Vrije Universiteit Brussel

Actions of Compact Quantum Groups VI

Free and homogeneous actions II

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Outline

Homogeneous and free: Galois objects

From homogeneous to free and back

Homogeneous actions of $SU_q(2)$

Galois objects

Definition

$\mathbb{X} \underset{\alpha}{\curvearrowleft} \mathbb{G}$ *Galois object* (or *quantum torsor*) if

1. α *free,*
2. α *homogeneous,*
3. $C(\mathbb{X}) \neq \{0\}$.

Lemma

If $X \underset{\alpha}{\curvearrowleft} G$ *Galois object, then* $X \cong G$ *equivariantly.*

No longer true in quantum case!

Quantum torus

Example (Quantum torus)

Let $\theta \in [0, 2\pi]$. Put

$$C(\mathbb{T}^2_\theta) = C^*(U, V \mid U, V \textit{ unitary}, UV = e^{i\theta} VU.\}.$$

Then free and homogeneous $\mathbb{T}^2_\theta \curvearrowleft \mathbb{T}^2$ by

$$\alpha_{(w,z)}(U) = wU, \quad \alpha_{(w,z)} V = zV.$$

Remarks:

- Check that $C(\mathbb{T}^2_q)$ not trivial.
- Instance of general construction: 2-cocycles on discrete quantum groups.

Twisting procedure

Theorem (Bichon-De Rijdt-Vaes)

1. *There is a one-to-one-correspondence between (classes of)*
   - *Galois objects* $\mathbb{X}$ *for* $\mathbb{G}$*,*
   - *Fiber functors* F *on* $\mathrm{Rep}(\mathbb{G})$ *(into Hilbert spaces).*
2. *Let* $\mathbb{X} \underset{\alpha}{\curvearrowleft} \mathbb{G}$ *Galois object. Then* $\exists!\mathbb{H}$ *such that*
   - $\mathbb{H} \underset{\beta}{\curvearrowright} \mathbb{X}$ *is (left) Galois object,*
   - α *and* β *commute.*

Remark:

- Abstractly: $\mathbb{H}$ from Tannaka-Krein on F.
- Concretely: $C(\mathbb{H}_{\mathrm{red}}) \subseteq C(\mathbb{X}_{\mathrm{red}}) \otimes C(\mathbb{X}_{\mathrm{red}})^{\mathrm{op}}$.
- One says $\mathbb{G}$ and $\mathbb{H}$ monoidally equivalent.

Another look at quantum $SU(2)$ and $O^+(n)$

Definition

Take $F \in GL_n(\mathbb{C})$ *with* $F\bar{F} \in \mathbb{R}$. *Then*

$$C(O_u^+(F)) = C^*(u_{ij} \mid 1 \leq i,j \leq n,\, U \text{ unitary}, F\overline{U}F^{-1} = U)$$

becomes compact quantum group for

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

Example

1. *For* $F = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix}$, $C(O_u^+(F)) = C(SU_q(2))$.
2. *For* $F = I_n$, $C(O_u^+(I_n)) = C(O_n^+)$.

Classification of all Galois objects of $SU_q(2)$

Notation

For $F \in GL_n(\mathbb{C})$ with $F\bar{F} \in \mathbb{R}$, write

$$c_F = -\text{sign}(F\bar{F})Tr(F^*F).$$

Remark: Always $|c_F| \geq 2$.

Theorem (Bichon-De Rijdt-Vaes)

- $\{O^+(F)\}$ *is complete w.r.t. monoidal equivalence.*
- $O^+(F_1) \underset{\textit{mon. eq.}}{\cong} O^+(F_2)$ *iff* $c_{F_1} = c_{F_2}$.
- $O^+(F) \underset{\textit{mon. eq.}}{\cong} SU_q(2)$ *for* $q + q^{-1} = c_F$.

