INTRODUCTION TO DISCRETE QUANTUM GROUPS

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Abstract. These are notes from an introductory lecture at the graduate school "Topological Quantum Groups" in Będlewo (June 28–July 11, 2015). The notes focus on introductory material about discrete quantum groups from the point of view of the theory of compact and locally compact quantum groups.

1. Compact quantum groups

Let $\mathbb{G}$ be a compact quantum group defined as the "virtual object" corresponding to a unital C*-algebra denoted $\mathrm{C}(\mathbb{G})$ equipped with a comultiplication (i.e. a coassociative unital $*$-homomorphism) $\Delta_{\mathbb{G}}$ satisfying appropriate density conditions. Let $\mathrm{Irr}(\mathbb{G})$ denote the set of equivalence classes of irreducible representations of $\mathbb{G}$. For each $\alpha \in \mathrm{Irr}(\mathbb{G})$ we choose a unitary representative U^α of α, so

$$U^\alpha \in \mathrm{B}(\mathscr{H}_\alpha) \otimes (\mathbb{G}),$$

where $\mathscr{H}_\alpha$ is a finite dimensional Hilbert space. Denote the dimension of $\mathscr{H}_\alpha$ by n_α. Upon Choosing an orthonormal basis in $\mathscr{H}_\alpha$ we can express U^α as

$$U^\alpha = \sum_{i,j=1}^{n_\alpha} e_{i,j} \otimes u^\alpha_{i,j},$$

where $\{e_{i,j}\}_{i,j=1,\ldots,n_\alpha}$ is the corresponding basis of matrix units in $\mathrm{B}(\mathscr{H}_\alpha)$. In other words U^α becomes a unitary matrix

$$U^\alpha = \begin{bmatrix} u^\alpha_{1,1} & \cdots & u^\alpha_{1,n_\alpha} \\ \vdots & \ddots & \vdots \\ u^\alpha_{n_\alpha,1} & \cdots & u^\alpha_{n_\alpha,n_\alpha} \end{bmatrix}.$$

The fact that each U^α is a representation of $\mathbb{G}$ can be expressed either by saying that

$$\Delta_{\mathbb{G}}(u^\alpha_{i,j}) = \sum_{k=1}^{n_\alpha} u^\alpha_{i,k} \otimes u^\alpha_{k,j}, \qquad i,j = 1,\ldots,n_\alpha$$

or, using the *leg numbering notation*, that

$$(\mathrm{id} \otimes \Delta_{\mathbb{G}})U^\alpha = U^\alpha_{12}U^\alpha_{13},$$

where

$$U^\alpha_{12} = U \otimes \mathbb{1} \in \mathrm{B}(\mathscr{H}) \otimes \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$$

and

$$U^\alpha_{13} = \sum_{i,j=1}^{n_\alpha} e_{i,j} \otimes \mathbb{1} \otimes u^\alpha_{i,j} \in \mathrm{B}(\mathscr{H}) \otimes \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G}).$$

Recall from previous lectures that $\mathrm{Pol}(\mathbb{G})$ defined as the linear span of the set

$$\left\{ u^\alpha_{i,j} \,\middle|\, \alpha \in \mathrm{Irr}(\mathbb{G}),\ i,j = 1,\ldots,n_\alpha \right\}$$

is a dense unital $*$-subalgebra of $\mathrm{C}(\mathbb{G})$ and, moreover, with comultiplication inherited from $\mathrm{C}(\mathbb{G})$, the $*$-algebra $\mathrm{Pol}(\mathbb{G})$ becomes a Hopf $*$-algebra. The antipode of $\mathrm{Pol}(\mathbb{G})$ will be denoted by S.

1.1. **Haar measure and irreducible representations.** Let us denote the Haar measure (Haar state) of $\mathbb{G}$ by $\boldsymbol{h}$.

Proposition 1.1. *For any $\alpha \in \operatorname{Irr}(\mathbb{G})$ there exists a unique $F_\alpha \in \mathrm{B}(\mathscr{H}_\alpha)$ such that*
- ▶ *F_α is invertible,*
- ▶ $(F_\alpha \otimes \mathbb{1})U^\alpha = \big((\mathrm{id} \otimes S^2)U^\alpha\big)(F_\alpha \otimes \mathbb{1})$,
- ▶ $\operatorname{Tr}(F_\alpha) = \operatorname{Tr}(F_\alpha^{-1}) > 0$.

