Vrije Universiteit Brussel

Actions of Compact Quantum Groups IV

Coactions, smash products and crossed products

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Outline

$\mathbb{G}$-coactions

Definition

Left $\mathscr{O}(\mathbb{G})$-module *-algebra $(\mathscr{O}(\mathbb{X}), \triangleright)$:

- *-algebra $\mathscr{O}(\mathbb{X})$,
- unital $\mathscr{O}(\mathbb{G})$-module structure $\triangleright$,
- $h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$,
- $(h \triangleright a)^* = S(h)^* \triangleright a^*$.

Terminology

Left $\mathscr{O}(\mathbb{G})$-module *-algebra structure on $\mathscr{O}(\mathbb{X})$

$$\updownarrow$$

Left $\mathbb{G}$-coaction $\mathbb{X} \curvearrowleft \widehat{\mathbb{G}}$.

Examples

Example

Γ *discrete group,* $C(\widehat{\Gamma}) = C^*(\Gamma)$, $\mathscr{O}(\widehat{\Gamma}) = \mathbb{C}[\Gamma]$*:*

Left $\widehat{\Gamma}$*-coaction on* $C_0(\mathbb{X}) \leftrightarrow$ *Left* Γ*-action on* $C_0(\mathbb{X})$.

For $\mathbb{X} = X$*:*

Left $\widehat{\Gamma}$*-coaction on* $C_0(X) \leftrightarrow$ *Right* Γ*-action* $X \curvearrowleft \Gamma$.

Example (Conjugate coaction)

$\mathbb{G} \curvearrowleft \widehat{\mathbb{G}}$*:*

$$h \triangleright f = h_{(1)} f S(h_{(2)}).$$

Lemma on boundedness

Lemma

1. $$\mathscr{O}(\mathbb{G})\text{-module }^*\text{-algebra } (C_0(\mathbb{X}), \triangleright)$$
$$\Downarrow$$
$$\forall h \in \mathscr{O}(\mathbb{G}), \quad l_h : C_0(\mathbb{X}) \to C_0(\mathbb{X}),\ a \mapsto h \triangleright a \ \text{ bounded.}$$
2. $\mathscr{O}(\mathbb{X})$ *which admits universal C*-envelope* $C_0(\mathbb{X}_u)$*:*
$$\mathscr{O}(\mathbb{G})\text{-module }^*\text{-algebra } (\mathscr{O}(\mathbb{X}), \triangleright)$$
$$\Downarrow$$
$$\mathscr{O}(\mathbb{G})\text{-module }^*\text{-algebra } (C_0(\mathbb{X}_u), \triangleright).$$

Proof of boundedness lemma

1. - Take π unitary representation.
   - Identity $\mathcal{H} = \mathbb{C}^n$.
   - Let $U = U^\pi \in B(\mathbb{C}^n) \otimes \mathscr{O}(\mathbb{G}) \cong M_n(\mathscr{O}(\mathbb{G}))$.
   - Then *-homomorphism

$$\gamma_{\mathcal{U}} : C_0(\mathbb{X}) \to M_n(C_0(\mathbb{X})), \quad x \mapsto U \triangleright x.$$

   - Hence $\|U \triangleright x\| \leq \|x\|$.
   - Hence, with $U = (u_{ij})_{ij}$, also $\|u_{ij} \triangleright x\| \leq \|x\|$.
   - All u_{ij}^π generate $\mathscr{O}(\mathbb{G})$.
2. Exercise

Algebraic smash product with coaction

Definition

Assume $\mathbb{X} \curvearrowleft \widehat{\mathbb{G}}$. *Algebraic smash product,*

$$\mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}}) = \mathscr{O}(\mathbb{X}) \# \mathscr{O}(\mathbb{G}),$$

- *Vector space* $\mathscr{O}(\mathbb{X}) \underset{\mathrm{alg}}{\otimes} \mathscr{O}(\mathbb{G})$,
- *Product* $(a \otimes h)(b \otimes g) = a(h_{(1)} \triangleright b) \otimes h_{(2)} g$,
- **-structure* $(a \otimes h)^* = (h^*_{(1)} \triangleright a^*) \otimes h^*_{(2)}$.

Check associativity!

Calculations in the smash product

Notation

Write

$$ah = a \otimes h = \text{‘}(a \otimes 1)(1 \otimes h)\text{’}.$$

Write

$$ha = (h_{(1)} \triangleright a) \otimes h_{(2)} = \text{‘}(1 \otimes h)(a \otimes 1)\text{’}.$$

Lemma

For all $a \in \mathscr{O}(\mathbb{X})$, $h \in \mathscr{O}(\mathbb{G})$*:*

1. $(ah)^* = h^* a^*$
2. $ha = (h_{(1)} \triangleright a) h_{(2)}$.
3. $ah = h_{(2)}(S^{-1}(h_{(1)}) \triangleright a)$.

