Vrije Universiteit Brussel

# Actions of Compact Quantum Groups III

## Reduced and universal actions

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# Outline

# Universal completions

**Proposition (H. Li)**

*Let* $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$. *Then* $\mathscr{O}_{\mathbb{G}}(\mathbb{X})$ *admits universal C*-completion* $C_0(\mathbb{X}_u)$.

# Proof

- Take $T \in \mathrm{Mor}(\pi, \alpha)$.
- Let $\{e_i\}$ o.n. basis of $\mathcal{H}_\pi$, $\delta_\pi(e_i) = \sum_j e_j \otimes u_{ji}$.
- Then

$$\begin{aligned} \alpha\Big(\sum_i T(e_i)T(e_i)^*\Big) &= \sum_{i,j,k} T(e_j)T(e_k)^* \otimes u_{ji}u_{ki}^* \\ &= \sum_j T(e_j)T(e_j)^* \otimes 1. \end{aligned}$$

- Hence $x_T = \sum_i T(e_i)T(e_i)^* \in C_0(\mathbb{X}/\mathbb{G})$, so

$$\|\lambda(T(e_i))\| \leq \|x_T\|, \quad \lambda \text{ *-representation of } \mathscr{O}_{\mathbb{G}}(\mathbb{X}).$$

- $\mathscr{O}_{\mathbb{G}}(\mathbb{X}) = \mathrm{span}\{T\xi \mid \pi, T \in \mathrm{Mor}(\pi, \alpha), \xi \in \mathcal{H}_\pi\}$, so

$$\forall a \in \mathscr{O}_{\mathbb{G}}(\mathbb{X}), \quad \|a\| = \sup\{\lambda(a) \mid \lambda \text{ *-representation}\} < \infty.$$

# Universal coaction

**Theorem (H. Li)**

*Let* $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$. *Then* $\alpha_{\mathrm{alg}}$ *extends to injective right coaction*

$$\alpha_u : C_0(\mathbb{X}_u) \to C_0(\mathbb{X}_u) \otimes C(\mathbb{G}_u).$$

*Moreover*

- $C_0(\mathbb{Y}_u) = C_0(\mathbb{Y})$,
- $\mathscr{O}_{\mathbb{G}_u}(\mathbb{X}_u) = \mathscr{O}_{\mathbb{G}}(\mathbb{X})$.

Remark: We will use the corresponding result for $\mathbb{X} = \mathbb{G}$.

# Proof (Part I)

(Cf. universal construction first lecture.)

- Existence $\alpha_u$: trivial.
- Coaction property: trivial.
- Density condition: via Hopf algebra theory (antipode)

$$\alpha(\mathscr{O}_{\mathbb{G}}(\mathbb{X}))(1 \otimes \mathscr{O}(\mathbb{G})) = \mathscr{O}_{\mathbb{G}}(\mathbb{X}) \underset{\text{alg}}{\otimes} \mathscr{O}(\mathbb{G}).$$

- Injectivity: counit extends to $C(\mathbb{G}_u)$.

# Proof (Part II)

- Let $\lambda_u : C_0(\mathbb{X}_u) \to C_0(\mathbb{X})$, $\lambda_u : C(\mathbb{G}_u) \to C(\mathbb{G})$.
- Then $(\lambda_u \otimes \lambda_u) \circ \alpha_u = \alpha \circ \lambda_u$.
- Hence: $\lambda_u(C_0(\mathbb{X}_u))_\pi = C_0(\mathbb{X})_\pi$.
- To show: $\lambda_u$ injective on each $C_0(\mathbb{X}_u)_\pi$.
- If $a_n \to b \in C_0(\mathbb{Y}_u)$ with $a_n \in \mathscr{O}_{\mathbb{G}}(\mathbb{X})$,

$$b_n = E_{\mathbb{Y}}(a_n) = (\mathrm{id} \otimes \varphi)\alpha(a_n) \to (\mathrm{id} \otimes \varphi)\alpha_u(b) = b.$$