In fact, Galois object between $O^+(F_1)$ and $O^+(F_2)$

$$C(O_u^+(F_1, F_2)) = C^*\left(u_{ij} \mid \begin{array}{l} 1 \leq i \leq \text{dim}(F_1), 1 \leq j \leq \text{dim}(F_2) \\ U \textit{ unitary}, F_1\overline{U}F_2^{-1} = U \end{array}\right).$$

Morita base change

Lemma

Let $\mathbb{X} \curvearrowleft \mathbb{G}$ free, $\mathbb{Y} = \mathbb{X}/\mathbb{G}$. Assume $p \in M(C_0(\mathbb{Y}))$ full projection:

$$[C_0(\mathbb{Y})pC_0(\mathbb{Y})] = C_0(\mathbb{Y}).$$

Then, with $C_0(\mathbb{X}_p) = pC_0(\mathbb{X})p$, free action $\mathbb{X}_p \curvearrowleft \mathbb{G}$ by

$$\alpha_p : C_0(\mathbb{X}_p) \to C_0(\mathbb{X}_p) \otimes C(\mathbb{G}), \quad a \mapsto \alpha(a).$$

Moreover, with $C_0(\mathbb{Y}_p) = pC_0(\mathbb{Y})p$,

$$\mathbb{X}_p/\mathbb{G} = \mathbb{Y}_p.$$

Remarks:

- $C_0(\mathbb{Y}_p)$ and $C_0(\mathbb{Y})$ (strongly) Morita equivalent ('non-commutative isomorphism $\mathbb{Y}_p \cong \mathbb{Y}$').
- Then also $C_0(\mathbb{X}_p)$ and $C_0(\mathbb{X})$ Morita equivalent.

Proof

- Well-defined coaction: clear.
- Free: using $C_0(\mathbb{X}) = [C_0(\mathbb{Y})C_0(\mathbb{X})C_0(\mathbb{Y})]$,

$$
\begin{aligned}
&[\alpha_p(C_0(\mathbb{X}_p))(C_0(\mathbb{X}_p)\otimes 1)] \\
&\quad = \quad [(p\otimes 1)\alpha(C_0(\mathbb{X}))(pC_0(\mathbb{X})p\otimes 1)] \\
&\quad = \quad [(p\otimes 1)\alpha(C_0(\mathbb{X}))(C_0(\mathbb{Y})pC_0(\mathbb{Y})C_0(\mathbb{X})p\otimes 1)] \\
&\quad = \quad [(p\otimes 1)\alpha(C_0(\mathbb{X}))(C_0(\mathbb{Y})C_0(\mathbb{X})p\otimes 1)] \\
&\quad = \quad [(p\otimes 1)\alpha(C_0(\mathbb{X}))(C_0(\mathbb{X})p\otimes 1)] \\
&\quad = \quad [pC_0(\mathbb{X})p\otimes C(\mathbb{G})] \\
&\quad = \quad C_0(\mathbb{X}_p)\otimes C(\mathbb{G}).
\end{aligned}
$$

- $X_p/\mathbb{G} = \mathbb{Y}_p$ as exercise.

Reduction to free actions

<u>Recall</u>: $\mathbb{X} \curvearrowleft \mathbb{G}$ homogeneous, then $C_0(\mathbb{X} \rtimes \mathbb{G}) \cong \oplus_{i\in I} B_0(\mathcal{H}_i)$.

Corollary (Wassermann construction)

Consider $e_{00}^{(i)}$ fixed matrix unit in $B_0(\mathcal{H}_i)$, and put

$$p = \oplus e_{00}^{(i)} \in M(\oplus_{i\in I} B_0(\mathcal{H}_i)).$$

- *p full projection for $\oplus_{i\in I} B_0(\mathcal{H}_i)$.*
- *$\mathbb{X}_{\text{free}} \curvearrowleft \mathbb{G}$ free with $C_0(\mathbb{X}_{\text{free}}) = pC_0(X \rtimes \mathbb{G} \rtimes \widehat{\mathbb{G}})p$.*
- *$C_0(\mathbb{X}_{\text{free}}/\mathbb{G}) = c_0(I)$.*

Proof.

Use $(\mathbb{X} \rtimes \mathbb{G} \rtimes \widehat{\mathbb{G}})/\mathbb{G} = \mathbb{X} \rtimes \mathbb{G}$. □

From free to homogeneous and back

Lemma

$\mathbb{X} \curvearrowleft \mathbb{G}$ with $C_0(\mathbb{X}/\mathbb{G}) = c_0(I)$, and $C(\mathbb{X}_i) = \delta_i C_0(\mathbb{X})\delta_i$. Then $\mathbb{X}_i \curvearrowleft \mathbb{G}$ homogeneous.

