# Compact Quantum Groups and Free Combinatorics 

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## Section 1

## The Origin of Freeness: Free Group Factors

## Let us Look on Moments

Free (non-commutative) probability theory investigates
operators on Hilbert spaces
by looking at
moments of those operators

Many methods and concepts for understanding those moments are inspired by analogues from

## classical probability theory

## Some Basic Notations

## Definition

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, i.e.,

- $\mathcal{A}$ is a unital algebra
- $\varphi: \mathcal{A} \rightarrow \mathbb{C}$ is a unital linear functional, i.e. $\varphi(1)=1$

Consider (non-commutative) random variables $a_{1}, \ldots, a_{n} \in \mathcal{A}$. Expressions of the form

$$
\varphi\left(a_{i(1)} \cdots a_{i(k)}\right) \quad(k \in \mathbb{N}, 1 \leq i(1), \ldots, i(k) \leq n)
$$

are called moments of $a_{1}, \ldots, a_{n}$.

## Moments of Generators Determine vN-Algebra

Let $\mathcal{A}, \mathcal{B}$ be two von Neumann algebras with

- $\mathcal{A}=\mathrm{vN}\left(a_{1}, \ldots, a_{n}\right), \quad$ and $\quad \mathcal{B}=\mathrm{vN}\left(b_{1}, \ldots, b_{n}\right)$ with selfadjoint generators $a_{i}$ and $b_{i}$
- $\varphi: \mathcal{A} \rightarrow \mathbb{C}$ and $\psi: \mathcal{B} \rightarrow \mathbb{C}$ are faithful and normal states
- for all $k \in \mathbb{N}$ and $1 \leq i(1), \ldots, i(k) \leq n$ :

$$
\varphi\left(a_{i(1)} \cdots a_{i(k)}\right)=\psi\left(b_{i(1)} \cdots b_{i(k)}\right)
$$

Then

$$
\mathcal{A} \cong \mathcal{B} \quad \text { via } \quad a_{i} \mapsto b_{i} \quad(i=1, \ldots, n)
$$

## Consequence: Moments Can be Uselful

- All questions on operators, which depend only on the generated operator algebra, ...
... like: spectrum, polar decomposition, existence of hyperinvariant subspaces, inequalities for $L^{p}$-norms...
... can in principle be answered by the knowldege of the moments of the operators with respect to a faithful normal state
- This insight is in general not very helpful, since moments are usually quite complicated
- However, in many special (and interesting) situations moments have a special structure

This is the realm of free probability theory

## Measure Theory Versus Probability Theory

Difference between measure theory and classical probability theory is essentially given by notion of

## independence

Difference between von Neumann algebra theory and free probability theory is essentially given by notion of

## freeness or free independence

Freeness describes special structure of moments arising from group von Neumann algebras $L(G)$, if $G$ is the free product of subgroups

## Group von Neumann Algebra $L(G)$

## Definition

Let $G$ be a discrete group. The corresponding group von Neumann algebra is

$$
L(G):={\overline{\mathbb{C}} \bar{G}^{\text {STOP }}}^{\text {TOP }}
$$

$\uparrow$
representation of the group algebra acting on the group by left multiplication

If $G$ is i.c.c. (all non-trivial conjugacy classes are infinite), then $L(G)$ is a $\mathrm{II}_{1}$ factor.
In particular, the neutral element $e$ of $G$ induces a trace $\tau$ on $L(G)$, which is faithful and normal, via

$$
\tau(a):=\langle a e, e\rangle
$$

## Hyperfinite and Free Group Factors

$$
G \text { amenable } \quad \Longrightarrow \quad \begin{gathered}
L(G) \\
\text { hyperfinite } \|_{1} \text {-Faktor }
\end{gathered}
$$

$$
\begin{gathered}
G=\mathbb{F}_{n} \\
\text { free group on } \\
n \text { generators }
\end{gathered} \quad \Longrightarrow \quad \begin{gathered}
L\left(\mathbb{F}_{n}\right) \text { is } \\
\text { not hyperfinite } \\
\text { (Murray/von Neumann) }
\end{gathered}
$$

Voiculescu's philosophy: The free group factors $L\left(\mathbb{F}_{n}\right)$ are the next interesting class of von Neumann algebras after the hyperfinite one

## The Structure of the Free Group Factors

Free probability theory was created

- in order to understand $L\left(\mathbb{F}_{n}\right)$ and similar von Neumann algebras;
- in particular, to attack the most famous (and still open!!!) problem in this context:
(Isomorphism problem of the free group factors:)
Is it true or false that

$$
L\left(\mathbb{F}_{n}\right) \cong L\left(\mathbb{F}_{m}\right) \quad \text { for } n \neq m(n, m \geq 2)
$$

## Transfering Freeness from $G$ to $L(G)$

$$
\begin{gathered}
G=G_{1} * G_{2} \\
\downarrow \\
\mathbb{C} G=\mathbb{C} G_{1} * \mathbb{C} G_{2} \\
\downarrow ? \\
L(G)=L\left(G_{1}\right) * L\left(G_{2}\right)
\end{gathered}
$$

free product of groups
free product
of algebras
erc

## Algebraic Freeness of Subgroups

$G_{1}, G_{2}$ are free in $G$ (as subgroups) means:

$$
\left.\begin{array}{c}
g_{i} \in G_{j(i)} \\
g_{i} \neq e \quad \forall i \\
j(1) \neq j(2) \neq \cdots \neq j(k)
\end{array}\right\} \Longrightarrow g_{1} \cdots g_{k} \neq e
$$

This algebraic formulation can be extented to finite sums (as in $\mathbb{C} G$ ), but not to infinite sums (as in $L(G)$ ).

## Reformulation in Terms of the Trace

We can reformulate the freeness of the subgroups also in terms of $\tau$ :

$$
\left.\begin{array}{c}
g_{i} \in G_{j(i)} \\
\tau\left(g_{i}\right)=0 \quad \forall i \\
j(1) \neq \cdots \neq j(k)
\end{array}\right\} \Longrightarrow \tau\left(g_{1} \cdots g_{k}\right)=0
$$

This characterisation goes over to finite as well as to infinite sums (note that $\tau$ is normal).

This motivated Voiculescu to make the following definition.

## The Fundamental Notion: Freeness

## Definition (Voiculescu 1985)

Let $\mathcal{A}$ be a unital algebra and $\varphi: \mathcal{A} \rightarrow \mathbb{C}$ a unital linear functional. Subalgebras $\mathcal{A}_{1}, \ldots, \mathcal{A}_{n} \subset \mathcal{A}$ are free (w.r.t. $\varphi$ ), if:

$$
\left.\begin{array}{c}
a_{i} \in \mathcal{A}_{j(i)} \\
\varphi\left(a_{i}\right)=0 \quad \forall i \\
j(1) \neq \cdots \neq j(k)
\end{array}\right\} \Longrightarrow \varphi\left(a_{1} \cdots a_{k}\right)=0
$$

- Freeness is a special structure of the mixed moments in elements from $\mathcal{A}_{1}, \ldots, \mathcal{A}_{n}$.
- This structure should be seen and investigated in analogy to the classical concept of "independence".


## Section 2

## Freeness

## Some History



1985 Voiculescu introduces "freeness" in the context of isomorphism problem of free group factors
1991 Voiculescu discovers relation with random matrices (which leads, among others, to deep results on free group factors)
1994 Speicher develops combinatorial theory of freeness, based on "free cumulants"
later ... many new results on operator algebras, eigenvalue distribution of random matrices, and much more ....

## Definition of Freeness

## Definition

- Let $(\mathcal{A}, \varphi)$ be non-commutative probability space, i.e., $\mathcal{A}$ is a unital algebra and $\varphi: \mathcal{A} \rightarrow \mathbb{C}$ is unital linear functional (i.e., $\varphi(1)=1$ )
- Unital subalgebras $\mathcal{A}_{i}(i \in I)$ are free or freely independent, if $\varphi\left(a_{1} \cdots a_{n}\right)=0$ whenever

$$
\begin{aligned}
& a_{i} \in \mathcal{A}_{j(i)}, \quad j(i) \in I \quad \forall i, \\
& j(1) \neq j(2) \neq \cdots \neq j(n) \\
& \varphi\left(a_{i}\right)=0 \quad \forall i
\end{aligned}
$$

- Random variables $x_{1}, \ldots, x_{n} \in \mathcal{A}$ are free, if their generated unital subalgebras $\mathcal{A}_{i}:=\operatorname{algebra}\left(1, x_{i}\right)$ are so.


## What is Freeness?

Freeness between $x$ and $y$ is an infinite set of equations relating various moments in $x$ and $y$ :

$$
\varphi\left(p_{1}(x) q_{1}(y) p_{2}(x) q_{2}(y) \cdots\right)=0
$$

Basic observation: freeness between $x$ and $y$ is actually a rule for calculating mixed moments in $x$ and $y$ from the moments of $x$ and the moments of $y$ :

$$
\varphi\left(x^{m_{1}} y^{n_{1}} x^{m_{2}} y^{n_{2}} \cdots\right)=\operatorname{polynomial}\left(\varphi\left(x^{i}\right), \varphi\left(y^{j}\right)\right)
$$

## Example

If $x$ and $y$ are free, then we have

$$
\varphi\left(x^{m} y^{n}\right)=\varphi\left(x^{m}\right) \cdot \varphi\left(y^{n}\right)
$$

## Example

$$
\varphi\left(\left(x^{m}-\varphi\left(x^{m}\right) 1\right)\left(y^{n}-\varphi\left(y^{n}\right) 1\right)\right)=0
$$

thus

$$
\begin{gathered}
\varphi\left(x^{m} y^{n}\right)-\varphi\left(x^{m} \cdot 1\right) \varphi\left(y^{n}\right)-\varphi\left(x^{m}\right) \varphi\left(1 \cdot y^{n}\right)+\varphi\left(x^{m}\right) \varphi\left(y^{n}\right) \varphi(1 \cdot 1)=0 \\
\text { and hence } \quad \varphi\left(x^{m} \boldsymbol{y}^{n}\right)=\varphi\left(x^{m}\right) \cdot \varphi\left(\boldsymbol{y}^{n}\right)
\end{gathered}
$$

## Example

$$
\varphi((x-\varphi(x) 1) \cdot(y-\varphi(y) 1) \cdot(x-\varphi(x) 1) \cdot(y-\varphi(y) 1))=0
$$

which results in

$$
\begin{aligned}
\varphi(x y x y)=\varphi(x x) \cdot \varphi(y) \cdot \varphi(y)+ & \varphi(x) \cdot \varphi(x) \cdot \varphi(y y) \\
& -\varphi(x) \cdot \varphi(y) \cdot \varphi(x) \cdot \varphi(y)
\end{aligned}
$$

## Example

If $x$ and $y$ are free, then we have

$$
\begin{aligned}
\varphi\left(x^{m} y^{n}\right) & =\varphi\left(x^{m}\right) \cdot \varphi\left(y^{n}\right) \\
\varphi\left(x^{m_{1}} y^{n} x^{m_{2}}\right) & =\varphi\left(x^{m_{1}+m_{2}}\right) \cdot \varphi\left(y^{n}\right)
\end{aligned}
$$

but also

$$
\varphi(x y x y)=\varphi\left(x^{2}\right) \cdot \varphi(y)^{2}+\varphi(x)^{2} \cdot \varphi\left(y^{2}\right)-\varphi(x)^{2} \cdot \varphi(y)^{2}
$$

Freeness is a rule for calculating mixed moments, analogous to the concept of independence for random variables. This is the reason that it is also called "free independence".

## Example

If $x$ and $y$ are free, then we have

$$
\begin{aligned}
\varphi\left(x^{m} y^{n}\right) & =\varphi\left(x^{m}\right) \cdot \varphi\left(y^{n}\right) \\
\varphi\left(x^{m_{1}} y^{n} x^{m_{2}}\right) & =\varphi\left(x^{m_{1}+m_{2}}\right) \cdot \varphi\left(y^{n}\right)
\end{aligned}
$$

but also

$$
\varphi(x y x y)=\varphi\left(x^{2}\right) \cdot \varphi(y)^{2}+\varphi(x)^{2} \cdot \varphi\left(y^{2}\right)-\varphi(x)^{2} \cdot \varphi(y)^{2}
$$

Free independence is a rule for calculating mixed moments, analogous to the concept of independence for random variables.
Note: free independence is a different rule from classical independence; free independence occurs typically for non-commuting random variables, like operators on Hilbert spaces or (random) matrices.