  But $b_n \in C_0(\mathbb{Y})$ C*-algebra, so $b \in C_0(\mathbb{Y})$.
- Assume $a \in C_0(\mathbb{X}_u)_\pi, \lambda_u(a) = 0$. Then

$$0 = \alpha(\lambda_u(a^*a)) = (\lambda_u \otimes \lambda_u)\alpha_u(a^*a) \in C(\mathbb{X}_u) \underset{\mathrm{alg}}{\otimes} \mathscr{O}(\mathbb{G}).$$

- Apply $(\mathrm{id} \otimes \varphi)$,

$$\lambda_u\big(E_{\mathbb{Y}_u}(a^*a)\big) = 0 \quad \Rightarrow \quad E_{\mathbb{Y}_u}(a^*a) = 0.$$

- But $E_{\mathbb{Y}_u}$ faithful on $\mathscr{O}_{\mathbb{G}_u}(\mathbb{X}_u)$, so $a = 0$.

# Right C*-algebra valued inner products

## Definition

*Let $C_0(\mathbb{X})$ C*-algebra. Let $\Gamma(\mathbb{E})$ (unital) right $C_0(\mathbb{X})$-module.*

***Right $C_0(\mathbb{X})$-valued inner product** on $\Gamma(\mathbb{E})$:*

$$\langle \cdot , \cdot \rangle : \Gamma(\mathbb{E}) \times \Gamma(\mathbb{E}) \to C_0(\mathbb{X}), \quad (s,t) \to \langle s,t \rangle$$

*s.t.*

- $\langle \cdot , \cdot \rangle$ *linear in second, anti-linear in first argument.*
- $\langle s, ta \rangle = \langle s,t \rangle a$,
- $\langle s,t \rangle^* = \langle t,s \rangle$,
- $\langle s,s \rangle \geq 0$,
- $\langle s,s \rangle = 0 \ \Rightarrow \ s = 0$.

# Right pre-Hilbert modules

## Definition

*Right pre-Hilbert $C_0(\mathbb{X})$-module:*

- *$C^*$-algebra $C_0(\mathbb{X})$,*
- *right $C_0(\mathbb{X})$-module $\Gamma(\mathbb{E})$*
- *right $C_0(\mathbb{X})$-valued inner product on $\Gamma(\mathbb{E})$.*

## Lemma

*If $\Gamma(\mathbb{E})$ right pre-Hilbert $C_0(\mathbb{X})$-module, then norm*

$$\|s\| = \|\langle s, s\rangle\|^{1/2}, \quad s \in \Gamma(\mathbb{E}).$$

## Definition

*$\Gamma(\mathbb{E})$ right Hilbert $C_0(\mathbb{X})$-module if $\Gamma(\mathbb{E})$ complete.*

$\rightsquigarrow$ $\Gamma(\mathbb{E})$ Right pre-Hilbert $\Rightarrow$ completion $\overline{\Gamma(\mathbb{E})}$ Hilbert.

# Hilbert bundles

## Example (Classical bundles)

*$X$ compact Hausdorff space, $E \underset{\pi}{\twoheadrightarrow} X$ locally trivial Hilbert bundle:*

- *$E$ is locally compact Hausdorff space,*
- *each $E_x = \pi^{-1}(x)$ is finite dimensional Hilbert space.*
- *the map*

$$E \underset{X}{\times} E = \{(e, f) \mid \pi(e) = \pi(f)\} \to \mathbb{C}, \quad (e, f) \mapsto \langle e, f \rangle$$

  *is continuous.*
- *$E$ is locally trivial: $\pi^{-1}(U) \cong U \times \mathbb{C}^n$,*

*Then*

$$\Gamma(\mathbb{E}) = \Gamma(E) = \{continuous\ sections\ X \to E\}$$

*is Hilbert $C(X)$-module by*

$$(sf)(x) = s(x)f(x), \quad \langle s, t \rangle(x) = \langle s(x), t(x) \rangle.$$

# Example

## Example (Trivial Hilbert modules)

*Let* $C_0(\mathbb{X})$ *C*-algebra,* $I$ *set.*

$$\begin{aligned}\Gamma(\mathbb{E}) &= l^2(I, C_0(\mathbb{X})) \\ &= \{(a_i)_{i\in I} \mid \sum_i a_i^* a_i \ \textit{norm-convergent}\}\end{aligned}$$