Proof.

Exercise (use $h_{(2)} S^{-1}(h_{(1)}) = \epsilon(h) 1_{\mathbb{G}}$). □

Universal smash product with coaction

From now on, we suppose $\mathbb{X}$ corresponds to C*-algebra $C_0(\mathbb{X})$.

Lemma

$\mathbb{X} \curvearrowleft \widehat{\mathbb{G}}$. *Then* $\mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}})$ *has universal C*-envelope* $C_0(\mathbb{X} \rtimes_u \widehat{\mathbb{G}})$.

Proof.

If π non-degenerate *-representation:

$$\|\pi(ah)\| \leq \|a\|\|h\|_u.$$

□

<u>Question</u>: $\mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}}) \subseteq C_0(\mathbb{X} \rtimes_u \widehat{\mathbb{G}})$?

Dual action

Proposition

$\mathbb{X} \curvearrowleft \widehat{\mathbb{G}}$. *Then* $\left(\mathbb{X} \rtimes \widehat{\mathbb{G}}\right) \curvearrowleft \mathbb{G}$ *by Hopf* **-algebra coaction*

$$\alpha : \mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}}) \to \mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}}) \otimes \mathscr{O}(\mathbb{G}), \quad ah \mapsto ah_{(1)} \otimes h_{(2)}.$$

Proof.

Exercise. □

Corollary

$\left(\mathbb{X} \rtimes_u \widehat{\mathbb{G}}\right) \curvearrowleft \mathbb{G}$.

A Hilbert module identification

Lemma

Put

$$E_{\mathbb{X}} : \mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}}) \to C_0(\mathbb{X}), \quad x \mapsto (\mathrm{id} \otimes \varphi)\alpha(x).$$

Then

$$\langle x, y \rangle_{\mathbb{X}} = E_{\mathbb{X}}(x^* y)$$

pre-Hilbert $C_0(\mathbb{X})$*-module structure on* $\mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}})$*, and*

$$L^2_{\mathbb{X}}(\mathbb{X} \rtimes \widehat{\mathbb{G}}) \cong L^2(\mathbb{G}) \otimes C_0(\mathbb{X}).$$

Proof

- $E_{\mathbb{X}}$ well-defined:

$$\begin{aligned}(\mathrm{id}\otimes\varphi)(\alpha(ha)) &= (\mathrm{id}\otimes\varphi)(\Delta(h)(a\otimes 1)) \\ &= \varphi(h)a.\end{aligned}$$

- For

$$V : \mathscr{O}(\mathbb{X}\rtimes\widehat{\mathbb{G}}) \to L^2(\mathbb{G})\otimes C_0(\mathbb{X}), \quad ha\mapsto h\otimes a,$$

$$\begin{aligned}\langle ha, gb\rangle_{\mathbb{X}} &= (\mathrm{id}\otimes\varphi)((a^*\otimes 1)\Delta(h^*g)(b\otimes 1)) \\ &= \varphi(h^*g)a^*b \\ &= \langle h\otimes a, g\otimes b\rangle \\ &= \langle V(ha), V(gb)\rangle.\end{aligned}$$

Reduced smash product with coaction

Theorem

*Left multiplication extends to injective *-homomorphism*

$$\lambda_{\mathrm{red}} : \mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}}) \to \mathcal{L}(L^2_{\mathbb{X}}(\mathbb{X} \rtimes \widehat{\mathbb{G}})).$$

Corollary

$\mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}}) \subseteq C_0(\mathbb{X} \rtimes_u \widehat{\mathbb{G}})$.

Proof

- - For $a \in C_0(\mathbb{X})$, $x \in \mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}})$,

    $$E_{\mathbb{X}}(x^*a^*ax) \leq \|a^*a\| E_{\mathbb{X}}(x^*x)$$

    by positivity of $E_{\mathbb{X}}$.
  - Hence $\|L_a x\|_{\mathbb{X}} \leq \|a\| \|x\|_{\mathbb{X}}$ for $L_a x = ax$.
  - $L_a^* = L_{a^*}$.
- For $g \in \mathscr{O}(\mathbb{X})$, $VL_g = (L_g \otimes 1)V$.
- - Assume $\lambda_{\mathrm{red}}(x) = 0$.
  - Then $\langle xx^*, xx^* \rangle_{\mathbb{X}} = 0$, hence $xx^* = 0$.
  - Applying $E_{\mathbb{X}}$, $\langle x^*, x^* \rangle_{\mathbb{X}} = 0$, hence $x = 0$.