## Where Does Freeness Show Up?

- generators of the free group in the corresponding free group von Neumann algebras $L\left(\mathbb{F}_{n}\right)$
- creation and annihilation operators on full Fock spaces
- for many classes of random matrices


## Section 3

## The Emergence of the Combinatorics of Freeness

## Motivation for the Combinatorics of Freeness: the Free (and Classical) CLT

Consider $a_{1}, a_{2}, \cdots \in(\mathcal{A}, \varphi)$ which are

- identically distributed
- centered and normalized: $\varphi\left(a_{i}\right)=0$ and $\varphi\left(a_{i}^{2}\right)=1$
- either classically independent or freely independent

What can we say about

$$
S_{n}:=\frac{a_{1}+\cdots+a_{n}}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} ? ? ?
$$

## Definition

We say that $S_{n}$ converges (in distribution) to $s$ if

$$
\lim _{n \rightarrow \infty} \varphi\left(S_{n}^{m}\right)=\varphi\left(s^{m}\right) \quad \forall m \in \mathbb{N}
$$

## Calculation of Moments of $S_{n}$

We have

$$
\begin{aligned}
\varphi\left(S_{n}^{m}\right) & =\frac{1}{n^{m / 2}} \varphi\left[\left(a_{1}+\cdots a_{n}\right)^{m}\right] \\
& =\frac{1}{n^{m / 2}} \sum_{i(1), \ldots, i(m)=1}^{n} \varphi\left[a_{i(1)} \cdots a_{i(m)}\right]
\end{aligned}
$$

## Basic Observation

Note:

$$
\varphi\left[a_{i(1)} \cdots a_{i(m)}\right]=\varphi\left[a_{j(1)} \cdots a_{j(m)}\right]
$$

whenever

$$
\operatorname{ker} i=\operatorname{ker} j
$$

## Example

For example, $i=(1,3,1,5,3)$ and $j=(3,4,3,6,4)$ :

$$
\varphi\left[a_{1} a_{3} a_{1} a_{5} a_{3}\right]=\varphi\left[a_{3} a_{4} a_{3} a_{6} a_{4}\right]
$$

because independence/freeness allows to express

$$
\begin{aligned}
& \varphi\left[a_{1} a_{3} a_{1} a_{5} a_{3}\right]=\text { polynomial }\left(\varphi\left(a_{1}\right), \varphi\left(a_{1}^{2}\right), \varphi\left(a_{3}\right), \varphi\left(a_{3}^{2}\right), \varphi\left(a_{5}\right)\right) \\
& \varphi\left[a_{3} a_{4} a_{3} a_{6} a_{4}\right]=\text { polynomial }\left(\varphi\left(a_{3}\right), \varphi\left(a_{3}^{2}\right), \varphi\left(a_{4}\right), \varphi\left(a_{4}^{2}\right), \varphi\left(a_{6}\right)\right) \\
& \text { and } \quad \varphi\left(a_{1}\right)=\varphi\left(a_{3}\right), \quad \varphi\left(a_{1}^{2}\right)=\varphi\left(a_{3}^{2}\right) \\
& \varphi\left(a_{3}\right)=\varphi\left(a_{4}\right), \quad \varphi\left(a_{3}^{2}\right)=\varphi\left(a_{4}^{2}\right), \quad \varphi\left(a_{5}\right)=\varphi\left(a_{6}\right)
\end{aligned}
$$

We put

$$
\kappa_{\pi}:=\varphi\left[a_{1} a_{3} a_{1} a_{5} a_{3}\right] \quad \text { where } \quad \pi:=\operatorname{ker} i=\operatorname{ker} j=\{\{1,3\},\{2,5\},\{4\}\}
$$

$\pi \in \mathcal{P}(5)$ is a partition of $\{1,2,3,4,5\}$.

## Calculation of Moments of $S_{n}$

## Thus

$$
\begin{aligned}
\varphi\left(S_{n}^{m}\right) & =\frac{1}{n^{m / 2}} \sum_{i(1), \ldots, i(m)=1}^{n} \varphi\left[a_{i(1)} \cdots a_{i(m)}\right] \\
& =\frac{1}{n^{m / 2}} \sum_{\pi \in \mathcal{P}(m)} \kappa_{\pi} \cdot \#\{i: \operatorname{ker} i=\pi\}
\end{aligned}
$$

Note:

$$
\#\{i: \operatorname{ker} i=\pi\}=n(n-1) \cdots(n-\# \pi-1) \sim n^{\# \pi}
$$

So

$$
\varphi\left(S_{n}^{m}\right) \sim \sum_{\pi \in \mathcal{P}(m)} \kappa_{\pi} \cdot n^{\# \pi-m / 2}
$$

## No Singletons in the Limit

Consider $\pi \in \mathcal{P}(m)$ with singleton:

$$
\pi=\{\ldots,\{k\}, \ldots\}
$$

thus

$$
\begin{aligned}
\kappa_{\pi} & =\varphi\left(a_{i(1)} \cdots a_{i(k)} \cdots a_{i(m)}\right) \\
& =\varphi\left(a_{i(1)} \cdots a_{i(k-1)} a_{i(k+1)} \cdots a_{i(m)}\right) \cdot \underbrace{\varphi\left(a_{i(k)}\right)}_{=0} \\
& =0
\end{aligned}
$$

We used: If $\{x, y\}$ and $a$ are free/independent, then:

$$
\varphi(x a y)=\varphi(x y) \varphi(a)
$$

## No Singletons in the Limit

Consider $\pi \in \mathcal{P}(m)$ with singleton:

$$
\pi=\{\ldots,\{k\}, \ldots\}
$$

thus

$$
\begin{aligned}
\kappa_{\pi} & =\varphi\left(a_{i(1)} \cdots a_{i(k)} \cdots a_{i(m)}\right) \\
& =\varphi\left(a_{i(1)} \cdots a_{i(k-1)} a_{i(k+1)} \cdots a_{i(m)}\right) \cdot \underbrace{\varphi\left(a_{i(k)}\right)}_{=0} \\
& =0
\end{aligned}
$$

Thus: $\kappa_{\pi}=0$ if $\pi$ has singleton.

## Only Pairings Survive in the Limit

So we have

$$
\begin{aligned}
\kappa_{\pi} \neq 0 & \Longrightarrow \quad \pi=\left\{V_{1}, \ldots, V_{r}\right\} \text { with } \# V_{j} \geq 2 \forall j \\
& \Longrightarrow \quad r=\# \pi \leq \frac{m}{2}
\end{aligned}
$$

So in

$$
\varphi\left(S_{n}^{m}\right) \sim \sum_{\pi \in \mathcal{P}(m)} \kappa_{\pi} \cdot n^{\# \pi-m / 2}
$$

only those $\pi$ survive for $n \rightarrow \infty$ with

- $\pi$ has no singleton, i.e., no block of size 1
- $\pi$ has exactly $m / 2$ blocks

Such $\pi$ are exactly those, where each block has size 2, i.e.,

$$
\pi \in \mathcal{P}_{2}(m):=\{\pi \in \mathcal{P}(m) \mid \pi \text { is pairing }\}
$$

## Limit Moments are Given by Summation over Pairings

Thus we have:

$$
\lim _{n \rightarrow \infty} \varphi\left(S_{n}^{m}\right)=\sum_{\pi \in \mathcal{P}_{2}(m)} \kappa_{\pi}
$$

- This gives in particular: odd moments are zero (because no pairings of odd number of elements), thus limit distribution is symmetric
- What are the even moments?

This depends on the $\kappa_{\pi}$ 's.
The actual value of those is now different for the classical and the free case!

## Classical CLT: Assume $a_{i}$ are Independent

If the $a_{i}$ commute and are independent, then

$$
\kappa_{\pi}=\varphi\left(a_{i(1)} \cdots a_{i(2 k)}\right)=1 \quad \forall \pi \in \mathcal{P}_{2}(2 k)
$$

## Example

$$
\varphi\left(a_{1} a_{2} a_{3} a_{3} a_{2} a_{1}\right)=1=\varphi\left(a_{1} a_{2} a_{2} a_{3} a_{1} a_{3}\right)
$$

Thus

$$
\lim _{n \rightarrow \infty} \varphi\left(S_{n}^{m}\right)=\# \mathcal{P}_{2}(m)= \begin{cases}0, & m \text { odd } \\ (m-1)(m-3) \cdots 5 \cdot 3 \cdot 1, & m \text { even }\end{cases}
$$

Those limit moments are the moments of a Gaussian distribution of variance 1.

## Free CLT: Assume $a_{i}$ are Free

If the $a_{i}$ are free, then, for $\pi \in \mathcal{P}_{2}(2 k)$,

$$
\kappa_{\pi}= \begin{cases}0, & \pi \text { is crossing } \\ 1, & \pi \text { is non-crossing }\end{cases}
$$

## Example

- non-crossing $\pi$

$$
\begin{aligned}
\kappa_{\{1,6\},\{2,5\},\{3,4\}}=\varphi\left(a_{1} a_{2} a_{3} a_{3} a_{2} a_{1}\right) & =\varphi\left(a_{3} a_{3}\right) \cdot \varphi\left(a_{1} a_{2} a_{2} a_{1}\right) \\
& =\varphi\left(a_{3} a_{3}\right) \cdot \varphi\left(a_{2} a_{2}\right) \cdot \varphi\left(a_{1} a_{1}\right) \\
& =1
\end{aligned}
$$

- crossing $\pi$

$$
\kappa_{\{1,5\},\{2,3\},\{4,6\}\}}=\varphi\left(a_{1} a_{2} a_{2} a_{3} a_{1} a_{3}\right)=\varphi\left(a_{2} a_{2}\right) \cdot \underbrace{\varphi\left(a_{1} a_{3} a_{1} a_{3}\right)}_{0}=0
$$

## Free CLT: Assume $a_{i}$ are Free

Notation
Put

$$
N C_{2}(m):=\left\{\pi \in \mathcal{P}_{2}(m) \mid \pi \text { is non-crossing }\right\}
$$

Thus

$$
\lim _{n \rightarrow \infty} \varphi\left(S_{n}^{m}\right)=\# N C_{2}(m)= \begin{cases}0, & m \text { odd } \\ c_{k}=\frac{1}{k+1}\binom{2 k}{k}, & m=2 k \text { even }\end{cases}
$$

Those limit moments are the moments of a semicircular distribution of variance 1,

$$
\lim _{n \rightarrow \infty} \varphi\left(S_{n}^{m}\right)=\frac{1}{2 \pi} \int_{-2}^{2} t^{m} \sqrt{4-t^{2}} d t
$$

## How to Recognize the Catalan Numbers $c_{k}$

Notation
Put

$$
c_{k}:=\# N C_{2}(2 k)
$$

Basic Observation
We have

$$
c_{k}=\sum_{\pi \in N C(2 k)} 1=\sum_{i=1}^{k} \sum_{\pi=\{1,2 i\} \cup \pi_{1} \cup \pi_{2}} 1=\sum_{i=1}^{k} c_{i-1} c_{k-i}
$$

This recursion, together with $c_{0}=1, c_{1}=1$, determines the sequence of Catalan numbers:

$$
\left\{c_{k}\right\}=1,1,2,5,14,42,132,429, \ldots
$$

## Intermezzo: <br> One Slide on Random Matrices

## Convergence of Eigenvalue Distribution of Gaussian Random Matrices to Semicircle



## Section 4

## Free Cumulants

## Understanding the Freeness Rule: the Idea of Cumulants

- write moments in terms of other quantities, which we call free cumulants
- freeness is much easier to describe on the level of free cumulants: vanishing of mixed cumulants
- relation between moments and cumulants is given by summing over non-crossing or planar partitions


## Non-Crossing Partitions

## Definition

A partition of $\{1, \ldots, n\}$ is a decomposition $\pi=\left\{V_{1}, \ldots, V_{r}\right\}$ with

$$
V_{i} \neq \emptyset, \quad V_{i} \cap V_{j}=\emptyset \quad(i \neq y), \quad \bigcup_{i} V_{i}=\{1, \ldots, n\}
$$

The $V_{i}$ are the blocks of $\pi \in \mathcal{P}(n)$. $\pi$ is non-crossing if we do not have

$$
p_{1}<q_{1}<p_{2}<q_{2}
$$

such that $p_{1}, p_{2}$ are in same block, $q_{1}, q_{2}$ are in same block, but those two blocks are different.

$$
\mathbf{N C}(\mathbf{n}):=\{\text { non-crossing partitions of }\{1, \ldots, n\}\}
$$

$N C(n)$ is actually a lattice with refinement order.