*is Hilbert* $C_0(\mathbb{X})$*-module by*

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix} \cdot a = \begin{pmatrix} a_1 a \\ a_2 a \\ \vdots \end{pmatrix}, \quad \langle s, t \rangle = s^* t.$$

# Tensor product with Hilbert space

## Example (Tensor product with Hilbert space)

*Let*

- $C_0(\mathbb{X})$ *$C^*$-algebra,*
- $\Gamma(\mathbb{E})$ *right Hilbert* $C_0(\mathbb{X})$*-module,*
- $\mathcal{H}$ *Hilbert space.*

*Then right pre-Hilbert* $C_0(\mathbb{X})$*-module*

$$\Gamma(\mathbb{E}) \underset{\text{alg}}{\otimes} \mathcal{H}$$

*with inner product*

$$\langle s \otimes \xi, t \otimes \eta \rangle = \langle \xi, \eta \rangle \langle s, t \rangle.$$

$\Rightarrow$ *Completion* $\Gamma(\mathbb{E}) \otimes \mathcal{H}$.

When $\Gamma(\mathbb{E}) = C_0(\mathbb{X})$ and $\mathcal{H} = l^2(I)$,

$$l^2(I, C_0(\mathbb{X})) \cong C_0(\mathbb{X}) \otimes l^2(I).$$

# Hilbert modules from conditional expectations

## Example

*Let* $C_0(\mathbb{X})$ *C*-algebra. Let* $E_{\mathbb{Y}}$ *faithful conditional expectation,*

$$E_{\mathbb{Y}} : C_0(\mathbb{X}) \to C_0(\mathbb{Y}) \subseteq C_0(\mathbb{X}).$$

*Then* $C_0(\mathbb{X})$ *pre-Hilbert* $C_0(\mathbb{Y})$*-module by*

$$\langle a, b\rangle_{\mathbb{Y}} = E_{\mathbb{Y}}(a^*b).$$

Remark: For $E_{\mathbb{Y}}$ not faithful: first divide out submodule $\{a \in C_0(\mathbb{X}) \mid E_{\mathbb{Y}}(a^*a) = 0\}$.

## Notation

$L^2_{\mathbb{Y}}(\mathbb{X})$*: completed Hilbert* $C_0(\mathbb{Y})$*-module of* $(C_0(\mathbb{X}), \langle \cdot\,, \cdot\rangle_{\mathbb{Y}})$

# Adjointable maps

## Definition

*Let $\Gamma(\mathbb{E})$ and $\Gamma(\mathbb{F})$ Hilbert $C_0(\mathbb{X})$-modules.*

*Linear map $T : \Gamma(\mathbb{E}) \to \Gamma(\mathbb{F})$ **adjointable** if $\exists T^* : \Gamma(\mathbb{F}) \to \Gamma(\mathbb{E})$ s.t.*

$$\langle s, Tt \rangle = \langle T^* s, t \rangle, \qquad \forall s, t.$$

*Then*

$$\mathcal{L}(\Gamma(\mathbb{E}), \Gamma(\mathbb{F})) = \{T : \Gamma(\mathbb{E}) \to \Gamma(\mathbb{F}) \mid T \ adjointable\}.$$

# Properties of adjointable maps

## Lemma

- *Adjointable maps are bounded ($\Leftarrow$ Banach-Steinhaus).*
- $T$ *adjointable* $\Rightarrow$ $T$ *module map,* $T(\xi a) = T(\xi)a$.
- $\mathcal{L}(\Gamma(\mathbb{E}), \Gamma(\mathbb{F}))$ *is a Banach space.*
- $\mathcal{L}(\Gamma(\mathbb{E})) = \mathcal{L}(\Gamma(\mathbb{E}), \Gamma(\mathbb{E}))$ *is $C^*$-algebra.*
- $U : \Gamma(\mathbb{E}) \to \Gamma(\mathbb{F})$ *surjective linear isometry iff* $U \in \mathcal{L}(\Gamma(\mathbb{E}), \Gamma(\mathbb{F}))$ *and unitary.*

Remark: $U$ linear isometry $\not\Rightarrow U \in \mathcal{L}(\Gamma(\mathbb{E}), \Gamma(\mathbb{F}))$.

