi)\alpha(a),$$

- *range* $C_0(\mathbb{Y})$,
- *idempotent,*
- *completely positive,*
- *bimodular:*

$$E_{\mathbb{Y}}(bac) = bE_{\mathbb{Y}}(a)c, \qquad a \in C_0(\mathbb{X}), b, c \in C_0(\mathbb{Y}),$$

- *non-degenerate:* $[C_0(\mathbb{X})C_0(\mathbb{Y})] = C_0(\mathbb{X})$.

Non-degeneracy: 'Every point of $\mathbb{X}$ is in an orbit (point of $\mathbb{Y}$)'.

Examples

Examples

- $G \overset{\alpha}{\curvearrowright} C_0(\mathbb{X})$

$$E_{\mathbb{Y}}(a) = \int_G \alpha_g(a) \mathrm{d}\mu(g).$$

- $X \overset{\alpha}{\curvearrowleft} G$: *integration over orbits,*

$$E_Y(f)(xG) = \int_G f(xg) \mathrm{d}\mu(g).$$

Example

$S^1 \curvearrowright \mathcal{O}_n$:

$$\begin{aligned} E_{\mathbb{Y}}(V_{i_1} \dots V_{i_N} V_{j_1}^* \dots V_{j_M}^*) &= \int_{S^1} z^{N-M} (V_{i_1} \dots V_{i_N} V_{j_1}^* \dots V_{j_M}^*) \mathrm{d}z \\ &= \delta_{M,N} V_{i_1} \dots V_{i_N} V_{j_1}^* \dots V_{j_N}^*. \end{aligned}$$

Proof (of properties $E_{\mathbb{Y}}$)

- Range $\subseteq C_0(\mathbb{Y})$:

$$\begin{aligned} \alpha(E_{\mathbb{Y}}(a)) &= \alpha\big((\mathrm{id}\otimes\varphi)\alpha(a)\big) \\ &= (\mathrm{id}\otimes\mathrm{id}\otimes\varphi)((\alpha\otimes\mathrm{id})\alpha(a)) \\ &= (\mathrm{id}\otimes\mathrm{id}\otimes\varphi)((\mathrm{id}\otimes\Delta)\alpha(a)) \\ &= (\mathrm{id}\otimes\varphi)(\alpha(a))\otimes 1_{\mathbb{G}} \\ &= E_{\mathbb{Y}}(a)\otimes 1_{\mathbb{G}}. \end{aligned}$$

- Trivially, $E_{\mathbb{Y}}(b) = b$ for $b \in C_0(\mathbb{Y})$.
- Trivially, $E_{\mathbb{Y}}$ completely positive (state + *-homs c.p.).
- Trivially, $E_{\mathbb{Y}}$ $C_0(\mathbb{Y})$-bimodular.
- Non-degenerate: u_α bounded approximate unit $C_0(\mathbb{X})$,

$$\forall b \in C_0(\mathbb{X}), \quad E_{\mathbb{Y}}(u_\alpha)b = (\mathrm{id}\otimes\varphi)(\alpha(u_\alpha)(b\otimes 1)) \to b,$$

since $b\otimes 1 \in [\alpha(C_0(\mathbb{X}))(1\otimes C(\mathbb{G}))]$.

Results on $\mathbb{G}$-representations

Definition

- π *indecomposable*: $\pi \ncong \pi_1 \oplus \pi_2$.
- π *irreducible*: $T \in \mathrm{Mor}(\pi', \pi)$, $T^*T = \mathrm{id} \Rightarrow TT^* = \mathrm{id}$ *or* 0.

Proposition

Let $\mathbb{G}$ compact quantum group.

- *$\mathbb{G}$-representation indecomposable $\Leftrightarrow$ irreducible.*
- *$\mathbb{G}$-representation $\cong$ direct sum irreducibles.*

Isotypical components

Definition

$\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$, π $\mathbb{G}$*-representation.*

Intertwiner space $\mathrm{Mor}(\pi, \alpha)$*:*

$$\mathrm{Mor}(\pi, \alpha) = \{T : \mathcal{H}_\pi \to C_0(\mathbb{X}) \mid \alpha(T\xi) = (T \otimes \mathrm{id})\delta_\pi(\xi)\}.$$

π *irreducible:* π*-isotypical component (or* π*-spectral subspace)*

$$C_0(\mathbb{X})_\pi = \{T\xi \mid \xi \in \mathcal{H}_\pi, T \in \mathrm{Mor}(\pi, \alpha)\} \subseteq C_0(\mathbb{X}).$$

Note: Each $C_0(\mathbb{X})_\pi$ is $C_0(\mathbb{Y})$-bimodule.

