Vrije Universiteit Brussel

Actions of Compact Quantum Groups V

Free and homogeneous actions I

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Outline

Free actions

Definition

$X \overset{\alpha}{\curvearrowleft} G$ *free if*

$$\forall x \in X, \quad G_x = \{g \in G \mid xg = x\} = \{e_G\}.$$

A C*-algebraic characterisation

Lemma

$X \overset{\alpha}{\curvearrowleft} G$ free iff

$$[(C_0(X) \otimes 1_G)\alpha(C_0(X))] = C_0(X) \otimes C(G).$$

Proof.

α free iff

$$\mathrm{Can} : X \times G \mapsto X \times X, \quad (x,g) \mapsto (x,xg)$$

injective iff $\mathrm{Can} : C_0(X) \otimes C_0(X) \to C_0(X) \otimes C(G)$,

$$F = f \otimes h \mapsto \mathrm{Can}(F) = (f \otimes 1_G)\alpha(h), \quad (x,g) \mapsto F(x,xg)$$

surjective. □

Freeness for compact quantum group actions

Definition (Ellwood)

Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$*. Then* α ***free*** *if*

$$[(C_0(\mathbb{X}) \otimes 1_{\mathbb{G}})\alpha(C_0(\mathbb{X}))] = C_0(\mathbb{X}) \otimes C(\mathbb{G}).$$

C*-correspondences

Definition

$C_0(\mathbb{X})$-$C_0(\mathbb{Y})$-correspondence:

- *right Hilbert $C_0(\mathbb{Y})$-module $\Gamma(\mathbb{E})$,*
- *non-degenerate *-representation*

$$\lambda : C_0(\mathbb{X}) \to \mathcal{L}(\Gamma(\mathbb{E})).$$

Interior tensor product

Definition (Interior tensor product)

Assume

- $(\Gamma(\mathbb{E}), \lambda)$ *is* $C_0(\mathbb{X})$-$C_0(\mathbb{Y})$ *correspondence.*
- $(\Gamma(\mathbb{F}), \tau)$ *is* $C_0(\mathbb{Y})$-$C_0(\mathbb{Z})$ *correspondence.*

Then $C_0(\mathbb{X})$-$C_0(\mathbb{Z})$ *correspondence* $(\Gamma(\mathbb{E}) \underset{C_0(\mathbb{Y})}{\otimes} \Gamma(\mathbb{F}), \lambda \underset{C_0(\mathbb{Y})}{\otimes} \mathrm{id})$*:*

separation-completion of

$$\Gamma(\mathbb{E}) \underset{\mathrm{alg}}{\otimes} \Gamma(\mathbb{F}), \qquad \langle s \otimes u, t \otimes v \rangle = \langle u, \tau(\langle s, t \rangle) v \rangle.$$

The Galois isometry

Lemma

Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$*. Then* $\exists$ *isometry, Galois (or canonical) isometry,*

$$\mathcal{G}_\alpha : L^2_{\mathbb{Y}}(\mathbb{X}) \underset{C_0(\mathbb{Y})}{\otimes} L^2_{\mathbb{Y}}(\mathbb{X}) \to L^2_{\mathbb{Y}}(\mathbb{X}) \otimes L^2(\mathbb{G}), \quad a \otimes b \mapsto \alpha(a)(b \otimes 1_{\mathbb{G}}).$$

Proof.

$$\begin{aligned} &\langle \alpha(c)(d \otimes 1), \alpha(a)(b \otimes 1) \rangle \\ &\quad = (E_{\mathbb{Y}} \otimes \varphi)((d^* \otimes 1)\alpha(c^* a)((b \otimes 1)) \\ &\quad = E_{\mathbb{Y}}(d^* E_{\mathbb{Y}}(c^* a) b) \\ &\quad = \langle c \otimes d, a \otimes b \rangle. \end{aligned}$$

□

What's Galois got to do with it?

Theorem (Chase-Harrison-Rosenberg)

Let

- $E \subseteq F$ *finite field extension,*
- $G = \mathrm{Aut}_E(F)$.

Then

- $H = \mathrm{Map}(G, E)$ *Hopf algebra over* E,
- *Hopf algebraic coaction*

$$\alpha : F \to F \underset{E}{\otimes} H, \quad \alpha(f)(g) = \alpha_g(f),$$

and $E \subseteq F$ *Galois if and only if the following map is bijective,*

$$F \underset{E}{\otimes} F \to F \underset{E}{\otimes} H, \quad a \otimes b \mapsto \alpha(a)(b \otimes 1).$$

Unitarity of the Galois map

Theorem (DC-Yamashita; Baum-DC-Hajac)

Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$. *The following are equivalent.*

1. *The action is free.*
2. *The Galois map is unitary.*
3. $C_0(\mathbb{X} \rtimes \mathbb{G}) \overset{\cong}{\rightarrow} \mathcal{K}(L^2_{\mathbb{Y}}(\mathbb{X}))$.