## Moments and Cumulants

## Definition

For unital linear functional

$$
\varphi: \mathcal{A} \rightarrow \mathbb{C}
$$

we define cumulant functionals $\kappa_{n}$ (for all $n \geq 1$ )

$$
\kappa_{n}: \mathcal{A}^{n} \rightarrow \mathbb{C}
$$

as multi-linear functionals by moment-cumulant relation

$$
\varphi\left(a_{1} \cdots a_{n}\right)=\sum_{\pi \in N C(n)} \kappa_{\pi}\left[a_{1}, \ldots, a_{n}\right]
$$

Note: classical cumulants are defined by a similar formula, where only $N C(n)$ is replaced by $\mathcal{P}(n)$

## Example ( $n=1$ )

$$
\varphi\left(a_{1}\right)=\kappa_{1}\left(a_{1}\right)
$$



## Example ( $n=2$ )

$$
\begin{aligned}
\varphi\left(a_{1} a_{2}\right)= & \kappa_{2}\left(a_{1}, a_{2}\right) \\
& +\kappa_{1}\left(a_{1}\right) \kappa_{1}\left(a_{2}\right)
\end{aligned}
$$

and thus

$$
\kappa_{2}\left(a_{1}, a_{2}\right)=\varphi\left(a_{1} a_{2}\right)-\varphi\left(a_{1}\right) \varphi\left(a_{2}\right)
$$

## Example ( $n=3$ )

$$
\begin{aligned}
\varphi\left(a_{1} a_{2} a_{3}\right)= & \kappa_{3}\left(a_{1}, a_{2}, a_{3}\right) \\
& +\kappa_{1}\left(a_{1}\right) \kappa_{2}\left(a_{2}, a_{3}\right) \\
& +\kappa_{2}\left(a_{1}, a_{2}\right) \kappa_{1}\left(a_{3}\right) \\
& +\kappa_{2}\left(a_{1}, a_{3}\right) \kappa_{1}\left(a_{2}\right) \\
& +\kappa_{1}\left(a_{1}\right) \kappa_{1}\left(a_{2}\right) \kappa_{1}\left(a_{3}\right)
\end{aligned}
$$


and thus

$$
\begin{aligned}
& \kappa_{3}\left(a_{1}, a_{2}, a_{3}\right)=\varphi\left(a_{1} a_{2} a_{3}\right)-\varphi\left(a_{1}\right) \varphi\left(a_{2} a_{3}\right)-\varphi\left(a_{2}\right) \varphi\left(a_{1} a_{3}\right) \\
&-\varphi\left(a_{3}\right) \varphi\left(a_{1} a_{2}\right)+2 \varphi\left(a_{1}\right) \varphi\left(a_{2}\right) \varphi\left(a_{3}\right)
\end{aligned}
$$

## Example $(n=4)$

$$
\begin{aligned}
& \varphi\left(a_{1} a_{2} a_{3} a_{4}\right)=\quad 山+\| \amalg+\amalg^{1} \downarrow+山^{1}+山 \mid \\
& +\sqcup \sqcup+\sqcup \sqcup+\|\Delta+|\sqcup|+\sqcup\| \\
& +|\square+\|!+\| 1|+||| | \\
& =\kappa_{4}\left(a_{1}, a_{2}, a_{3}, a_{4}\right)+\kappa_{1}\left(a_{1}\right) \kappa_{3}\left(a_{2}, a_{3}, a_{4}\right) \\
& +\kappa_{1}\left(a_{2}\right) \kappa_{3}\left(a_{1}, a_{3}, a_{4}\right)+\kappa_{1}\left(a_{3}\right) \kappa_{3}\left(a_{1}, a_{2}, a_{4}\right) \\
& +\kappa_{3}\left(a_{1}, a_{2}, a_{3}\right) \kappa_{1}\left(a_{4}\right)+\kappa_{2}\left(a_{1}, a_{2}\right) \kappa_{2}\left(a_{3}, a_{4}\right) \\
& +\kappa_{2}\left(a_{1}, a_{4}\right) \kappa_{2}\left(a_{2}, a_{3}\right)+\kappa_{1}\left(a_{1}\right) \kappa_{1}\left(a_{2}\right) \kappa_{2}\left(a_{3}, a_{4}\right) \\
& +\kappa_{1}\left(a_{1}\right) \kappa_{2}\left(a_{2}, a_{3}\right) \kappa_{1}\left(a_{4}\right)+\kappa_{2}\left(a_{1}, a_{2}\right) \kappa_{1}\left(a_{3}\right) \kappa_{1}\left(a_{4}\right) \\
& +\kappa_{1}\left(a_{1}\right) \kappa_{2}\left(a_{2}, a_{4}\right) \kappa_{1}\left(a_{3}\right)+\kappa_{2}\left(a_{1}, a_{4}\right) \kappa_{1}\left(a_{2}\right) \kappa_{1}\left(a_{3}\right) \\
& +\kappa_{2}\left(a_{1}, a_{3}\right) \kappa_{1}\left(a_{2}\right) \kappa_{1}\left(a_{4}\right)+\kappa_{1}\left(a_{1}\right) \kappa_{1}\left(a_{2}\right) \kappa_{1}\left(a_{3}\right) \kappa_{1}\left(a_{4}\right)
\end{aligned}
$$

## Freeness $\hat{=}$ Vanishing of Mixed Cumulants

Theorem (Speicher 1994)
The fact that $x_{1}, \ldots, x_{m}$ are free is equivalent to the fact that

$$
\kappa_{n}\left(x_{i(1)}, \ldots, x_{i(n)}\right)=0
$$

whenever

- $1 \leq i(1), \ldots, i(n) \leq m$
- there are $p, q$ such that $i(p) \neq i(q)$ (in particular, $n \geq 2$ )


## Example

If $x$ and $y$ are free then: $\varphi(x y x y)=$

$$
\kappa_{1}(x) \kappa_{1}(x) \kappa_{2}(y, y)+\kappa_{2}(x, x) \kappa_{1}(y) \kappa_{1}(y)+\kappa_{1}(x) \kappa_{1}(y) \kappa_{1}(x) \kappa_{1}(y)
$$



## Factorization of Non-Crossing Moments

## Example

The iteration of the rule

$$
\varphi(a x b)=\varphi(a b) \varphi(x) \quad \text { if }\{a, b\} \text { and } x \text { free }
$$

leads to the factorization of all "non-crossing" moments in free variables

$$
\begin{gathered}
x_{1} \quad x_{2} \quad x_{3} \quad x_{3} \quad x_{2} \quad x_{4} \quad x_{5} \quad x_{5} \quad x_{2} \quad x_{1} \\
\varphi\left(x_{1} x_{2} x_{3} x_{3} x_{2} x_{4} x_{5} x_{5} x_{2} x_{1}\right) \\
=\varphi\left(x_{1} x_{1}\right) \cdot \varphi\left(x_{2} x_{2} x_{2}\right) \cdot \varphi\left(x_{3} x_{3}\right) \cdot \varphi\left(x_{4}\right) \cdot \varphi\left(x_{5} x_{5}\right)
\end{gathered}
$$

## Section 5

## Operator-Valued Extension of Free Probability

## Definition

Let $\mathcal{B} \subset \mathcal{A}$. A linear map $E: \mathcal{A} \rightarrow \mathcal{B}$ is a conditional expectation if

$$
E[b]=b \quad \forall b \in \mathcal{B}
$$

and

$$
E\left[b_{1} a b_{2}\right]=b_{1} E[a] b_{2} \quad \forall a \in \mathcal{A}, \quad \forall b_{1}, b_{2} \in \mathcal{B}
$$

An operator-valued probability space consists of $\mathcal{B} \subset \mathcal{A}$ and a conditional expectation $E: \mathcal{A} \rightarrow \mathcal{B}$

## Example (Classical conditional expectation)

Let $\mathfrak{M}$ be a $\sigma$-algebra and $\mathfrak{N} \subset \mathfrak{M}$ be a sub- $\sigma$-algebra. Then

- $\mathcal{A}=L^{\infty}(\Omega, \mathfrak{M}, P)$
- $\mathcal{B}=L^{\infty}(\Omega, \mathfrak{N}, P)$
- $E[\cdot \mid \mathfrak{N}]$ is the classical conditional expectation from the bigger onto the smaller $\sigma$-algebra.


## Operator-Valued Freeness

## Definition

Consider an operator-valued probability space $(\mathcal{A}, E: \mathcal{A} \rightarrow \mathcal{B})$. The operator-valued distribution of $x \in \mathcal{A}$ is given by all operator-valued moments

$$
E\left[x b_{1} x b_{2} \cdots b_{n-1} x\right] \in \mathcal{B} \quad\left(n \in \mathbb{N}, b_{1}, \ldots, b_{n-1} \in \mathcal{B}\right)
$$

Random variables $x_{i} \in \mathcal{A}(i \in I)$ are free with respect to $E$ (or free with amalgamation over $\mathcal{B}$ ) if

$$
E\left[a_{1} \cdots a_{n}\right]=0
$$

whenever

- $a_{i} \in \mathcal{B}\left\langle x_{j(i)}\right\rangle$ are polynomials in some $x_{j(i)}$ with coefficients from $\mathcal{B}$
- $j(1) \neq j(2) \neq \cdots \neq j(n)$
- $E\left[a_{i}\right]=0$ for all $i$


## Operator-Valued Freeness: NC Moments

Note: random variables $x$ and scalars $b$ from $\mathcal{B}$ do not commute in general!

## Example

Still one has factorizations of all non-crossing moments in free variables.

| $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{3}$ | $x_{2}$ | $x_{4}$ | $x_{5}$ | $x_{5}$ | $x_{2}$ | $x_{1}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |

$$
\begin{aligned}
& E\left[x_{1} x_{2} x_{3} x_{3} x_{2} x_{4} x_{5} x_{5} x_{2} x_{1}\right] \\
& \quad=E\left[x_{1} \cdot E\left[x_{2} \cdot E\left[x_{3} x_{3}\right] \cdot x_{2} \cdot E\left[x_{4}\right] \cdot E\left[x_{5} x_{5}\right] \cdot x_{2}\right] \cdot x_{1}\right]
\end{aligned}
$$

## Operator-Valued Freeness: NC Moments

Operator-valued freeness works mostly like ordinary freeness, one only has to take care of the order of the variables; in all expressions they have to appear in their original order!

## Example

Still one has factorizations of all non-crossing moments in free variables.


$$
\begin{aligned}
& E\left[x_{1} x_{2} x_{3} x_{3} x_{2} x_{4} x_{5} x_{5} x_{2} x_{1}\right] \\
& \qquad=E\left[x_{1} \cdot E\left[x_{2} \cdot E\left[x_{3} x_{3}\right] \cdot x_{2} \cdot E\left[x_{4}\right] \cdot E\left[x_{5} x_{5}\right] \cdot x_{2}\right] \cdot x_{1}\right]
\end{aligned}
$$

## Operator-Valued Freeness: Crossing Moments

For "crossing" moments one has analogous formulas as in scalar-valued case, modulo respecting the order of the variables ...

## Example

The formula for free $x_{1}$ and $x_{2}$

$$
\begin{aligned}
& \varphi\left(x_{1} x_{2} x_{1} x_{2}\right)=\varphi\left(x_{1} x_{1}\right) \varphi\left(x_{2}\right) \varphi\left(x_{2}\right)+\varphi\left(x_{1}\right) \varphi\left(x_{1}\right) \varphi\left(x_{2} x_{2}\right) \\
&-\varphi\left(x_{1}\right) \varphi\left(x_{2}\right) \varphi\left(x_{1}\right) \varphi\left(x_{2}\right)
\end{aligned}
$$

has now to be written as

$$
\begin{aligned}
E\left[x_{1} x_{2} x_{1} x_{2}\right]=E\left[x_{1} E\left[x_{2}\right] x_{1}\right] \cdot E\left[x_{2}\right]+E\left[x_{1}\right] & \cdot E\left[x_{2} E\left[x_{1}\right] x_{2}\right] \\
& -E\left[x_{1}\right] E\left[x_{2}\right] E\left[x_{1}\right] E\left[x_{2}\right]
\end{aligned}
$$

## Operator-Valued Free Cumulants

## Definition

Consider an operator-valued probability space $E: \mathcal{A} \rightarrow \mathcal{B}$.
We define operator-valued free cumulants

$$
\kappa_{n}^{\mathcal{B}}: \mathcal{A}^{n} \rightarrow \mathcal{B}
$$

by

$$
E\left[a_{1} \cdots a_{n}\right]=\sum_{\pi \in N C(n)} \kappa_{\pi}^{\mathcal{B}}\left[a_{1}, \ldots, a_{n}\right]
$$

- arguments of $\kappa_{\pi}^{\mathcal{B}}$ are distributed according to blocks of $\pi$
- but now: cumulants are nested inside each other according to nesting of blocks of $\pi$