The Podleś subalgebra

Theorem (Podleś)

Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$. *Then* **-algebra (unital if* $\mathbb{X}$ *compact)*

$$\mathscr{O}_{\mathbb{G}}(\mathbb{X}) = \textit{linear span}\ \{C_0(\mathbb{X})_\pi \mid \pi\ \textit{irreducible}\} \subseteq C_0(\mathbb{X}).$$

Definition

$\mathscr{O}_{\mathbb{G}}(\mathbb{X})$ *Podleś subalgebra of* $C_0(\mathbb{X})$.

Example

Example

$\mathscr{O}(\mathbb{G}) = \mathscr{O}_{\mathbb{G}}(\mathbb{G})$ *for* $\mathbb{G} \overset{\Delta}{\curvearrowleft} \mathbb{G}$.

Proof.

- $\subseteq$ For $\omega \in \mathcal{H}_\pi^*$, put

$$R_\omega : \mathcal{H} \to C(\mathbb{G}), \quad \xi \mapsto (\omega \otimes \mathrm{id})\delta_\pi(\xi).$$

Then

$$\begin{aligned}(R_\omega \otimes \mathrm{id})\delta_\pi(\xi) &= (\omega \otimes \mathrm{id} \otimes \mathrm{id})((\delta_\pi \otimes \mathrm{id})\delta_\pi(\xi)) \\ &= (\omega \otimes \mathrm{id} \otimes \mathrm{id})((\mathrm{id} \otimes \Delta)\delta_\pi(\xi)) \\ &= \Delta(R_\omega(\xi)),\end{aligned}$$

hence $R_\omega \in \mathrm{Mor}(\pi, \Delta)$, and

$$\delta_\pi : \mathcal{H}_\pi \mapsto \mathcal{H}_\pi \otimes C(\mathbb{G})_\pi.$$

- $\supseteq$ $\mathscr{O}(\mathbb{G})_\pi$ unitary $\mathbb{G}$-representation for Δ and $\langle g, h \rangle = \varphi(g^* h)$.

□

Tensor products of unitary left comodules

Notation (Unsummed Heyneman-Sweedler notation)

For $h \in \mathscr{O}(\mathbb{G})$*:*

$$\Delta(h) = h_{(1)} \otimes h_{(2)}, \quad \Delta^{(2)}(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}, \ldots$$

For π *and* $\xi \in \mathcal{H}_\pi$*:*

$$\delta_\pi(\xi) = \xi_{(0)} \otimes \xi_{(1)}, \quad (\mathrm{id} \otimes \Delta)\delta_\pi(\xi) = \xi_{(0)} \otimes \xi_{(1)} \otimes \xi_{(2)}, \ldots$$

Definition

π_1, π_2 $\mathbb{G}$*-representations: tensor product* $\pi_1 \otimes \pi_2$,

$$(\mathcal{H}_1 \otimes \mathcal{H}_2, \delta_{\pi_1 \otimes \pi_2}), \quad (\delta_{\pi_1 \otimes \pi_2})(\xi \otimes \eta) = \xi_{(0)} \otimes \eta_{(0)} \otimes \xi_{(1)}\eta_{(1)}.$$

Contragredient representation

Lemma

π $\mathbb{G}$-representation:

- $\langle\langle \xi^*, \eta^* \rangle\rangle = (\mathrm{Tr} \otimes \varphi)(\delta(\xi)\delta(\eta)^*)$ *inner product on* $\mathcal{H}_\pi^*$.
- $\delta_\pi^c : \mathcal{H}_\pi^* \to \mathcal{H}_\pi^* \otimes C(\mathbb{G})$,

$$\delta_\pi^c(\xi^*) = \delta_\pi(\xi)^*$$

unitary $\mathbb{G}$-representation.