Remark: Last condition: 'saturatedness' (Rieffel)

Example: Action by compact quantum subgroup

Example

Let $\mathbb{H} \subseteq \mathbb{G}$ *compact quantum subgroup:*

$$\pi : C(\mathbb{G}) \twoheadrightarrow C(\mathbb{H}), \quad (\pi \otimes \pi) \circ \Delta_{\mathbb{G}} = \Delta_{\mathbb{H}} \circ \pi.$$

Then free action

$$\alpha = (\mathrm{id} \otimes \pi) \circ \Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{H}).$$

Proof.

Exercise. □

Example: free action on smash product

Lemma

Let $\mathbb{X} \curvearrowleft \widehat{\mathbb{G}}$. *Then* $(\mathbb{X} \rtimes \widehat{\mathbb{G}}) \curvearrowleft \mathbb{G}$ *free.*

Proof.

$$\begin{aligned}
[\alpha(C_0(\mathbb{X} \rtimes \widehat{\mathbb{G}}))(C_0(\mathbb{X} \rtimes \widehat{\mathbb{G}}) \otimes 1_{\mathbb{G}})] &\supseteq \alpha(\mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}}))(\mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}}) \otimes 1_{\mathbb{G}}) \\
&\supseteq (\mathscr{O}(\mathbb{X}) \otimes 1_{\mathbb{G}})\Delta(\mathscr{O}(\mathbb{G}))(\mathscr{O}(\mathbb{G})\mathscr{O}(\mathbb{X}) \otimes 1_{\mathbb{G}}) \\
&= \mathscr{O}(\mathbb{X})\mathscr{O}(\mathbb{G})\mathscr{O}(\mathbb{X}) \otimes \mathscr{O}(\mathbb{G}) \\
&= \mathscr{O}(\mathbb{X} \rtimes \widehat{\mathbb{G}}) \otimes \mathscr{O}(\mathbb{G}).
\end{aligned}$$

□

Corollary (Takesaki-Takai duality)

$C_0(\mathbb{X} \rtimes \widehat{\mathbb{G}} \rtimes \mathbb{G}) \cong B_0(L^2(\mathbb{G})) \otimes C_0(\mathbb{X})$ *(since* $L^2_{\mathbb{X}}(\mathbb{X} \rtimes \widehat{\mathbb{G}}) = L^2(\mathbb{G}) \otimes C_0(\mathbb{X})$*).*

Homogeneous actions

Definition

$\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$ *homogeneous (or ergodic) if*

$$\alpha(x) = x \otimes 1_{\mathbb{G}} \leftrightarrow x \in \mathbb{C}1_{\mathbb{X}}.$$

$\Rightarrow C(\mathbb{X})$ unital.

Lemma

If $X \overset{\alpha}{\curvearrowleft} G$ homogeneous, then α transitive (in the ordinary sense).

Proof.

$C(X)^G = C(X/G)$. □

Homogeneity and reduced and universal actions

Lemma

If $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$ *homogeneous, then* α_u *and* α_{red} *homogeneous.*

Proof.

$\mathbb{Y} = \mathbb{X}/\mathbb{G} = \mathbb{X}_u/\mathbb{G}_u = \mathbb{X}_{\mathrm{red}}/\mathbb{G}_{\mathrm{red}}$.

□

Associated von Neumann algebra

Definition

Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$ *homogeneous. Then* *invariant state* $\varphi_{\mathbb{X}}$ *on* $C(\mathbb{X})$*,*

$$\forall a \in C_0(\mathbb{X}), \quad \varphi_{\mathbb{X}}(a)1_{\mathbb{X}} = (\mathrm{id} \otimes \varphi)\alpha(a).$$

Lemma

Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$ *homogeneous.*

- $\varphi_{\mathbb{X}}$ *is invariant:*

$$\forall a \in C(\mathbb{X}), \quad (\varphi_{\mathbb{X}} \otimes \mathrm{id}_{\mathbb{G}})\alpha(a) = \varphi_{\mathbb{X}}(a)1_{\mathbb{G}}.$$

- *With* $L^\infty(\mathbb{X}) = C(\mathbb{X}_{\mathrm{red}})'' \subseteq B(L^2(\mathbb{X}, \varphi_{\mathbb{X}}))$*, normal coassociative* **-homomorphism*

$$\alpha_{\mathrm{vN}} : L^\infty(\mathbb{X}) \to L^\infty(\mathbb{X})\overline{\otimes}L^\infty(\mathbb{G}).$$

Actions of quotient type

Definition (Podleś)

Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$. *One calls* α *of* ***quotient type*** *if* $\exists \mathbb{H} \subseteq \mathbb{G}$ *and*

$$\theta : C(\mathbb{X}) \cong C(\mathbb{H}\backslash\mathbb{G}) = \{f \in C(\mathbb{G}) \mid (\pi \otimes \mathrm{id})\Delta(f) = 1_{\mathbb{H}} \otimes f\}$$

such that

$$(\theta \otimes \mathrm{id}) \circ \alpha = \Delta \circ \theta.$$

Remarks:

- Quotient type $\Rightarrow$ Homogeneous.
- G, X classical: Homogeneous $\Rightarrow$ Quotient type.
- In general: Homogeneous $\not\Rightarrow$ Quotient type.