## Operator-Valued Free Cumulants

## Example

$$
\pi=\{\{1,10\},\{2,5,9\},\{3,4\},\{6\},\{7,8\}\} \in N C(10)
$$



$$
\begin{aligned}
\kappa_{\pi}^{\mathcal{B}}\left[a_{1}, \ldots,\right. & \left.a_{10}\right] \\
& =\kappa_{2}^{\mathcal{B}}\left(a_{1} \cdot \kappa_{3}^{\mathcal{B}}\left(a_{2} \cdot \kappa_{2}^{\mathcal{B}}\left(a_{3}, a_{4}\right), a_{5} \cdot \kappa_{1}^{\mathcal{B}}\left(a_{6}\right) \cdot \kappa_{2}^{\mathcal{B}}\left(a_{7}, a_{8}\right), a_{9}\right), a_{10}\right)
\end{aligned}
$$

## Vanishing of Mixed Cumulants Characterizes Freeness

## Definition

We define operator-valued free cumulants $\kappa_{n}^{\mathcal{B}}: \mathcal{A}^{n} \rightarrow \mathcal{B}$ by

$$
E\left[a_{1} \cdots a_{n}\right]=\sum_{\pi \in N C(n)} \kappa_{\pi}^{\mathcal{B}}\left[a_{1}, \ldots, a_{n}\right]
$$

As in the scalar-valued case the following are equivalent:

- $x_{1}, \ldots, x_{m}$ are free over $\mathcal{B}$
- for all $n, 1 \leq i(1), \ldots, i(n) \leq m$ with $i(p) \neq i(q)$ for some $p, q$, and all $b_{1}, \ldots, b_{n-1} \in \mathcal{B}$ we have

$$
\kappa_{n}^{\mathcal{B}}\left(x_{i(1)} b_{1}, x_{i(2)} b_{2}, \ldots, x_{i(n-1)} b_{n-1}, x_{i(n)}\right)=0
$$

## Section 6

## Non-Commutative de Finetti Theorem, Quantum Permutation Group and Non-Crossing Partitions

## Classical Exchangeable Random Variables

Consider probability space $(\Omega, \mathfrak{A}, P)$. Denote expectation by $\varphi$,

$$
\varphi(Y)=\int_{\Omega} Y(\omega) d P(\omega)
$$

## Definition

We say that random variables $X_{1}, X_{2}, \ldots$ are exchangeable if their joint distribution is invariant under finite permutations, i.e. if

$$
\varphi\left(X_{i(1)} \cdots X_{i(n)}\right)=\varphi\left(X_{\pi(i(1))} \cdots X_{\pi(i(n))}\right)
$$

for all $n \in \mathbb{N}$, all $i(1), \ldots, i(n) \in \mathbb{N}$, and all permutations $\pi$

## Example

$$
\varphi\left(X_{1}^{n}\right)=\varphi\left(X_{7}^{n}\right), \quad \varphi\left(X_{1}^{3} X_{3}^{7} X_{4}\right)=\varphi\left(X_{8}^{3} X_{2}^{7} X_{9}\right)
$$

## Tail $\sigma$-Algebra

## Example

- Independent and identically distributed random variables are exchangeable.
- Note that the $X_{i}$ might all contain some common component; e.g., if all $X_{i}$ are the same, then clearly $X, X, X, X, X \ldots$ is exchangeable.

Theorem of de Finetti says that an infinite sequence of exchangeable random variables is independent modulo its common part.

## Tail $\sigma$-Algebra

## Example

- Independent and identically distributed random variables are exchangeable.
- Note that the $X_{i}$ might all contain some common component; e.g., if all $X_{i}$ are the same, then clearly $X, X, X, X, X \ldots$ is exchangeable.

Formalize common part via tail $\sigma$-algebra of the sequence $X_{1}, X_{2}, \ldots$

$$
\mathfrak{A}_{\text {tail }}:=\bigcap_{i \in \mathbb{N}} \sigma\left(X_{k} \mid k \geq i\right)
$$

Denote by $E$ the conditional expectation onto this tail $\sigma$-algebra

$$
E: L^{\infty}(\Omega, \mathfrak{A}, P) \rightarrow L^{\infty}\left(\Omega, \mathfrak{A}_{\text {tail }}, P\right)
$$

## Classical de Finetti Theorem

## Definition

$$
\begin{gathered}
\mathfrak{A}_{\text {tail }}:=\bigcap_{i \in \mathbb{N}} \sigma\left(X_{k} \mid k \geq i\right) \\
E: L^{\infty}(\Omega, \mathfrak{A}, P) \rightarrow L^{\infty}\left(\Omega, \mathfrak{A}_{\text {tail }}, P\right)
\end{gathered}
$$

Theorem (de Finetti 1931, Hewitt, Savage 1955)
The following are equivalent for an infinite sequence of random variables:

- the sequence is exchangeable
- the sequence is independent and identically distributed with respect to the conditional expectation $E$ onto the tail $\sigma$-algebra of the sequence

$$
E\left[X_{1}^{m(1)} X_{2}^{m(2)} \cdots X_{n}^{m(n)}\right]=E\left[X_{1}^{m(1)}\right] \cdot E\left[X_{2}^{m(2)}\right] \cdots E\left[X_{n}^{m(n)}\right]
$$

## Non-commutative Random Variables

## Replace now

random variables $\rightarrow$ operators on Hilbert spaces
expectation $\rightarrow$ state on the algebra generated by those operators

## Setting

In the following our setting will be a non-commutative $W^{*}$-probability space $(\mathcal{A}, \varphi)$, i.e.,

- $\mathcal{A}$ is von Neumann algebra (i.e., weakly closed subalgebra of bounded operators on Hilbert space)
- $\varphi: \mathcal{A} \rightarrow \mathbb{C}$ is faithful state on $\mathcal{A}$, i.e.,

$$
\begin{gathered}
\varphi\left(a a^{*}\right) \geq 0, \quad \text { for all } a \in \mathcal{A} \\
\varphi\left(a a^{*}\right)=0 \quad \text { if and only if } a=0
\end{gathered}
$$

## Exchangeable NC Random Variables

## Definition

Non-commutative random variables $x_{1}, x_{2}, \cdots \in \mathcal{A}$ are exchangeable if

$$
\varphi\left(x_{i(1)} \cdots x_{i(n)}\right)=\varphi\left(x_{\pi(i(1))} \cdots x_{\pi(i(n))}\right)
$$

for all $n \in \mathbb{N}$, all $i(1), \ldots, i(n) \in \mathbb{N}$, and all permutations $\pi$.

## Question

Does exchangeability imply anything like independence in this general non-commutative setting?

## Answer

Only partially. Exchangeability gives, by work of Koestler, some weak form of independence (special factorization properties), but does not fully determine all mixed moments ... there are too many possibilities out in the non-commutative world, and exchangeability is a too weak condition!

## However ...

Invariance under permutations is in a sense also a commutative concept ... ... and should be replaced by a non-commutative counterpart in the non-commutative world!

## permutation group $\longrightarrow$ quantum permutation group

Recall: classical permutation group

$$
S_{k} \hat{=}\{k \times k \text { permutation matrices }\}
$$

Dualize

$$
C\left(S_{k}\right)=\left\{f: S_{k} \rightarrow \mathbb{C} ; g \mapsto\left(\left(u_{i j}(g)\right)_{i, j=1}^{k}\right\}\right.
$$

## Classical Permutation Group $S_{k}$

Then $C\left(S_{k}\right)$ is the universal commutative $C^{*}$-algebra generated by $u_{i j}$ $(i, j=1, \ldots, k)$, subject to the relations

$$
u_{i j}^{*}=u_{i j}=u_{i j}^{2} \quad \forall i, j, \quad \sum_{j} u_{i j}=1=\sum_{j} u_{j i} \quad \forall i
$$

$\operatorname{alg}\left(u_{i j} \mid i, j=1, \ldots, k\right)$ is a Hopf algebra (which is dense in $\left.C\left(S_{k}\right)\right)$ with

$$
\begin{array}{rlr}
\Delta u_{i j}=\sum_{k} u_{i k} \otimes u_{k j} & \text { coproduct } \\
\varepsilon\left(u_{i j}\right)=\delta_{i j} & \text { co-unit } \\
S\left(u_{i j}\right)=u_{j i} & \text { antipode }
\end{array}
$$

## Quantum Permutation Group

## Definition (Wang 1998)

The quantum permutation group $A_{s}(k)$ is given by the universal unital $C^{*}$-algebra generated by $u_{i j}(i, j=1, \ldots, k)$ subject to the relations

- $u_{i j}^{2}=u_{i j}=u_{i j}^{*}$ for all $i, j=1, \ldots, k$
- each row and column of $u=\left(u_{i j}\right)_{i, j=1}^{k}$ is a partition of unity:

$$
\sum_{j=1}^{k} u_{i j}=1 \quad \forall i \quad \text { and } \quad \sum_{i=1}^{k} u_{i j}=1 \quad \forall j
$$

(note: elements within a row or within a column are orthogonal)
$A_{s}(k)$ is a compact quantum group in the sense of Woronowicz.

## Notation

We write: $A_{s}(k)=C\left(S_{k}^{+}\right)$

$$
\begin{aligned}
& S_{k}^{+} \hat{=}\{\text { quantum permutations }\} \\
& \hat{=}\left\{u=\left(u_{i j}\right) \mid u_{i j}\right. \text { operators on Hilbert space } \\
&\text { satisfying these relations }\}
\end{aligned}
$$

If

$$
u_{1}=\left(u_{i j}^{(1)}\right)_{i, j=1}^{k} \quad \text { and } \quad u_{2}=\left(u_{i j}^{(2)}\right)_{i, j=1}^{k}
$$

are quantum permutations, then so is

$$
u_{1} \odot u_{2}:=\left(\sum_{k} u_{i k}^{(1)} \otimes u_{k j}^{(2)}\right)_{i, j=1}^{k}
$$

## Quantum Permutations

## Example

Examples of $u=\left(u_{i j}\right)_{i, j=1}^{k}$ satisfying these relations are:

- permutation matrices
- basic non-commutative example is of the form (for $k=4$ ):

$$
\left(\begin{array}{cccc}
p & 1-p & 0 & 0 \\
1-p & p & 0 & 0 \\
0 & 0 & q & 1-q \\
0 & 0 & 1-q & 1
\end{array}\right)
$$

for (in general, non-commuting) projections $p$ and $q$

- $S_{2}^{+}=S_{2}$,
- $S_{3}^{+}=S_{3}$,
- but $S_{k}^{+} \neq S_{k}$ for $k \geq 4$


## Quantum Exchangeability

## Definition

A sequence $x_{1}, \ldots, x_{k}$ in $(\mathcal{A}, \varphi)$ is quantum exchangeable if its distribution does not change under the action of quantum permutations $S_{k}^{+}$, i.e., if we have:
Let a quantum permutation $u=\left(u_{i j}\right) \in C\left(S_{k}^{+}\right)$act on $\left(x_{1}, \ldots, x_{k}\right)$ by

$$
y_{i}:=\sum_{j} u_{i j} \otimes x_{j} \quad \in \quad C\left(S_{k}^{+}\right) \otimes \mathcal{A}
$$

Then

- $\left(x_{1}, \ldots, x_{k}\right) \in(\mathcal{A}, \varphi)$
- $\left(y_{1}, \ldots, y_{k}\right) \in\left(C\left(S_{k}^{+}\right) \otimes \mathcal{A}\right.$, id $\left.\otimes \varphi\right)$
have the same distribution, i.e.,

$$
\varphi\left(x_{i(1)} \cdots x_{i(n)}\right) \cdot 1_{C\left(S_{k}^{+}\right)}=\operatorname{id} \otimes \varphi\left(y_{i(1)} \cdots y_{i(n)}\right)
$$

## Quantum Exchangeability

Equality of distributions

$$
\varphi\left(x_{i(1)} \cdots x_{i(n)}\right) \cdot 1_{C\left(S_{k}^{+}\right)}=\mathrm{id} \otimes \varphi\left(y_{i(1)} \cdots y_{i(n)}\right)
$$

means explicitly that

$$
\varphi\left(x_{i(1)} \cdots x_{i(n)}\right) \cdot 1=\sum_{j(1), \ldots, j(n)=1}^{k} u_{i(1) j(1)} \cdots u_{i(n) j(n)} \varphi\left(x_{j(1)} \cdots x_{j(n)}\right)
$$

for all $u=\left(u_{i j}\right)_{i, j=1}^{k}$ which satisfy the defining relations for $A_{s}(k)$.

- In particular: quantum exchangeable $\Longrightarrow$ exchangeable
- Commuting variables are usually not quantum exchangeable


## F.I.D. Variables are Quantum Exchangeable

## Proposition

Consider $x_{1}, \ldots, x_{k} \in(\mathcal{A}, \varphi)$ which are free and identically distributed. Then $x_{1}, \ldots, x_{k}$ are quantum exchangeable.