Definition

$\pi^c = \big((\mathcal{H}_\pi^*, \langle\langle \cdot, \cdot \rangle\rangle), \delta_\pi^c\big)$ *contragredient of* π.

Proof $\mathscr{O}_{\mathbb{G}}(\mathbb{X})$ is unital *-algebra

- Unitality: trivial left representation

$$\eta : \mathbb{C} \to \mathbb{C} \otimes C(\mathbb{G}), \quad 1 \mapsto 1 \otimes 1_{\mathbb{G}}.$$

- By linearity and semisimplicity:

$$\mathscr{O}_{\mathbb{G}}(\mathbb{X}) = \{T\xi \mid \pi, \xi \in \mathcal{H}_\pi, T \in \mathrm{Mor}(\pi, \alpha)\} \subseteq C_0(\mathbb{X}).$$

- If $a = T\xi, b = S\eta$, then with m multiplication map,

$$ab = m(T\xi \otimes S\eta),$$

  where $m \circ (T \otimes S)$ in $\mathrm{Mor}(\pi_1 \otimes \pi_2, \alpha)$ since α hom.
- If $a = T\xi$, then

$$a^* = T^c(\xi^*),$$

  where $T^c : \eta^* \mapsto (T\eta)^*$ is in $\mathrm{Mor}(\pi^c, \alpha)$ since α *-preserving.

An algebraisation of actions

Proposition

Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$. Then α restricts to Hopf *-algebraic right coaction

$$\alpha_{\mathrm{alg}} : \mathscr{O}_{\mathbb{G}}(\mathbb{X}) \to \mathscr{O}_{\mathbb{G}}(\mathbb{X}) \underset{\mathrm{alg}}{\otimes} \mathscr{O}(\mathbb{G}).$$

This means:

$$(\alpha_{\mathrm{alg}} \otimes \mathrm{id})\alpha_{\mathrm{alg}} = (\mathrm{id} \otimes \Delta)\alpha_{\mathrm{alg}},$$

$$(\mathrm{id} \otimes \epsilon)\alpha_{\mathrm{alg}} = \mathrm{id}_{C(\mathbb{X})},$$

with ϵ counit $\mathscr{O}(\mathbb{G})$.

Proof

- For $a = T\xi$ with $\xi \in \mathcal{H}_\pi$, $T \in \mathrm{Mor}(\pi, \alpha)$:
  - $a \in C_0(\mathbb{X})_\pi$ and

$$\alpha(a) = \alpha(T\xi) = (T \otimes \mathrm{id})\delta_\pi(\xi) \in C_0(\mathbb{X})_\pi \underset{\mathrm{alg}}{\otimes} C(\mathbb{G}).$$

  - Hence

$$\alpha(a) \in C_0(\mathbb{X})_\pi \underset{\mathrm{alg}}{\otimes} C(\mathbb{G})_\pi \subseteq \mathscr{O}_{\mathbb{G}}(\mathbb{X}) \underset{\mathrm{alg}}{\otimes} \mathscr{O}(\mathbb{G}).$$

- α_{alg} coaction property: immediate.
- α_{alg} counital: for $a = T\xi$ with $\xi \in \mathcal{H}_\pi$, $T \in \mathrm{Mor}(\pi, \alpha)$:

$$(\mathrm{id} \otimes \epsilon)\alpha(T\xi) = T((\mathrm{id} \otimes \epsilon)\delta_\pi(\xi)) = T\xi.$$

Density

Theorem (Podleś)

Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$. Then $\mathscr{O}_{\mathbb{G}}(\mathbb{X})$ is dense in $C_0(\mathbb{X})$.

Remark: density $\mathscr{O}(\mathbb{G})$ in $C(\mathbb{G})$ used in proof.