# Left Hilbert modules

## Definition

*Left pre-Hilbert $C_0(\mathbb{X})$-module:*

- *left $C_0(\mathbb{X})$-module $\Gamma(\mathbb{E})$,*
- *left $C_0(\mathbb{X})$-valued inner product on $\Gamma(\mathbb{E})$,*
    - $\langle \cdot , \cdot \rangle$ *linear in first, anti-linear in second argument.*
    - $\langle as, t \rangle = a\langle s, t \rangle$,
    - $\langle s, t \rangle^* = \langle t, s \rangle$,
    - $\langle s, s \rangle \geq 0$,
    - $\langle s, s \rangle = 0 \;\Rightarrow\; s = 0$.

# Examples

## Example

*For $E_{\mathbb{Y}}$ conditional expectation,*

$$\langle x, y\rangle_{\mathbb{Y}} = E_{\mathbb{Y}}(xy^*).$$

## Example

*Let $\Gamma(\mathbb{E})$ right Hilbert $C(\mathbb{X})$-module. Then*

$$\Gamma(\mathbb{E}^*) = \Gamma(\mathbb{E})^* = \mathcal{L}(\Gamma(\mathbb{E}), C(\mathbb{X})) = \{L_\xi^* : \eta \mapsto \langle \xi, \eta\rangle \mid \xi \in \Gamma(\mathbb{E})\}$$

*left Hilbert $C_0(\mathbb{X})$-module by*

$$(aL)(s) = a(L(s)), \quad \langle L, M\rangle = LM^*,$$

*where we use $\mathcal{L}(C(\mathbb{X})) \cong C(\mathbb{X})$ by $T \mapsto T(1_{\mathbb{X}})$.*

# An equivariance property

## Lemma

*Let* $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$ *with* $\mathbb{Y} = \mathbb{X}/\mathbb{G}$. *Then*

$$(E_{\mathbb{Y}} \otimes \mathrm{id})\alpha(a) = E_{\mathbb{Y}}(a) \otimes 1, \quad a \in C_0(\mathbb{X}).$$

## Proof.

We have

$$\begin{aligned}(E_{\mathbb{Y}} \otimes \mathrm{id})\alpha(a) &= (\mathrm{id} \otimes \varphi \otimes \mathrm{id})((\alpha \otimes \mathrm{id})\alpha(a)) \\ &= (\mathrm{id} \otimes (\varphi \otimes \mathrm{id}) \circ \Delta))(\alpha(a)) \\ &= (\mathrm{id} \otimes \varphi)\alpha(a) \otimes 1_{\mathbb{G}} \\ &= E_{\mathbb{Y}}(a) \otimes 1_{\mathbb{G}}.\end{aligned}$$

□

# The implementing unitary

## Lemma

*Let* $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$ *with* $\mathbb{Y} = \mathbb{X}/\mathbb{G}$. *Then*

$$\mathscr{O}_{\mathbb{G}}(\mathbb{X}) \underset{\mathrm{alg}}{\otimes} \mathscr{O}(\mathbb{G}) \to \mathscr{O}_{\mathbb{G}}(\mathbb{X}) \underset{\mathrm{alg}}{\otimes} \mathscr{O}(\mathbb{G}), \quad a \otimes g \mapsto \alpha(a)(1 \otimes g)$$

*completes to a unitary map*

$$U_\alpha : L^2_{\mathbb{Y}}(\mathbb{X}) \otimes L^2(\mathbb{G}) \to L^2_{\mathbb{Y}}(\mathbb{X}) \otimes L^2(\mathbb{G}).$$

## Proof.

- Isometric:

$$\begin{aligned} &\langle \alpha(a)(1 \otimes g), \alpha(b)(1 \otimes h)\rangle \\ &\quad = (E_{\mathbb{Y}} \otimes \varphi)((1_{\mathbb{X}} \otimes g^*)\alpha(a^* b)(1_{\mathbb{X}} \otimes h)) \\ &\quad = \varphi(g^* h) E_{\mathbb{Y}}(a^* b) \\ &\quad = \langle a \otimes g, b \otimes h\rangle. \end{aligned}$$

- Surjective: range dense by algebraic surjectivity.