## Proof

We have to show equality of moments of $x_{i}$ 's and of $y_{i}$ 's. This is the same, by moment-cumulant formula, as showing for all $n \in \mathbb{N}$ and all $\pi \in N C(n)$

$$
\mathrm{id} \otimes \kappa_{\pi}\left(y_{i(1)}, \ldots, y_{i(n)}\right)=\kappa_{\pi}\left(x_{i(1)}, \ldots, i_{i(n)}\right)
$$

Consider $n=3$ and $\pi=\lfloor 1$. Then we have

$$
L H S=\sum_{j(1), j(2), j(3)} u_{i(1) j(1)} u_{i(2) j(2)} u_{i(3) j(3)} \cdot \kappa_{\pi}\left(x_{j(1)}, x_{j(2)}, x_{j(3)}\right)
$$

## F.I.D. Variables are Quantum Exchangeable

## Proposition

Consider $x_{1}, \ldots, x_{k} \in(\mathcal{A}, \varphi)$ which are free and identically distributed. Then $x_{1}, \ldots, x_{k}$ are quantum exchangeable.

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$$

Consider $n=3$ and $\pi=\lfloor 1$. Then we have

$$
L H S=\sum_{j(1), j(2), j(3)} u_{i(1) j(1)} u_{i(2) j(2)} u_{i(3) j(3)} \cdot \underbrace{\kappa_{\pi}\left(x_{j(1)}, x_{j(2)}, x_{j(3)}\right)}_{\kappa_{2}\left(x_{j(1)}, x_{j(3)}\right) \cdot \kappa_{1}\left(x_{j(2)}\right)}
$$

## F.I.D. Variables are Quantum Exchangeable

## Proposition

Consider $x_{1}, \ldots, x_{k} \in(\mathcal{A}, \varphi)$ which are free and identically distributed. Then $x_{1}, \ldots, x_{k}$ are quantum exchangeable.

## Proof

We have to show equality of moments of $x_{i}$ 's and of $y_{i}$ 's. This is the same, by moment-cumulant formula, as showing for all $n \in \mathbb{N}$ and all $\pi \in N C(n)$

$$
\mathrm{id} \otimes \kappa_{\pi}\left(y_{i(1)}, \ldots, y_{i(n)}\right)=\kappa_{\pi}\left(x_{i(1)}, \ldots, i_{i(n)}\right)
$$

Consider $n=3$ and $\pi=\lfloor 1$. Then we have

$$
L H S=\sum_{j(1), j(2), j(3)} u_{i(1) j(1)} u_{i(2) j(2)} u_{i(3) j(3)} \cdot \kappa_{2}\left(x_{j(1)}, x_{j(3)}\right) \cdot \underbrace{\kappa_{1}\left(x_{j(2)}\right)}_{\kappa_{1}(x)}
$$

## F.I.D. Variables are Quantum Exchangeable

## Proposition

Consider $x_{1}, \ldots, x_{k} \in(\mathcal{A}, \varphi)$ which are free and identically distributed. Then $x_{1}, \ldots, x_{k}$ are quantum exchangeable.

## Proof

We have to show equality of moments of $x_{i}$ 's and of $y_{i}$ 's. This is the same, by moment-cumulant formula, as showing for all $n \in \mathbb{N}$ and all $\pi \in N C(n)$

$$
\mathrm{id} \otimes \kappa_{\pi}\left(y_{i(1)}, \ldots, y_{i(n)}\right)=\kappa_{\pi}\left(x_{i(1)}, \ldots, i_{i(n)}\right)
$$

Consider $n=3$ and $\pi=\lfloor 1$. Then we have

$$
L H S=\sum_{j(1), j(2), j(3)} u_{i(1) j(1)} \underbrace{u_{i(2) j(2)}}_{\sum_{j(2)} \rightarrow 1} u_{i(3) j(3)} \cdot \kappa_{2}\left(x_{j(1)}, x_{j(3)}\right) \cdot \underbrace{\kappa_{1}\left(x_{j(2)}\right)}_{\kappa_{1}(x)}
$$

## F.I.D. Variables are Quantum Exchangeable

## Proposition

Consider $x_{1}, \ldots, x_{k} \in(\mathcal{A}, \varphi)$ which are free and identically distributed. Then $x_{1}, \ldots, x_{k}$ are quantum exchangeable.

## Proof

We have to show equality of moments of $x_{i}$ 's and of $y_{i}$ 's. This is the same, by moment-cumulant formula, as showing for all $n \in \mathbb{N}$ and all $\pi \in N C(n)$

$$
\mathrm{id} \otimes \kappa_{\pi}\left(y_{i(1)}, \ldots, y_{i(n)}\right)=\kappa_{\pi}\left(x_{i(1)}, \ldots, i_{i(n)}\right)
$$

Consider $n=3$ and $\pi=\lfloor$. Then we have

$$
L H S=\sum_{j(1), j(3)} u_{i(1) j(1)} u_{i(3) j(3)} \cdot \kappa_{2}\left(x_{j(1)}, x_{j(3)}\right) \cdot \kappa_{1}(x)
$$

## F.I.D. Variables are Quantum Exchangeable

## Proposition

Consider $x_{1}, \ldots, x_{k} \in(\mathcal{A}, \varphi)$ which are free and identically distributed. Then $x_{1}, \ldots, x_{k}$ are quantum exchangeable.

## Proof

We have to show equality of moments of $x_{i}$ 's and of $y_{i}$ 's. This is the same, by moment-cumulant formula, as showing for all $n \in \mathbb{N}$ and all $\pi \in N C(n)$

$$
\mathrm{id} \otimes \kappa_{\pi}\left(y_{i(1)}, \ldots, y_{i(n)}\right)=\kappa_{\pi}\left(x_{i(1)}, \ldots, i_{i(n)}\right)
$$

Consider $n=3$ and $\pi=\lfloor 1$. Then we have

$$
L H S=\sum_{j(1), j(3)} u_{i(1) j(1)} u_{i(3) j(3)} \cdot \underbrace{\kappa_{2}\left(x_{j(1)}, x_{j(3)}\right)}_{\delta_{j(1) j(3)} \cdot \kappa_{2}(x, x)} \cdot \kappa_{1}(x)
$$

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$$
L H S=\sum_{j(1)} \underbrace{u_{i(1) j(1)} u_{i(3) j(1)}}_{\delta_{i(1) i(3)} u_{i(1) j(1)}} \cdot \kappa_{2}(x, x) \cdot \kappa_{1}(x)
$$

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$$

Consider $n=3$ and $\pi=\lfloor$. Then we have

$$
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$$

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Consider $x_{1}, \ldots, x_{k} \in(\mathcal{A}, \varphi)$ which are free and identically distributed. Then $x_{1}, \ldots, x_{k}$ are quantum exchangeable.

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$$
\mathrm{id} \otimes \kappa_{\pi}\left(y_{i(1)}, \ldots, y_{i(n)}\right)=\kappa_{\pi}\left(x_{i(1)}, \ldots, i_{i(n)}\right)
$$

Consider $n=3$ and $\pi=\lfloor 1$. Then we have

$$
L H S=\kappa_{2}\left(x_{i(1)}, x_{i(3)}\right) \cdot \kappa_{1}\left(x_{i(2)}\right)
$$

## F.I.D. Variables are Quantum Exchangeable

## Proposition

Consider $x_{1}, \ldots, x_{k} \in(\mathcal{A}, \varphi)$ which are free and identically distributed. Then $x_{1}, \ldots, x_{k}$ are quantum exchangeable.

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We have to show equality of moments of $x_{i}$ 's and of $y_{i}$ 's. This is the same, by moment-cumulant formula, as showing for all $n \in \mathbb{N}$ and all $\pi \in N C(n)$

$$
\mathrm{id} \otimes \kappa_{\pi}\left(y_{i(1)}, \ldots, y_{i(n)}\right)=\kappa_{\pi}\left(x_{i(1)}, \ldots, i_{i(n)}\right)
$$

Consider $n=3$ and $\pi=\lfloor$. Then we have

$$
L H S=\kappa_{\pi}\left(x_{i(1)}, x_{i(2)}, x_{i(3)}\right)=R H S
$$

## Implications of Quantum Exchangeability

## Question

What does quantum exchangeability for an infinite sequence $x_{1}, x_{2}, \ldots$ imply?

As before, constant sequences are trivially quantum exchangeable, thus we have to take out the common part of all the $x_{i}$.
erc

## Implications of Quantum Exchangeability

## Question

What does quantum exchangeability for an infinite sequence $x_{1}, x_{2}, \ldots$ imply?

## Definition

Define the tail algebra of the sequence:

$$
\mathcal{A}_{\text {tail }}:=\bigcap_{i \in \mathbb{N}} \mathrm{vN}\left(x_{k} \mid k \geq i\right),
$$

then there exists conditional expectation $E: \mathrm{vN}\left(x_{i} \mid i \in \mathbb{N}\right) \rightarrow \mathcal{A}_{\text {tail }}$.

## Question

Does quantum exchangeability imply an independence like property for this $E$ ?

## Non-commutative de Finetti Theorem

## Theorem (Köstler, Speicher 2009)

The following are equivalent for an infinite sequence of non-commutative random variables:

- the sequence is quantum exchangeable
- the sequence is free and identically distributed with respect to the conditional expectation $E$ onto the tail-algebra of the sequence


## Proof

Consider first non-crossing moments like $E\left[x_{9} x_{7} x_{2} x_{7} x_{9}\right]=? ? ?$ Because of exchangeability we have

$$
\begin{aligned}
? ? ? & =\frac{E\left[x_{9} x_{7} x_{10} x_{7} x_{9}\right]+E\left[x_{9} x_{7} x_{11} x_{7} x_{9}\right]+\cdots+E\left[x_{9} x_{7} x_{9+N} x_{7} x_{9}\right]}{N} \\
& =E\left[x_{9} x_{7} \cdot \frac{1}{N} \sum_{i=1}^{N} x_{9+i} \cdot x_{7} x_{9}\right]
\end{aligned}
$$

However, by the mean ergodic theorem,

$$
\lim _{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^{N} x_{9+i}=E\left[x_{9}\right]=E\left[x_{2}\right]
$$

Thus

$$
E\left[x_{9} x_{7} x_{2} x_{7} x_{9}\right]=E\left[x_{9} x_{7} E\left[x_{2}\right] x_{7} x_{9}\right] .
$$

## Proof

$$
E\left[x_{9} x_{7} x_{2} x_{7} x_{9}\right]=E\left[x_{9} x_{7} E\left[x_{2}\right] x_{7} x_{9}\right] .
$$

Do now the same for $x_{7} E\left[x_{2}\right] x_{7}$.

$$
\begin{aligned}
E\left[x_{9} x_{7} E\left[x_{2}\right] x_{7} x_{9}\right] & =\lim _{N \rightarrow \infty} E\left[x_{9}\left(\frac{1}{N} \sum_{i=1}^{N} x_{13+i} E\left[x_{2}\right] x_{13+i}\right) x_{9}\right] \\
& =E\left[x_{9} E\left[x_{7} E\left[x_{2}\right] x_{7}\right] x_{9}\right]
\end{aligned}
$$

So we get

$$
E\left[x_{9} x_{7} x_{2} x_{7} x_{9}\right]=E\left[x_{9} E\left[x_{7} E\left[x_{2}\right] x_{7}\right] x_{9}\right]
$$

## Proof

In the same way one gets factorizations for all non-crossing moments in an iterative way (always work on interval blocks)

$$
\pi=\{\{1,10\},\{2,5,9\},\{3,4\},\{6\},\{7,8\}\} \in N C(10)
$$

| $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{3}$ | $x_{2}$ | $x_{4}$ | $x_{5}$ | $x_{5}$ | $x_{2}$ | $x_{1}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |



$$
\begin{aligned}
& E\left[x_{1} x_{2} x_{3} x_{3} x_{2} x_{4} x_{5} x_{5} x_{2} x_{1}\right] \\
& \quad=E\left[x_{1} \cdot E\left[x_{2} \cdot E\left[x_{3} x_{3}\right] \cdot x_{2} \cdot E\left[x_{4}\right] \cdot E\left[x_{5} x_{5}\right] \cdot x_{2}\right] \cdot x_{1}\right]
\end{aligned}
$$

## Proof

- Thus exchangeability implies factorizations for all non-crossing terms (Köstler 2008).
- For commuting variables this determines everything.
- However, for non-commuting variables there are many more expressions which cannot be treated like this.