Lemma

Let $\mathbb{X} \curvearrowleft \mathbb{G}$ homogeneous.

- *$C_0(\mathbb{X} \rtimes \mathbb{G}) \twoheadrightarrow B(L^2_{\mathbb{Y}}(\mathbb{X})) \Rightarrow$ distinguished block $B_0(\mathcal{H}_{i_0}) \subseteq C_0(\mathbb{X} \rtimes \mathbb{G})$.*
- *Associated projection $\delta_{i_0} \in c_0(I) \subseteq C_0(\mathbb{X}_{\text{free}})$ is full.*

$\Rightarrow C_0(\mathbb{X}_{\text{free}}) \underset{\text{Morita}}{\cong} C_0(\mathbb{X})$.

Theorem

$\mathbb{G}$ CQG. The above gives one-to-one correspondence between

- *(Irreducible) free actions $\mathbb{X}' \curvearrowleft \mathbb{G}$ with $\mathbb{X}'/\mathbb{G}$ classical discrete set (up to iso)*
- *Homogeneous actions $\mathbb{X} \curvearrowleft \mathbb{G}$ (up to 'equivariant Morita equivalence').*

$\Rightarrow$ classifying homogeneous actions = classifying certain free actions.

Free actions and fiber functors

Definition

I a set. Monoidal category $({}_I\mathrm{Hilb}_I, \boxtimes)$:

- *Objects: I-bigraded Hilbert spaces, $\mathcal{H} = \bigoplus_{k,l} {}_k\mathcal{H}_l$,*
- *Tensor product:*
$$ {}_k(\mathcal{H} \boxtimes \mathcal{G})_l = \bigoplus_m {}_k\mathcal{H}_m \otimes {}_m\mathcal{G}_l. $$

Theorem (DC-Yamashita)

There is a one-to-one correspondence between

1. *Free actions $\mathbb{X} \curvearrowleft \mathbb{G}$ with $\mathbb{X}/\mathbb{G}$ classical discrete set I (up to isomorphism)*
2. *Tensor C^*-functors $\mathrm{Rep}_{\mathrm{fd}}(\mathbb{G}) \to {}_I\mathrm{Hilb}_I$ (up to 'equivalence').*

$\rightsquigarrow$ Concrete Tannaka-Krein reconstruction process.

Reduction scheme

Classifying homogeneous actions of $SU_q(2)$.

$\Updownarrow$

Classifying free actions of $SU_q(2)$ with discrete quotient space

$\Updownarrow$

Classifying Monoidal C*-functors $\mathrm{Rep}_{\mathrm{fd}}(SU_q(2)) \to {}_I\mathrm{Hilb}_I$.

But... $\mathrm{Rep}_{\mathrm{fd}}(SU_q(2))$ easy generators and relations...

Classifying Monoidal C*-functors $\mathrm{Rep}_{\mathrm{fd}}(SU_q(2)) \to {}_I\mathrm{Hilb}_I$.

$\Updownarrow$

Combinatorial data.

<u>Remark</u>:

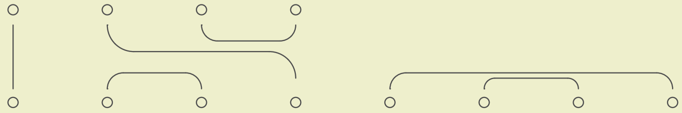
$\exists q, \mathrm{Rep}(O^+(F)) = \mathrm{Rep}(SU_q(2))$, so: classification homogeneous $\mathbb{X} \curvearrowleft O^+(F)$.

Representation category of $SU_q(2)$

Lemma

$\mathrm{Rep}_{\mathrm{fd}}(SU_q(2))$ *is 'completion' of tensor C*-category with*

- *Objects: finite ordinals*
- *Basis for morphisms: non-crossing 2-partitions*



and

- *Tensor product: horizontal juxtaposition*
- *Composition: vertical stacking with rule* $\bigcirc = -q - q^{-1}$.
- **-structure:* $\cap^* = -\mathrm{sgn}(q)\cup$.

Reciprocal random walks

Definition (DC-Yamashita)

Let $\delta \in \mathbb{R}_0$.