The reduced crossed product

Definition

$$C_0(\mathbb{X} \rtimes_{\mathrm{red}} \widehat{\mathbb{G}}) = \left[\lambda_{\mathrm{red}}\big(\mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}})\big)\right] \subseteq \mathcal{L}(L^2_{\mathbb{X}}(\mathbb{X} \rtimes \widehat{\mathbb{G}})).$$

Remark: Under

$$V : L^2_{\mathbb{X}}(\mathbb{X} \rtimes \widehat{\mathbb{G}}) \to L^2(\mathbb{G}) \otimes C_0(\mathbb{X}), \quad ha \mapsto h \otimes a,$$

$$V\lambda_{\mathrm{red}}(ha)V^*(g \otimes b) = hg_{(2)} \otimes \Big(S^{-1}(g_{(1)}) \triangleright a\Big)b.$$

Coinvariant C*-algebra for smash product

Lemma

Assume $\mathbb{X} \curvearrowleft \widehat{\mathbb{G}}$.

- $(\mathbb{X} \rtimes_u \widehat{\mathbb{G}})/\mathbb{G} = \mathbb{X}$,
- $(\mathbb{X} \rtimes_{\mathrm{red}} \widehat{\mathbb{G}})/\mathbb{G} = \mathbb{X}$
- $\mathscr{O}_{\mathbb{G}}(\mathbb{X} \rtimes_u \widehat{\mathbb{G}}) = \mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}})$.
- $\mathscr{O}_{\mathbb{G}}(\mathbb{X} \rtimes_{\mathrm{red}} \widehat{\mathbb{G}}) = \mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}})$.

Proof.

Exercise. □

The dual algebra of a CQG

Definition

$$\mathscr{O}(\widehat{\mathbb{G}}) = \{\varphi(\,\cdot\, h) \mid h \in \mathscr{O}(\mathbb{G})\} \subseteq \mathscr{O}(\mathbb{G})^* \text{ (Vector space dual).}$$

Lemma

1. $\mathscr{O}(\widehat{\mathbb{G}})$ **-algebra for*

$$(\omega \cdot \theta)(h) = (\omega \otimes \theta)\Delta(h), \quad (\omega^*)(h) = \overline{\omega(S(h)^*)}.$$

2.

$$\mathscr{O}(\widehat{\mathbb{G}}) \cong \oplus_\pi B(\mathcal{H}_\pi), \quad \omega \mapsto \sum_\pi (\mathrm{id} \otimes \omega)\delta_\pi.$$

A *-representation of $\mathscr{O}(\widehat{\mathbb{G}})$

Lemma

Let $\mathbb{X} \underset{\alpha}{\curvearrowleft} \mathbb{G}$*. There is a non-degenerate* **-representation*

$$l : \mathscr{O}(\widehat{\mathbb{G}}) \to \mathcal{L}(L^2_{\mathbb{Y}}(\mathbb{X})), \qquad a \mapsto l_\omega(a) = (\iota \otimes \omega)\alpha(a).$$

Recall: $L^2_{\mathbb{Y}}(\mathbb{X})$ completion $\mathscr{O}_{\mathbb{G}}(\mathbb{X})$ w.r.t.

$$\|a\|_{\mathbb{Y}} = \|E_{\mathbb{Y}}(a^*a)\|^{1/2}.$$

Proof

- l_ω well-defined and bounded:

$$\begin{aligned}\|l_\omega(a)\|_{\mathbb{Y}}^2 &= \|E_{\mathbb{Y}}((\iota\otimes\omega)(\alpha(a))^*(\iota\otimes\omega)(\alpha(a)))\| \\ &\leq \|\omega\|\|E_{\mathbb{Y}}\left((\iota\otimes|\omega|)\alpha(a^*a)\right)\| \\ &= \|\omega\|^2\|E_{\mathbb{Y}}(a^*a)\| \\ &= \|\omega\|^2\|a\|_{\mathbb{Y}}^2\end{aligned}$$

- *-representation:
  - Representation: clear.
  - *-preserving:
    - in $L^2(\mathbb{G})$: $\langle g, l_\omega h\rangle = \langle l_{\omega^*}g, h\rangle$.
    - in general: use $\alpha(l_\omega a) = (\mathrm{id}\otimes l_\omega)\alpha(a)$.
- Non-degenerate: For $a\in\mathscr{O}_{\mathbb{G}}(\mathbb{X})$, $\exists\omega\in\mathscr{O}(\widehat{\mathbb{G}})$, $l_\omega a = a$.

A commutation relation

Lemma

For $a \in \mathscr{O}_{\mathbb{G}}(\mathbb{X})$ *and* $\omega \in \mathscr{O}(\widehat{\mathbb{G}})$,

$$l_\omega L_a = L_{a_{(0)}} l_{\omega(a_{(1)}\,\cdot\,)} \quad \in \mathcal{L}(L^2_{\mathbb{Y}}(\mathbb{X})).$$

Proof.