□

# The reduced C*-algebra

## Lemma

*The non-degenerate *-homomorphisms*

$$\pi_{\mathrm{red}} : C_0(\mathbb{X}) \to \mathcal{L}(L^2_{\mathbb{Y}}(\mathbb{X})), \quad \pi_{\mathrm{red}} : C(\mathbb{G}) \to B(L^2(\mathbb{G}))$$

*by left multiplication satisfy*

$$(\pi_{\mathrm{red}} \otimes \pi_{\mathrm{red}})(\alpha(a)) = U_\alpha(\pi_{\mathrm{red}}(a) \otimes 1)U_\alpha^*.$$

*Moreover, $\pi_{\mathrm{red}}$ is injective on $\mathscr{O}_{\mathbb{G}}(\mathbb{X})$.*

## Proof.

- $\pi_{\mathrm{red}}$ well-defined and non-degenerate: basic (positivity $E_{\mathbb{Y}}$).
- $U_\alpha$ implements $\alpha$: check on $\mathscr{O}_{\mathbb{G}}(\mathbb{X})$.
- $E_{\mathbb{Y}}$ faithful on $\mathscr{O}_{\mathbb{G}}(\mathbb{X})$, so $\pi_{\mathrm{red}}$ injective on $\mathscr{O}_{\mathbb{G}}(\mathbb{X})$. □

# The reduced coaction

## Theorem (H. Li)

*Let* $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$, $C_0(\mathbb{X}_{\mathrm{red}}) = \pi_{\mathrm{red}}(C_0(\mathbb{X}))$, $C(\mathbb{G}_{\mathrm{red}}) = \pi_{\mathrm{red}}(C(\mathbb{G}))$. *Then*

$$\alpha_{\mathrm{red}} : C_0(\mathbb{X}_{\mathrm{red}}) \to C_0(\mathbb{X}_{\mathrm{red}}) \otimes C(\mathbb{G}_{\mathrm{red}}) \subseteq \mathcal{L}(L^2_{\mathbb{Y}}(\mathbb{X}) \otimes L^2(\mathbb{G})),$$

$$a \mapsto U_\alpha(\pi_{\mathrm{red}}(a) \otimes 1)U_\alpha^*$$

*defines injective right coaction* $\mathbb{X}_{\mathrm{red}} \overset{\alpha_{\mathrm{red}}}{\curvearrowleft} \mathbb{G}_{\mathrm{red}}$.

*Moreover,* $\mathscr{O}_{\mathbb{G}_{\mathrm{red}}}(\mathbb{X}_{\mathrm{red}}) = \mathscr{O}_{\mathbb{G}}(\mathbb{X})$ *and* $C_0(\mathbb{Y}_{\mathrm{red}}) = C_0(\mathbb{Y})$.

# Proof

- $U_\alpha$ implements coaction $\alpha_{\mathrm{red}}$: check on $\mathscr{O}_{\mathbb{G}}(\mathbb{X})$.
- $\alpha_{\mathrm{red}}$ injective: obvious.
- For $\lambda_{\mathrm{red}} : C_0(\mathbb{X}) \to C_0(\mathbb{X}_{\mathrm{red}})$,

$$(\lambda_{\mathrm{red}} \otimes \lambda_{\mathrm{red}}) \circ \alpha = \alpha_{\mathrm{red}} \circ \lambda_{\mathrm{red}}.$$

- $\mathscr{O}_{\mathbb{G}_{\mathrm{red}}}(\mathbb{X}_{\mathrm{red}}) = \mathscr{O}_{\mathbb{G}}(\mathbb{X})$: faithfulness $E_{\mathbb{Y}}$ on $\mathscr{O}_{\mathbb{G}}(\mathbb{X})$.
- Hence $C_0(\mathbb{Y}_{\mathrm{red}}) = C_0(\mathbb{Y})$.