## Problem

Basic example: $\mathrm{E}\left[\mathrm{x}_{1} \mathrm{x}_{2} \mathrm{x}_{1} \mathrm{x}_{2}\right]=? ? ?$
To determine those we need quantum exchangeability!

## Proof: determine $E\left[x_{1} x_{2} x_{1} x_{2}\right]$

Assume, for convenience, that $E\left[x_{1}\right]=E\left[x_{2}\right]=0$. By quantum exchangeability we have

$$
E\left[x_{1} x_{2} x_{1} x_{2}\right]=\sum_{j(1), \ldots, j(4)=1}^{k} u_{1 j(1)} u_{2 j(2)} u_{1 j(3)} u_{2 j(4)} E\left[x_{j(1)} x_{j(2)} x_{j(3)} x_{j(4)}\right]
$$

$$
=\sum_{j(1) \neq j(2) \neq j(3) \neq j(4)} \cdots
$$

$$
=\underbrace{\sum_{j(1)=j(3) \neq j(2)=j(4)} u_{1 j(1)} u_{2 j(2)} u_{1 j(3)} u_{2 j(4)}}_{\neq 1 \text { for general }\left(u_{i j}\right)} E\left[x_{1} x_{2} x_{1} x_{2}\right]
$$

## Proof: determine $E\left[x_{1} x_{2} x_{1} x_{2}\right]$

Thus we have: if $E\left[x_{1}\right]=0=E\left[x_{2}\right]$, then $E\left[x_{1} x_{2} x_{1} x_{2}\right]=0$ This implies in general:

$$
\begin{aligned}
E\left[x_{1} x_{2} x_{1} x_{2}\right]=E\left[x_{1} E\left[x_{2}\right] x_{1}\right] \cdot E\left[x_{2}\right]+E\left[x_{1}\right] \cdot & E\left[x_{2} E\left[x_{1}\right] x_{2}\right] \\
& -E\left[x_{1}\right] E\left[x_{2}\right] E\left[x_{1}\right] E\left[x_{2}\right]
\end{aligned}
$$

.... which is the formula for variables which are free with respect to $E$.

## Proof

In general, one shows in the same way that

$$
E\left[p_{1}\left(x_{i(1)}\right) p_{2}\left(x_{i(2)}\right) \cdots p_{n}\left(x_{i(n)}\right)\right]=0
$$

whenever

- $n \in \mathbb{N}$ and $p_{1}, \ldots, p_{n} \in \mathcal{A}_{\text {tail }}\langle X\rangle$ are polynomials in one variable
- $i(1) \neq i(2) \neq i(3) \neq \cdots \neq i(n)$
- $E\left[p_{j}\left(x_{i(j)}\right)\right]=0$ for all $j=1, \ldots, n$

Thus, the $x_{i}$ are free w.r.t $E$ in the sense of Voiculescu's free probability theory.

## Non-commutative de Finetti Theorem

## Theorem (Köstler, Speicher 2009)

The following are equivalent for an infinite sequence of non-commutative random variables:

- the sequence is quantum exchangeable
- the sequence is free and identically distributed with respect to the conditional expectation $E$ onto the tail-algebra of the sequence

Thus, freeness arises very naturally from symmetry requirements, if one takes the quantum permutation group as the right analogue of the permutation group in the non-commutative world.

## Section 7

## More Quantum Symmetries in Free/Non-Commutative Probability

## What are Quantum Groups?

- are generalizations of groups $G$ (actually, of $C(G)$ )
- are supposed to describe non-classical symmetries
- are Hopf algebras, with some additional structure ...


## What are Quantum Groups?

Deformation of Classical Symmetries: $\mathrm{G} \rightsquigarrow \mathrm{G}_{\mathrm{q}}$

- quantum groups are often deformations $G_{q}$ of classical groups, depending on some parameter $q$, such that for $q \rightarrow 1$, they go to the classical group $G=G_{1}$
- $G_{q}$ and $G_{1}$ are incomparable, none is stronger than the other
$G_{1}$ is supposed to act on commuting variables
$G_{q}$ is the right replacement to act on $q$-commuting variables
Strengthening of Classical Symmetries: $\mathbf{G} \rightsquigarrow \mathrm{G}^{+}$
- there are situations where a classical group $G$ has a genuine non-commutative analogue $G^{+}$(no interpolations)
- $G^{+}$is "stronger" than $G: \quad G \subset G^{+}$
$G$ acts on commuting variables
$G^{+}$is the right replacement for acting on maximally non-commuting variables


## Orthogonal Hopf Algebras

We are interested in quantum versions of real compact matrix groups. Think of

- orthogonal matrices or permutation matrices

Such quantum versions are captured by the notion of orthogonal Hopf algebra.

## Definition

An orthogonal Hopf algebra is a $C^{*}$-algebra $A$, given with a system of $n^{2}$ self-adjoint generators $u_{i j} \in A(i, j=1, \ldots, n)$, subject to the following conditions:

- The inverse of $u=\left(u_{i j}\right)$ is the transpose matrix $u^{t}=\left(u_{j i}\right)$.
- $\Delta\left(u_{i j}\right)=\Sigma_{k} u_{i k} \otimes u_{k j}$ defines a morphism $\Delta: A \rightarrow A \otimes A$.
- $\varepsilon\left(u_{i j}\right)=\delta_{i j}$ defines a morphism $\varepsilon: A \rightarrow \mathbb{C}$.
- $S\left(u_{i j}\right)=u_{j i}$ defines a morphism $S: A \rightarrow A^{o p}$.


## Orthogonal Hopf Algebras

## Definition

An orthogonal Hopf algebra is a $C^{*}$-algebra $A$, given with a system of $n^{2}$ self-adjoint generators $u_{i j} \in A(i, j=1, \ldots, n)$, subject to the following conditions:

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- $\varepsilon\left(u_{i j}\right)=\delta_{i j}$ defines a morphism $\varepsilon: A \rightarrow \mathbb{C}$.
- $S\left(u_{i j}\right)=u_{j i}$ defines a morphism $S: A \rightarrow A^{o p}$.
- These are compact quantum groups in the sense of Woronowicz.
- In the spirit of non-commutative geometry, we are thinking of
$\mathbf{A}=\mathbf{C}\left(\mathbf{G}^{+}\right)$as the continuous functions, generated by the coordinate functions $u_{i j}$, on some (non-existing) quantum group $G^{+}$, replacing a classical group $G$.


## Definition (Quantum Orthogonal Group $O_{n}^{+}$(Wang 1995))

The quantum orthogonal group $A_{o}(n)=C\left(O_{n}^{+}\right)$is the universal unital $C^{*}$-algebra generated by selfadjoint $u_{i j}(i, j=1, \ldots, n)$ subject to the relation: $u=\left(u_{i j}\right)_{i, j=1}^{n}$ is an orthogonal matrix; i.e., for all $i, j$ we have

$$
\sum_{k=1}^{n} u_{i k} u_{j k}=\delta_{i j} \quad \text { and } \quad \sum_{k=1}^{n} u_{k i} u_{k j}=\delta_{i j}
$$

## Definition (Quantum Permutation Group $S_{n}^{+}$(Wang 1998))

The quantum permutation group $A_{s}(n)=C\left(S_{n}^{+}\right)$is the universal unital $C^{*}$-algebra generated by $u_{i j}(i, j=1, \ldots, n)$ subject to the relations

- $u_{i j}^{2}=u_{i j}=u_{i j}^{*}$ for all $i, j=1, \ldots, n$
- each row and column of $u=\left(u_{i j}\right)_{i, j=1}^{n}$ is a partition of unity:

$$
\sum_{j=1}^{n} u_{i j}=1 \quad \text { and } \quad \sum_{i=1}^{n} u_{i j}=1
$$

## Are there more of those?



## Questions

## Are there more of those?

$$
\begin{array}{lllll}
S_{n}^{+} & \subset & G_{n}^{+} & \subset & O_{n}^{+} \\
& & & & \\
& \cup & & \cup & \\
& & & & \cup \\
& & & & \\
S_{n} & \subset & G_{n} & \subset & O_{n}
\end{array}
$$

## Questions

- Are there more non-commutative versions $G_{n}^{+}$of classical groups $G_{n}$ ?


## Are there more of those?

| $S_{n}^{+}$ |  | $\subset$ |  |
| :---: | :---: | :---: | :---: |
|  |  |  | $O_{n}^{+}$ |
|  |  | $G_{n}^{*}$ |  |
|  |  |  |  |
|  |  |  |  |
| $S_{n}$ |  | $\subset$ |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

## Questions

- Are there more non-commutative versions $G_{n}^{+}$of classical groups $G_{n}$ ?
- Actually, are there more nice non-commutative quantum groups $G_{n}^{*}$ ?


## How can we describe and understand intermediate quantum groups

## Questions:

- Are there more non-commutative versions $G_{n}^{+}$of classical groups $G_{n}$ ?
- Actually, are there more nice non-commutative quantum groups $G_{n}^{*}$, stronger than $S_{n}$ ?

$$
\begin{gathered}
S_{n} \subset \mathbf{G}_{\mathbf{n}}^{*} \subset O_{n}^{+} \\
C\left(S_{n}\right) \leftarrow \mathbf{C}\left(\mathbf{G}_{\mathbf{n}}^{*}\right) \leftarrow C\left(O_{n}^{+}\right)
\end{gathered}
$$

Deal with quantum groups by looking on their representations!!!

## Section 8

## Describing Quantum Groups via Intertwiner Spaces

## Spaces of Intertwiners

## Definition

Associated to an orthogonal Hopf algebra $\left(A=C\left(G_{n}^{*}\right),\left(u_{i j}\right)_{i, j=1}^{n}\right)$ are the spaces of intertwiners:

$$
\mathbf{I}_{G_{n}^{*}}(k, l)=\left\{T:\left(\mathbb{C}^{n}\right)^{\otimes k} \rightarrow\left(\mathbb{C}^{n}\right)^{\otimes l} \mid T u^{\otimes k}=u^{\otimes l} T\right\}
$$

where $u^{\otimes k}$ is the $n^{k} \times n^{k}$ matrix $\left(u_{i_{1} j_{1}} \ldots u_{i_{k} j_{k}}\right)_{i_{1} \ldots i_{k}, j_{1} \ldots j_{k}}$.

$$
\begin{gathered}
u \in M_{n}(A) \quad u: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n} \otimes A \\
u^{\otimes k}:\left(\mathbb{C}^{n}\right)^{\otimes k} \rightarrow\left(\mathbb{C}^{n}\right)^{\otimes k} \otimes A
\end{gathered}
$$

Note: if $T \in \mathbf{I}_{G_{n}^{*}}(0, l)$, then $\xi:=T 1 \in\left(\mathbb{C}^{n}\right)^{\otimes l}$ is a fixed vector unter $u^{\otimes l}$ :

$$
T u^{\otimes 0}=u^{\otimes l} T \quad \Longrightarrow \quad \xi=T 1=u^{\otimes l} T 1=u^{\otimes l} \xi
$$

## $\mathbf{I}_{G_{n}^{*}}$ is Tensor Category with Duals

## Proposition

Collection of vector spaces $\mathbf{I}_{G_{n}^{*}}(k, l)$ has the following properties:

- $T, T^{\prime} \in \mathbf{I}_{G_{n}^{*}}$ implies $T \otimes T^{\prime} \in \mathbf{I}_{G_{n}^{*}}$.
- If $T, T^{\prime} \in \mathbf{I}_{G_{n}^{*}}$ are composable, then $T T^{\prime} \in \mathbf{I}_{G_{n}^{*}}$.
- $T \in \mathbf{I}_{G_{n}^{*}}$ implies $T^{*} \in \mathbf{I}_{G_{n}^{*}}$.
- $i d(x)=x$ is in $\mathbf{I}_{G_{n}^{*}}(1,1)$.
- $\xi=\sum e_{i} \otimes e_{i}$ is in $\mathbf{I}_{G_{n}^{*}}(0,2)$.