Exercise. □

Crossed product construction

Definition

Let $\mathbb{X} \underset{\alpha}{\curvearrowleft} \mathbb{G}$. *Crossed product* **-algebra:*

$$\mathscr{O}(\mathbb{X} \rtimes \mathbb{G}) = \mathscr{O}_{\mathbb{G}}(\mathbb{X}) \underset{\mathrm{alg}}{\otimes} \mathscr{O}(\widehat{\mathbb{G}})$$

with, writing $a\omega = a \otimes \omega$*,*

$$(a\omega)(b\theta) = ab_{(0)}\omega(b_{(1)} \cdot\,)\theta, \quad (a\omega)^* = \omega^* a^* = a^*_{(0)}\omega^*(a^*_{(1)} \cdot\,).$$

Check associativity!

Universal crossed product

Lemma

The universal C-envelope* $C(\mathbb{X} \rtimes_u \mathbb{G})$ *of* $\mathscr{O}(\mathbb{X} \rtimes \mathbb{G})$ *exists.*

Proof.

π non-degenerate *-representation $\Rightarrow \|\pi(\omega a)\| \leq \|\omega\|_{\mathscr{O}(\widehat{\mathbb{G}})} \|a\|_u$. □

Definition

$C_0(\mathbb{X} \rtimes_u \mathbb{G})$ *full (or universal) crossed product C*-algebra.*

Reduced crossed product

Definition

Reduced crossed product C^*-algebra $C_0(\mathbb{X} \rtimes_{\text{red}} \mathbb{G})$*:*

$$\left[L_{\alpha(a)}(1 \otimes l_\omega) \mid a \in C_0(\mathbb{X}), \omega \in \mathscr{O}(\widehat{G})\right] \subseteq \mathcal{L}(L^2_{\mathbb{Y}}(\mathbb{X}) \otimes L^2(\mathbb{G})).$$

Here $l_\omega(g) = (\text{id} \otimes \omega)\Delta(g)$ on $\mathscr{O}(\mathbb{G}) \subseteq L^2(\mathbb{G})$.

Lemma

$C_0(\mathbb{X} \rtimes_{\text{red}} \mathbb{G})$ *is C^*-algebra and* $\mathscr{O}(\mathbb{X} \rtimes \mathbb{G}) \subseteq C_0(\mathbb{X} \rtimes_{\text{red}} \mathbb{G})$ *densily.*

In fact, $C_0(\mathbb{X} \rtimes_u \mathbb{G}) \cong C_0(\mathbb{X} \rtimes_{\text{red}} \mathbb{G})$.

Proof

- Easy verification: *-homomorphism

$$\mathscr{O}(\mathbb{X} \rtimes \mathbb{G}) \to C_0(\mathbb{X} \rtimes_{\mathrm{red}} \mathbb{G}), \quad a\omega \mapsto L_{\alpha(a)}(1 \otimes l_\omega).$$

- Injective: $\sum_i L_{\alpha(a_i)}(1 \otimes l_{\omega_i}) = 0$, then

$$\forall h \in \mathscr{O}(\mathbb{G}), \quad \sum_i a_{i(0)} \otimes a_{i(1)} l_{\omega_i} h = 0,$$

hence

$$\forall h \in \mathscr{O}(\mathbb{G}), \quad \sum_i a_{i(0)} \otimes a_{i(1)} \otimes a_{i(2)} l_{\omega_i} h = 0,$$

and

$$\sum_i a_{i(0)} \otimes S(a_{i(1)}) a_{i(2)} l_{\omega_i} h = \sum_i a_i \otimes l_{\omega_i} h = 0.$$

Coaction on crossed product

Example

Let $\mathbb{X} \underset{\alpha}{\curvearrowleft} \mathbb{G}$. *Then* $\mathbb{G}$ *coacts on* $\mathbb{X} \rtimes \mathbb{G}$ *by*

$$h \triangleright (a\omega) = a\omega(\,\cdot h).$$

- Well-defined: on $\mathscr{O}(\mathbb{X} \rtimes \mathbb{G})$: OK.
- Unital module: OK.
- Preserves product: OK.
- Preserves *-structure: OK.
- Extends to $C_0(\mathbb{X} \rtimes_u \mathbb{G})$: OK.

Takesaki-Takai duality

Theorem

Let $\mathbb{X} \curvearrowleft \widehat{\mathbb{G}}$. *Then*

$$C_0(\mathbb{X} \rtimes \widehat{\mathbb{G}} \rtimes \mathbb{G}) \cong C_0(\mathbb{X}) \otimes B_0(L^2(\mathbb{G})).$$

Theorem

Let $\mathbb{X} \curvearrowleft \mathbb{G}$. *Then*

$$C_0(\mathbb{X} \rtimes \mathbb{G} \rtimes \widehat{\mathbb{G}}) \cong C_0(\mathbb{X}) \otimes B_0(L^2(\mathbb{G})).$$