A δ-reciprocal random walk consists of a quadruple $(\Gamma, w, \mathrm{sgn}, i)$ where

- $\Gamma = (\Gamma^{(0)}, \Gamma^{(1)}, s, t)$ is a graph with source and target maps s and t,
- w is a weight function $w\colon \Gamma^{(1)} \to \mathbb{R}_0^+$,
- sgn a sign function $\mathrm{sgn}\colon \Gamma^{(1)} \to \{\pm 1\}$,
- i is an involution $e \mapsto \overline{e}$ on $\Gamma^{(1)}$ interchanging source and target,

s.t.

- for all e, $w(e)w(\bar{e}) = 1$,
- for all e, $\mathrm{sgn}(e)\mathrm{sgn}(\bar{e}) = \mathrm{sgn}(\delta)$,
- for all v, $\sum_{s(e)=v} \frac{1}{|\delta|} w(e) = 1$.

Examples

- Action $SU_q(2)$ on non-standard Podleś sphere $S^2_{q,x}$

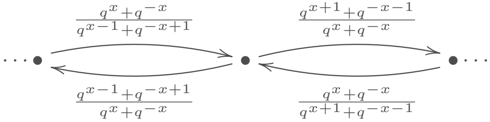


Figure: $\delta = -(q + q^{-1})$ $(q > 0, x \in \mathbb{R})$

- Action O_n^+ on S_+^{N-1}

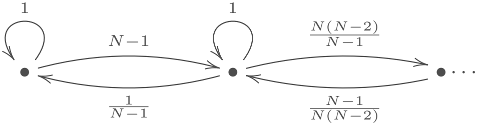


Figure: $\delta = N$

Abundance of reciprocal random walks

Lemma (DC-Yamashita)

Let $(\Gamma, w, \mathrm{sgn}, i)$ *δ-reciprocal random walk. Then* Γ *bounded degree:*

$$\sup_{v \in \Gamma^{(0)}} \#\{e \in \Gamma^{(1)} \mid s(e) = v\} < \infty.$$

Theorem (DC-Yamashita)

Γ *bounded degree* $\Rightarrow \exists \delta$ *and* δ*-reciprocal random walk on* Γ*.*

'Proof'.

By Frobenius-Perron theory. □

Theorem (Kronecker)

ADE-classification for 2*-reciprocal random walks.*

A one-to-one correspondence

Theorem (DC-Yamashita)

Fix $q \neq 0$, put $\delta_q = -q - q^{-1}$. There is (up to appropriate equivalence) a one-to-one correspondence between

- *Tensor C^*-functors $F : \mathrm{Rep} \to {}_I\mathrm{Hilb}_I$, and*
- *δ_q-reciprocal random walks $\Gamma = (\Gamma^{(0)}, \Gamma^{(1)}, s, t)$ with $\Gamma^{(0)} = I$.*

Construction of F from Γ:

- $F(\ ) = l^2(\Gamma^{(0)})$,
- $F(\bullet) = l^2(\Gamma^{(1)})$,
- $F(\cap)(\delta_v) = \sum_{s(e)=v} \mathrm{sgn}(e) w(e)^{1/2} \delta_e \otimes \delta_{\bar{e}}$.

Wassermann's theorem

Lemma

Let $H \subseteq G$ compact Hausdorff groups. Let π irreducible H-representation. Write

$$C(G, B(\mathcal{H}_\pi))^H = \{f \in C(G, B(\mathcal{H}_\pi)) \mid f(hg) = \pi(h)f(g)\pi(h)^*\}.$$

Then homogeneous action

$$G \overset{\alpha}{\curvearrowright} C(G, B(\mathcal{H}_\pi))^H, \quad \alpha_g(f)(g') = f(g'g).$$

Theorem (Wassermann)

$\mathbb{X} \curvearrowleft SU(2)$ homogeneous, then $\exists H \subseteq SU(2)$ and irreducible H-representation s.t.

$$C(\mathbb{X}) \underset{G\text{-equivariantly}}{\cong} C(G, B(\mathcal{H}_\pi))^H.$$

Further questions

Project

$\mathbb{X} \curvearrowleft SU_q(2)$ homogeneous.

1. *Analyse $C(\mathbb{X})$ and $L^\infty(\mathbb{X})$ in terms of their weighted graph.*
2. *Classify all $\mathbb{X}$ with $C(\mathbb{X})$ or $L^\infty(\mathbb{X})$ type I/simplicial/factorial.*