Let us check that

$$
\xi=\sum e_{i} \otimes e_{i} \in \mathbf{I}_{G_{n}^{*}}(0,2)
$$

## Proof: Why is $\xi=\sum_{\mathbf{i}} \mathbf{e}_{\mathbf{i}} \otimes \mathbf{e}_{\mathbf{i}} \in \mathbf{I}_{\mathbf{G}_{\mathbf{n}}^{*}}(\mathbf{0}, \mathbf{2})$

We have to see

$$
\left(u^{\otimes 2} \xi\right)_{i_{1}, i_{2}}=\xi_{i_{1}, i_{2}}
$$

$$
\begin{aligned}
\left(u^{\otimes 2 \sum_{i}} e_{i} \otimes e_{i}\right)_{i_{1}, i_{2}} & =\sum_{i} \sum_{j_{1}, j_{2}} u_{i_{1} j_{1}} u_{i_{2} j_{2}}\left(e_{i} \otimes e_{i}\right)_{j_{1}, j_{2}} \\
& =\sum_{i}^{\sum_{j_{1}, j_{2}} u_{i_{1} j_{1}} u_{i_{2} j_{2}} \delta_{i_{j}} \delta_{i j_{2}}} \\
& =\sum_{i} u_{i_{1} i} u_{i_{2} i}=\delta_{i_{1} i_{2}}=\left(\sum_{i} e_{i} e_{i}\right)_{i}
\end{aligned}
$$

## Tannaka-Krein for compact quantum groups

Theorem (Woronowicz 1988)
The compact quantum group $G_{n}^{*}$ can actually be rediscovered from its space of intertwiners.
There is a one-to-one correspondence between:

- orthogonal Hopf algebras $C\left(O_{n}^{+}\right) \rightarrow \mathbf{C}\left(\mathbf{G}_{\mathbf{n}}^{*}\right) \rightarrow C\left(S_{n}\right)$
- tensor categories with duals $\mathbf{I}_{O_{n}^{+}} \subset \mathbf{I}_{\mathbf{G}_{\mathbf{n}}^{*}} \subset \mathbf{I}_{S_{n}}$.


## How to Get Intertwiners

## Definition

We denote by $P(k, l)$ the set of partitions of the set with repetitions $\{1, \ldots, k, 1, \ldots, l\}$. Such a partition will be pictured as

$$
p=\begin{gathered}
1 \ldots k \\
\mathcal{P} \\
1 \ldots l
\end{gathered}
$$

where $\mathcal{P}$ is a diagram joining the elements in the same block of the partition.

## Example



## How to Get Intertwiners

## Definition

Associated to any partition $p \in P(k, l)$ is the linear map

$$
T_{p}:\left(\mathbb{C}^{n}\right)^{\otimes k} \rightarrow\left(\mathbb{C}^{n}\right)^{\otimes l}
$$

given by

$$
T_{p}\left(e_{i_{1}} \otimes \ldots \otimes e_{i_{k}}\right)=\sum_{j_{1} \ldots j_{l}} \delta_{p}(i, j) e_{j_{1}} \otimes \ldots \otimes e_{j_{l}}
$$

where $e_{1}, \ldots, e_{n}$ is the standard basis of $\mathbb{C}^{n}$, and where

$$
\delta_{p}(i, j)= \begin{cases}1, & \text { if all indices which are connected by } p \text { are the same } \\ 0, & \text { otherwise }\end{cases}
$$

## How to Get Intertwiners

Definition

$$
T_{p}:\left(\mathbb{C}^{n}\right)^{\otimes k} \rightarrow\left(\mathbb{C}^{n}\right)^{\otimes l}
$$

given by

$$
T_{p}\left(e_{i_{1}} \otimes \ldots \otimes e_{i_{k}}\right)=\sum_{j_{1} \ldots j_{l}} \delta_{p}(i, j) e_{j_{1}} \otimes \ldots \otimes e_{j_{l}}
$$

## Example

$$
\begin{gathered}
T_{\{| |\}}\left(e_{a} \otimes e_{b}\right)=e_{a} \otimes e_{b} \\
T_{\{|-|\}}\left(e_{a} \otimes e_{b}\right)=\delta_{a b} e_{a} \otimes e_{a} \\
T_{\left\{\begin{array}{c}
\sqcup \\
\mid ।
\end{array}\right\}} \begin{array}{l}
\left(e_{a} \otimes e_{b}\right)=\delta_{a b} \sum_{c d} e_{c} \otimes e_{d}
\end{array}
\end{gathered}
$$

## Intertwiners of (Quantum) Permutation and of (Quantum) Orthogonal Group

Question: What are the intertwiners?

| $S_{n}^{+}$ | $\subset$ | $O_{n}^{+}$ | $\mathbf{I}_{S_{n}^{+}}$ | $\supset$ | $\mathbf{I}_{O_{n}^{+}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\cup$ |  | $\cup$ | $\cap$ |  | $\cap$ |
| $S_{n}$ | $\subset$ | $O_{n}$ | $\mathbf{I}_{S_{n}}$ | $\supset$ | $\mathbf{I}_{O_{n}}$ |

First answer: Intertwiners of $S_{n}$

$$
\operatorname{span}\left(T_{p} \mid p \in P(k, l)\right)=\mathbf{I}_{S_{n}}(k, l)
$$

## Proof: Why is $T_{p}$ in $\mathbf{I}_{S_{n}}$ for all $p \in P$ ?

Take $u \hat{=} \pi$ permutation matrix, i.e., $u e_{i}=e_{\pi^{-1}(i)}$. Then

$$
\begin{aligned}
T_{p} u^{\otimes k} e_{i_{1}} \otimes & \cdots \otimes e_{i_{k}}=T_{p} e_{\pi^{-1}\left(i_{1}\right)} \otimes \cdots \otimes e_{\pi^{-1}\left(i_{k}\right)} \\
& =\sum_{j} \delta_{p}\left(\pi^{-1}\left(i_{1}\right), \ldots, \pi^{-1}\left(i_{k}\right), j_{1}, \ldots, j_{l}\right) e_{j_{1}} \otimes \cdots \otimes e_{j_{l}}
\end{aligned}
$$

and

$$
\begin{aligned}
u^{\otimes l} T_{p} e_{i_{1}} \otimes & \cdots \otimes e_{i_{k}}=u^{\otimes l} \sum_{r} \delta_{p}\left(i_{1}, \ldots, i_{k}, r_{1}, \ldots, r_{l}\right) e_{r_{1}} \otimes \cdots \otimes e_{r_{l}} \\
& =\sum_{r} \delta_{p}\left(i_{1}, \ldots, i_{k}, r_{1}, \ldots, r_{l}\right) e_{\pi^{-1}\left(r_{1}\right)} \otimes \cdots \otimes e_{\pi^{-1}\left(r_{l}\right)} \\
& =\sum_{j} \delta_{p}\left(i_{1}, \ldots, i_{k}, \pi\left(j_{1}\right), \ldots, \pi\left(j_{l}\right)\right) e_{j_{1}} \otimes \cdots \otimes e_{j_{l}}
\end{aligned}
$$

But $\quad \delta_{p}\left(\pi^{-1}\left(i_{1}\right), \ldots, \pi^{-1}\left(i_{k}\right), j_{1}, \ldots, j_{l}\right)=\delta_{p}\left(i_{1}, \ldots, i_{k}, \pi\left(j_{1}\right), \ldots, \pi\left(j_{l}\right)\right)$

## Intertwiners of (Quantum) Permutation and of (Quantum) Orthogonal Group

Let $N C(k, l) \subset P(k, l)$ be the subset of noncrossing partitions.
$\operatorname{span}\left(T_{p} \mid p \in N C(k, l)\right)=\mathbf{I}_{S_{n}^{+}}(k, l) \quad \supset \quad \mathbf{I}_{O_{n}^{+}}(k, l)=\operatorname{span}\left(T_{p} \mid p \in N C_{2}(k, l)\right)$
$\operatorname{span}\left(T_{p} \mid p \in P(k, l)\right)=\mathbf{I}_{S_{n}}(k, l) \quad \supset \quad \mathbf{I}_{O_{n}}(k, l)=\operatorname{span}\left(T_{p} \mid p \in P_{2}(k, l)\right)$

## Easy Quantum Groups

## Definition (Banica, Speicher 2009)

A quantum group $S_{n} \subset G_{n}^{*} \subset O_{n}^{+}$is called easy when its associated tensor category is of the form

$$
\begin{aligned}
\mathbf{I}_{S_{n}} & =\operatorname{span}\left(T_{p} \mid p \in P\right) \\
& \cup \\
\mathbf{I}_{\mathbf{G}_{\mathbf{n}}^{*}} & =\operatorname{span}\left(\mathbf{T}_{\mathbf{p}} \mid \mathbf{p} \in \mathbf{P}_{\mathbf{G}^{*}}\right) \\
\cup & \\
\mathbf{I}_{O_{n}^{+}} & =\operatorname{span}\left(T_{p} \mid p \in N C_{2}\right)
\end{aligned}
$$

for a certain collection of subsets $P_{G^{*}} \subset P$.

## What are we interested in?

- classification of easy (and more general) quantum groups
- understanding of meaning/implications of symmetry under such quantum groups; in particular, under quantum permutations $S_{n}^{+}$, or quantum rotations $O_{n}^{+}$
- treating series of such quantum groups (like $S_{n}^{+}$or $O_{n}^{+}$) as fundamental examples of non-commuting random matrices


## Section 9

## Easy Quantum Groups: Classification

## Classification Results for Easy Quantum Groups

## Theorem and Definition

The category of partitions $P_{G^{*}} \subset P$ for an easy quantum group $G_{n}^{*}$ must satisfy:

- $P_{G^{*}}$ is stable by tensor product.
- $P_{G^{*}}$ is stable by composition.
- $P_{G^{*}}$ is stable by involution.
- $P_{G^{*}}$ contains the "unit" partition $\mid$.
- $P_{G^{*}}$ contains the "duality" partition $\Pi$.


## Example of Composition $P(2,4) \times P(4,1) \rightarrow P(2,1)$

## Example



## Are there more of those easy quantum groups?

$$
\begin{array}{llll}
S_{n}^{+} & & \subset & \\
& & O_{n}^{+} \\
\cup & & & \\
& & & \\
& & & \\
S_{n} & & \subset & \\
& O_{n}
\end{array}
$$

## Are there more of those easy quantum groups?

$$
\begin{array}{cccccc}
S_{n}^{+} & \subset & G_{n}^{+} & \subset & O_{n}^{+} \\
& & & & \\
\cup & & \cup & & & \cup \\
& & & & & \\
S_{n} & \subset & G_{n} & \subset & O_{n}
\end{array}
$$

## Questions

- Are there more easy non-commutative versions $G_{n}^{+}$of easy classical groups $G_{n}$ ?


## Classification Results

## Theorem (Banica, Speicher 2009; Weber 2011)

## There are

- 7 Categories of Noncrossing Partitions and

$$
\left.\begin{array}{ccc}
\left.\begin{array}{c}
\text { singletons } \\
\text { pairings }
\end{array}\right\} & \supset\left\{\begin{array}{c}
\text { singletons } \\
\text { pairings } \\
\text { (even part })
\end{array}\right\} & \supset\left\{\begin{array}{c}
\text { singletons } \\
\text { pairings } \\
\text { resp. parity })
\end{array}\right\}
\end{array}\right)\left\{\begin{array}{c}
\text { all } \\
\text { pairings }
\end{array}\right\}
$$

## Classification Results

## Theorem (Banica, Speicher 2009; Weber 2011)

There are

- 6 Categories of Partitions containing Basic Crossing

$$
\begin{aligned}
& \left\{\begin{array}{c}
\text { singletons } \\
\text { pairings }
\end{array}\right\} \supset\left\{\begin{array}{c}
\text { singletons } \\
\text { pairings } \\
(\text { even part })
\end{array}\right\} \supset \supset \quad \supset\left\{\begin{array}{c}
\text { all } \\
\text { pairings }
\end{array}\right\} \\
& \left.\begin{array}{ccc}
\cap & \cap & \cap \\
\left\{\begin{array}{c}
\text { all } \\
\text { partitions }
\end{array}\right\} & \left\{\begin{array}{c}
\text { all partitions } \\
\text { (even part) }
\end{array}\right\} & \supset
\end{array} \begin{array}{c}
\text { blocks of } \\
\text { even size }
\end{array}\right\}
\end{aligned}
$$

## Classification Results

Theorem (Banica, Speicher 2009; Weber 2011)
...and thus there are

- 7 free easy quantum groups $S_{n}^{+} \subset G_{n}^{+} \subset O_{n}^{+}$and

| $B_{n}^{+}$ | $\subset$ | $B_{n}^{\prime+}$ | $\subset$ | $B_{n}^{\#+}$ | $\subset$ | $O_{n}^{+}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\cup$ |  | $\cup$ |  |  |  | $\cup$ |
| $S_{n}^{+}$ | $\subset$ | $S_{n}^{\prime+}$ |  | $\subset$ |  | $H_{n}^{+}$ |

## Classification Results

Theorem (Banica, Speicher 2009; Weber 2011)
...and thus there are

- 6 classical easy groups $S_{n} \subset G_{n} \subset O_{n}$

| $B_{n}$ | $\subset$ | $B_{n}^{\prime}$ | $\subset$ |  | $\subset$ | $O_{n}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\cup$ |  | $\cup$ |  |  |  | $\cup$ |
| $S_{n}$ | $\subset$ | $S_{n}^{\prime}$ |  | $\subset$ |  | $H_{n}$ |

## The easy classical groups

The easy classical groups are:

- $O_{n}$
- $S_{n}$
- $H_{n}=\mathbb{Z}_{2}$ 亿 $S_{n}$ : the hyperoctahedral group, consisting of monomial matrices with $\pm 1$ nonzero entries.
- $B_{n} \simeq O_{n-1}$ : the bistochastic group, consisting of orthogonal matrices having sum 1 in each row and each column.
- $S_{n}^{\prime}=\mathbb{Z}_{2} \times S_{n}$ : permutation matrices multiplied by $\pm 1$.
- $B_{n}^{\prime}=\mathbb{Z}_{2} \times B_{n}$ : bistochastic matrices multiplied by $\pm 1$.