More on unitary representations

Lemma

- *Any finite-dimensional $\mathscr{O}(\mathbb{G})$-comodule is unitarizable.*

  (Take any $\langle\cdot,\cdot\rangle$ and define $\langle\langle\xi,\eta\rangle\rangle = (\mathrm{id}\otimes\varphi)(\delta(\xi)^*\delta(\eta))$.)
- *$C(\mathbb{G})_\pi$ finite dimensional coalgebra.*

  (In fact $C(\mathbb{G})_\pi$ matrix coefficients δ_π.)
- *$C(\mathbb{G})_\pi$ for non-isomorphic π mutually $\perp$ for $\langle g,h\rangle = \varphi(g^*h)$.*

  (In fact $\xi\eta^* \mapsto (\mathrm{id}\otimes\varphi)(\delta(\xi)\delta(\eta)^*)$ in $\mathrm{Mor}(\pi_1,\pi_2)$.)
- *If $h \in C(\mathbb{G})_\pi$, $g \in C(\mathbb{G})$, then*

$$(\mathrm{id}\otimes\varphi)((1\otimes h^*)\Delta(g)) \in C(\mathbb{G})_\pi.$$

(In fact, follows from density $\mathscr{O}(\mathbb{G})$ and previous properties.)

Proof of density Podleś algebra

- $[\alpha(C_0(\mathbb{X}))(1 \otimes C(\mathbb{G}))] = C_0(\mathbb{X}) \otimes C(\mathbb{G}) \supseteq C_0(\mathbb{X}) \otimes \mathbb{C}$.
- $\mathscr{O}(\mathbb{G})$ dense in $C(\mathbb{G})$.
- $C_0(\mathbb{X}) = [(\mathrm{id} \otimes \varphi)((1 \otimes h^*)\alpha(a)) \mid a \in C_0(\mathbb{X}), \pi, h \in C(\mathbb{G})_\pi]$.
- For $a \in C_0(\mathbb{X})$, put

$$V_a = \{(\mathrm{id} \otimes \varphi)((1 \otimes h^*)\alpha(a)) \mid h \in C(\mathbb{G})_\pi\}.$$

Then (V_a, α) finite dimensional right $\mathscr{O}(\mathbb{G})$-comodule,

$$\alpha(V_a) \subseteq V_a \otimes C(\mathbb{G})_\pi.$$

Faithfulness of $E_{\mathbb{Y}}$ on $\mathscr{O}_{\mathbb{G}}(\mathbb{X})$

Lemma

Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$. *Then* $E_{\mathbb{Y}}$ *faithful on* $\mathscr{O}_{\mathbb{G}}(\mathbb{X})$*:*

$$\forall a \in \mathscr{O}_{\mathbb{G}}(\mathbb{X}), \quad E_{\mathbb{Y}}(a^*a) = 0 \quad \Rightarrow \quad a = 0.$$

Proof.

- Assume $a \in \mathscr{O}_{\mathbb{G}}(\mathbb{X})$, $E_{\mathbb{Y}}(a^*a) = 0$.
- For ω positive on $C_0(\mathbb{X})$:

$$0 = \omega(E_{\mathbb{Y}}(a^*a)) = \varphi((\omega \otimes \mathrm{id})\alpha(a^*a)).$$

- Since $(\omega \otimes \mathrm{id})\alpha(a^*a) \in \mathscr{O}(\mathbb{G})$ positive in $C(\mathbb{G})$,

$$(\omega \otimes \mathrm{id})\alpha(a^*a) = 0.$$

- Hence $\alpha(a^*a) = 0$, and $a = 0$.

□

An orthogonality result

Lemma

- $E_{\mathbb{Y}}(a^*b) = 0$ *if* $a \in C_0(\mathbb{X})_\pi$, $b \in C_0(\mathbb{X})_{\pi'}$, $\pi \not\cong \pi'$.
- $\mathscr{O}_{\mathbb{G}}(\mathbb{X}) = \sum_{\pi \, irrep}^{\oplus} C_0(\mathbb{X})_\pi$.

Proof.

Exercise. □

A nuisance

Question

Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$. *Is* $\mathrm{Ker}(\alpha) = \{0\}$*?*

Unfortunately, no.

Example (Sołtan)

Γ *non-amenable,* $C^*(\Gamma) \ncong C^*_{\mathrm{red}}(\Gamma)$. *Then*

$$\Delta : C^*(\Gamma) \to C^*(\Gamma) \otimes C^*_{\mathrm{red}}(\Gamma), \quad \lambda_g \mapsto \lambda_g \otimes \lambda_g$$

not injective by Fell absorption.