Example: Standard Podleś sphere

Definition

For $q \in [-1,1] \setminus \{0\}$: C-algebra $C(SU_q(2))$ as*

$$C^*(a, b \mid U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} a & -qb^* \\ b & a^* \end{pmatrix} \text{ unitary})$$

is CQG for 'matrix comultiplication'

$$\Delta(u_{ij}) = u_{i1} \otimes u_{1j} + u_{i2} \otimes u_{2j}.$$

Lemma

$S^1 \subseteq SU_q(2)$ by

$$\begin{pmatrix} a & -qb^* \\ b & a^* \end{pmatrix} \to \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}.$$

Definition

Standard Podleś sphere $S_q^2 = S^1 \backslash SU_q(2)$.

Concrete representation standard Podleś sphere

Lemma

$C(S_q^2)$ generated by $X = ab, Z = qb^*b$ and $Y = b^*a^*$.

Lemma

$C(S_q^2)$ universal C-algebra generated by X, Y, Z s.t.*

- - $X^* = Y$,
  - $Z^* = Z$,
- - $XZ = q^2 ZX$,
  - $YZ = q^{-2} ZY$,
- - $YX = q^{-1} Z - q^{-2} Z^2$,
  - $XY = q \ Z - q^2 \ Z^2$.

<u>Remark</u>: For $q = 1$, $|X|^2 + (Z - \frac{1}{2})^2 = \frac{1}{4}$.

Embeddable actions

Definition (Podleś)

Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$*. One calls* α *embeddable if* $\exists$ *faithful* **-homomorphism*

$$\theta : C(\mathbb{X}) \hookrightarrow C(\mathbb{G})$$

such that

$$(\theta \otimes \mathrm{id}) \circ \alpha = \Delta \circ \theta.$$

Remark:

- Embeddable $\Rightarrow$ Homogeneous.
- Quotient type $\Rightarrow$ Embeddable.
- Embeddable $\not\Rightarrow$ Quotient type (e.g. non-standard Podleś spheres)
- Homogeneous $\not\Rightarrow$ Embeddable.

Non-embeddable actions

Example

Let π irreducible left $\mathbb{G}$-representation.

$$\mathrm{Ad}_\pi : B(\mathcal{H}_\pi) \to B(\mathcal{H}_\pi) \otimes C(\mathbb{G}), \quad \xi\eta^* \to \delta_\pi(\xi)\delta_\pi(\eta)^*$$

homogeneous, but not embeddable for $\dim(\mathcal{H}_\pi) \geq 2$

Homogeneous actions with classical point

Lemma

$\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$ *homogeneous. TFAE:*

1. α_{red} *is embeddable.*
2. $C(\mathbb{X}_u)$ *has a character.*

Proof

$1. \Rightarrow 2.$ $\theta : C(\mathbb{X}_{\mathrm{red}}) \to C(\mathbb{G}_{\mathrm{red}})$ equivariant, so

$$\theta_{\mathrm{alg}} : \mathscr{O}_{\mathbb{G}}(\mathbb{X}) \to \mathscr{O}(\mathbb{G}) \quad \Rightarrow \quad \theta_u : C(\mathbb{X}_u) \to C(\mathbb{G}_u).$$

Then $\epsilon \circ \theta_u$ character.

$2. \Rightarrow 1.$

- If χ character, then equivariant *-homomorphism

$$\theta_u : C(\mathbb{X}_u) \to C(\mathbb{G}_u), \quad a \mapsto (\chi \otimes \mathrm{id}) \circ \alpha_u.$$

- Hence equivariant *-homomorphism

$$\theta_{\mathrm{alg}} : \mathscr{O}_{\mathbb{G}}(\mathbb{X}) \to \mathscr{O}(\mathbb{G}).$$

- Then

$$\varphi(\theta_{\mathrm{alg}}(a)^* \theta_{\mathrm{alg}}(a)) = \chi(E_{\mathbb{Y}}(a^* a)).$$

  But, by homogeneity, $E_{\mathbb{Y}}$ values in $\mathbb{C}1_{\mathbb{X}}$, so

$$E_{\mathbb{Y}}(a^* a) = \chi(E_{\mathbb{Y}}(a^* a))1_{\mathbb{X}}.$$

- Hence $\theta_r : C(\mathbb{X}_{\mathrm{red}}) \hookrightarrow C(\mathbb{G}_{\mathrm{red}})$ since $E_{\mathbb{Y}}$ faithful on $C(\mathbb{X}_{\mathrm{red}})$.

Boca's theorems

Theorem (Boca)

If $\mathbb{X} \underset{\alpha}{\curvearrowleft} \mathbb{G}$ *homogeneous, then all* $C(\mathbb{X})_\pi$ *finite dimensional.*

In fact, Boca gives concrete estimate in terms of 'quantum multiplicity'. Combined with Takesaki-Takai duality:

Theorem (Boca)

$\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$ *homogeneous* $\Rightarrow \exists$ *set* I *and Hilbert spaces* $\mathcal{H}_i$,

$$C_0(\mathbb{X} \rtimes \mathbb{G}) \cong \oplus_{i\in I} B_0(\mathcal{H}_i)$$