## Are there more of those easy quantum groups?

$$
\begin{array}{cccc}
S_{n}^{+} & & \subset & \\
& & O_{n}^{+} \\
\cup & & & \\
& & & \\
S_{n} & & \subset & \\
& O_{n}
\end{array}
$$

## Question

## Are there more of those easy quantum groups?

| $S_{n}^{+}$ |  | $\subset$ |  |
| :---: | :---: | :---: | :---: |
|  |  |  | $O_{n}^{+}$ |
|  |  |  | $G_{n}^{*}$ |
|  |  |  |  |
|  |  |  |  |
| $S_{n}$ |  | $\subset$ |  |
|  |  |  | $O_{n}$ |

## Question

- Are there more easy non-commutative quantum groups $G_{n}^{*}$, stronger than $S_{n}$ ?


## Classification of Easy Quantum Groups

- $\exists!7$ free easy QG's (categories noncrossing) [Banica, Speicher 09, Weber 13; (Banica, Bichon, Collins 07)]
- $\exists$ ! 6 easy groups (categ. containing $\times \in P(2,2), u_{i j} u_{k l}=u_{k l} u_{i j}$ ) [Banica, Speicher 09]
- $\exists$ ! 3 half-liberated easy QG's \& one infinite series (categories containing $* \in P(3,3), u_{i j} u_{k l} u_{s t}=u_{s t} u_{k l} u_{i j}$ ) [Banica, Curran, Speicher 10, Weber 13]
- $\exists$ ! 13 non-hyperoctahedral easy QG's
( $\sim$ categories containing singletons as blocks)
[Banica, Curran, Speicher 10, Weber 13]
- hyperoctahedral case: [Raum, Weber 12 \& 13]



## de Finetti Theorems

## Theorem (de Finetti 1931, Hewitt, Savage 1955)

The following are equivalent for an infinite sequence of classical, commuting random variables:

- the sequence is exchangeable (i.e., invariant under all $S_{n}$ )
- the sequence is independent and identically distributed with respect to the conditional expectation $E$ onto the tail $\sigma$-algebra of the sequence


## Theorem (Köstler, Speicher 2008)

The following are equivalent for an infinite sequence of non-commutative random variables:

- the sequence is quantum exchangeable (i.e., invariant under all $S_{n}^{+}$)
- the sequence is free and identically distributed with respect to the conditional expectation $E$ onto the tail-algebra of the sequence


## Section 10

## Haar State and Non-Commutative Random Matrices

## Non-Commutative Random Matrices

- there exists, as for any compact quantum group, a unique Haar state on the easy quantum groups, thus one can integrate/average over the quantum groups
- actually: for the easy quantum groups, there exist nice and "concrete" formula for the calculation of this state:

$$
\int_{G_{n}^{*}} u_{i_{1} j_{1}} \cdots u_{i_{k} j_{k}} d u=\sum_{\substack{p, q \in P_{G^{*}}(k) \\ p \leq \operatorname{ker} i \\ q \leq \operatorname{ker} j}} W_{n}(p, q)
$$

where $W_{n}$ is inverse of

$$
G_{n}(p, q)=n^{|p \vee q|}
$$

## Non-Commutative Random Matrices

- this allows the calculation of distributions of functions of our non-commutative random matrices $G_{n}^{*}$, in the limit $n \rightarrow \infty$
- in particular, in analogy to Diaconis\&Shashahani, we have results about the asymptotic distribution of $\operatorname{Tr}\left(u^{k}\right)$
- note: in the classical case, knowledge about traces of powers of the matrices is the same as knowledge about the eigenvalues of the matrices


## Weingarten Formula for Easy Quantum Groups

Denote by $D=(D(k))_{k \in \mathbb{N}}$ the category of partitions for the easy quantum group $G_{n}^{*}$; where $D(k):=D(0, k)$. Then

$$
\int_{G_{n}^{*}} u_{i_{1} j_{1}} \cdots u_{i_{k} j_{k}} d u=\sum_{\substack{p, q \in D(k) \\ p \leq \operatorname{ker} i \\ q \leq \operatorname{ker} j}} W_{n}(p, q),
$$

where $W_{k, n}=\left(W_{n}(p, q)\right)_{p, q \in D(k)}=G_{k, n}^{-1}$ is the inverse of the Gram matrix

$$
G_{k, n}=\left(G_{n}(p, q)\right)_{p, q \in D(k)} \quad \text { where } \quad G_{n}(p, q)=n^{|p \vee q|}
$$

Note: $p \vee q$ is always the supremum in the lattice of all partitions; i.e., $p \vee q$ is not necessarily in $D$

## Weingarten Formula for Easy Quantum Groups

Example: Integrate $u_{21} u_{23}$. Then $i=(2,2), j=(1,3)$, hence

$$
\operatorname{ker} i=\sqcup, \quad \operatorname{ker} j=| |
$$

and thus

$$
\int_{G_{n}} u_{21} u_{23} d u=W(\sqcup,| |)+W(| |,| |)
$$

Similarly,
$\int_{G_{n}} u_{23} u_{23} d u=W(\sqcup, \sqcup)+W(\sqcup,| |)+W(| |, \sqcup)+W(| |,| |)$

## Asymptotics of the Weingarten Formula

$$
\int_{G_{n}^{*}} u_{i_{1} j_{1}} \cdots u_{i_{k} j_{k}} d u=\sum_{\substack{p, q \in D(k) \\ p \leq \operatorname{ker} i \\ q \leq \operatorname{ker} j}} W_{n}(p, q)
$$

where $W_{k, n}=\left(W_{n}(p, q)\right)_{p, q \in D(k)}=G_{k, n}^{-1}$ is the inverse of the Gram matrix

$$
G_{k, n}=\left(G_{n}(p, q)\right)_{p, q \in D(k)} \quad \text { where } \quad G_{n}(p, q)=n^{|p \vee q|}
$$

We have the asymptotics

$$
W_{n}(p, q)=O\left(n^{|p \vee q|-|p|-|q|}\right)
$$

## Distribution of Traces of Powers

Let $G$ be an easy quantum group. Consider $s \in \mathbb{N}, k_{1}, \ldots, k_{s} \in \mathbb{N}$, $k:=\sum_{i=1}^{s} k_{i}$, and denote

$$
\gamma:=\left(1,2, \ldots, k_{1}\right)\left(k_{1}+1, k_{1}+2, \ldots, k_{1}+k_{2}\right) \cdots(\cdots, k) \in S_{k}
$$

Then we have, for any $n$ such that $G_{k n}$ is invertible:

$$
\int_{G_{n}} \operatorname{Tr}\left(u^{k_{1}}\right) \ldots \operatorname{Tr}\left(u^{k_{s}}\right) d u=\#\{p \in D(k) \mid p=\gamma(p)\}+O(1 / n) .
$$

If $G$ is a classical easy group, then this formula is exact, without any lower order corrections in $n$.

## Proof

$$
\begin{aligned}
& I:=\int_{G} \operatorname{Tr}\left(u^{k_{1}}\right) \ldots \operatorname{Tr}\left(u^{k_{s}}\right) d u \\
& =\sum_{i_{1} \ldots i_{k}} \int_{G}\left(u_{i_{1} i_{2}} \ldots u_{i_{k_{1}} i_{1}}\right) \ldots\left(u_{i_{k-k_{s}+1} i_{k-k_{s}+2}} \ldots u_{i_{k} i_{k-k_{s}+1}}\right) \\
& =\sum_{i_{1} \ldots i_{k}} \int_{G} u_{i_{1} i_{\gamma(1)}} \ldots u_{i_{k} i_{\gamma(k)}} \\
& =\sum^{n} \quad \sum \quad W_{k n}(p, q) \\
& i_{1} \ldots i_{k}=1 \quad p, q \in D_{k} \\
& p \leq \operatorname{ker} \mathbf{i}, q \leq \operatorname{ker} \mathbf{i} \circ \gamma \\
& =\sum_{i_{1} \ldots i_{k}=1}^{n} \sum_{\substack{p, q \in D_{k} \\
p \leq \operatorname{ker} \mathbf{i}, \gamma(q) \leq \operatorname{ker} \mathbf{i}}} W_{k n}(p, q)
\end{aligned}
$$

## Proof

$$
\begin{aligned}
I & =\sum_{i_{1} \ldots i_{k}=1}^{n} \sum_{\substack{p, q \in D_{k} \\
p \leq \operatorname{ker} \mathbf{i}, \gamma(q) \leq \operatorname{ker} \mathbf{i}}} W_{k n}(p, q) \\
& =\sum_{p, q \in D_{k}}^{n} \sum_{\substack{i_{1} \ldots i_{k}=1 \\
p \leq \operatorname{ker} \mathbf{i}, \gamma(q) \leq \operatorname{ker} \mathbf{i}}}^{n} W_{k n}(p, q) \\
& =\sum_{p, q \in D_{k}} n^{|p \vee \gamma(q)|} W_{k n}(p, q) \\
& =\sum_{p, q \in D_{k}} n^{|p \vee \gamma(q)|} n^{|p \vee q|-|p|-|q|}(1+O(1 / n))
\end{aligned}
$$

The leading order of $n^{|p \vee \gamma(q)|+|p \vee q|-|p|-|q|}$ is $n^{0}$, which is achieved if and erc only equivalently $p=q=\gamma(q)$.

## Proof

In the classical case, instead of using the approximation for $W_{n k}(p, q)$, we can write $n^{|p \vee \gamma(q)|}$ as $G_{n k}(\gamma(q), p)$.
(Note that this only makes sense if we know that $\gamma(q)$ is also an element in $D_{k}$; and this is only the case for the classical partition lattices.)
Then one can continue as follows:
$I=\sum_{p, q \in D_{k}} G_{n k}(\gamma(q), p) W_{k n}(p, q)=\sum_{q \in D_{k}} \delta(\gamma(q), q)=\#\left\{q \in D_{k} \mid q=\gamma(p)\right\}$.

## The Distribution of $u_{r}:=\lim _{n \rightarrow \infty} \operatorname{Tr}\left(u^{r}\right)$

| Variable | $O_{n}$ | $O_{n}^{+}$ |
| :--- | :--- | :--- |
| $u_{1}$ | real Gaussian | semicircular |
| $u_{2}$ | real Gaussian | semicircular |
| $u_{r}(r \geq 3)$ | real Gaussian | circular |


| Variable | $S_{n}$ | $S_{n}^{+}$ |
| :--- | :--- | :--- |
| $u_{1}$ | Poisson | free Poisson |
| $u_{2}-u_{1}$ | Poisson | semicircular |
| $u_{r}-u_{1}(r \geq 3)$ | sum of Poissons | circular |

## Something to Remember

Whereas $\operatorname{Tr}(u)$ and $\operatorname{Tr}\left(u^{2}\right)$ are selfadjoint, this is not true for $\operatorname{Tr}\left(u^{3}\right)$ in the general non-commutative situation!

$$
\begin{gathered}
u_{1}=\sum u_{i i}=u_{1}^{*} \\
u_{2}=\sum u_{i j} u_{j i}=\sum u_{j i} u_{i j}=u_{2}^{*} \\
u_{3}=\sum u_{i j} u_{j l} u_{l i} \neq \sum u_{l i} u_{j l} u_{i j}=u_{3}^{*}
\end{gathered}
$$

## Non-Commutative Random Matrices

- this allows the calculation of distributions of functions of our non-commutative random matrices $G_{n}^{*}$, in the limit $n \rightarrow \infty$
- in particular, in analogy to Diaconis\&Shashahani, we have results about the asymptotic distribution of $\operatorname{Tr}\left(u^{k}\right)$
- note: in the classical case, knowledge about traces of powers of the matrices is the same as knowledge about the eigenvalues of the matrices


## The Final Question

What actually are eigenvalues of a non-commutative matrix?

## The Final Question

What actually are eigenvalues of a non-commutative matrix?
"Whereof one cannot speak, thereof one must be silent"