Moreover the operator F_α is positive.

The proof of Proposition 1.1 can be found in [3]. Let us now recall some of the *Peter-Weyl-Woronowicz orthogonality relations*, namely that for each $\alpha \in \operatorname{Irr}(\mathbb{G})$ and all $i, j, k, l \in \{1, \ldots, n_\alpha\}$

$$\boldsymbol{h}\big(u^\alpha_{i,j} {u^\alpha_{k,l}}^*\big) = \delta_{i,k} \frac{[F_\alpha]_{l,j}}{\operatorname{Tr}(F_\alpha)}, \qquad \boldsymbol{h}\big({u^\alpha_{k,l}}^* u^\alpha_{i,j}\big) = \delta_{l,j} \frac{[F_\alpha^{-1}]_{i,k}}{\operatorname{Tr}(F_\alpha)},$$

where $[\,\cdot\,]_{a,b}$ denotes the (a, b)-matrix entry of an operator with respect to the fixed orthonormal basis of $\mathscr{H}_\alpha$.

1.2. **Building the dual of $\mathbb{G}$.** Let $\widehat{\mathsf{A}}$ be the C*-algebra defines as the c$_0$-direct sum

$$\widehat{\mathsf{A}} = \bigoplus_{\alpha \in \operatorname{Irr}(\mathbb{G})} \mathrm{B}(\mathscr{H}_\alpha).$$

Unless we are in the relatively simple situation when $\mathbb{G}$ is *finite* (i.e. $\dim \mathrm{C}(\mathbb{G}) < +\infty$), the C*-algebra $\widehat{\mathsf{A}}$ does not have a unit. Therefore we will be forced to deal with the *multiplier algebra* $\mathrm{M}(\widehat{\mathsf{A}})$ of $\widehat{\mathsf{A}}$. Due to relatively simple structure of $\widehat{\mathsf{A}}$, the multiplier algebra has a very convenient description: $\mathrm{M}(\widehat{\mathsf{A}})$ is the ℓ^∞-direct sum of the finite dimensional blocks $\mathrm{B}(\mathscr{H}_\alpha)$).

Now let

$$\mathrm{W} = \bigoplus_{\alpha \in \operatorname{Irr}(\mathbb{G})} U^\alpha.$$

It is not hard to see that W is a unitary element of $\mathrm{M}\big(\widehat{\mathsf{A}} \otimes \mathrm{C}(\mathbb{G})\big)$ and (suppressing the apparent difficulties with this formula) we have

$$(\mathrm{id} \otimes \Delta_{\mathbb{G}})\mathrm{W} = \mathrm{W}_{12}\mathrm{W}_{13}, \tag{1.1}$$

where again we used the leg numbering notation. Unitarity of W and (1.1) mean that W is an infinite dimensional representation of $\mathbb{G}$. It is a fact that such representations also decompose into direct sums of irreducible ones in an appropriate sense.

Finally let us define a $*$-subalgebra $\widehat{\mathscr{A}}$ of $\widehat{\mathsf{A}}$ as the algebraic direct sum

$$\widehat{\mathscr{A}} = \bigoplus_{\alpha \in \operatorname{Irr}(\mathbb{G})} \mathrm{B}(\mathscr{H}_\alpha).$$

This means that each element of $\widehat{\mathscr{A}}$ has only finitely many non-zero components in the direct summands of $\widehat{\mathsf{A}}$. The algebra $\widehat{\mathsf{A}}$ will play the role of the algebra of continuous functions vanishing at infinity on the dual of $\mathbb{G}$, while $\widehat{\mathscr{A}}$ will correspond to compactly supported functions. It is easy to see that $\widehat{\mathscr{A}}$ is dense in $\widehat{\mathsf{A}}$.

1.3. **Fourier transform.** Let us define a linear map $\mathscr{F}\colon \operatorname{Pol}(\mathbb{G}) \to \widehat{\mathscr{A}}$ by

$$\mathscr{F}(a) = (\mathrm{id} \otimes \boldsymbol{h})\big((\mathbb{1} \otimes a)\mathrm{W}^*\big). \tag{1.2}$$

It turns out that $\mathscr{F}$ is an isomorphism of vector spaces with inverse given by

$$\mathscr{F}^{-1}(x) = (\widehat{\boldsymbol{h}}_{\mathrm{L}} \otimes \mathrm{id})\big((x \otimes \mathbb{1})\mathrm{W}\big), \qquad x \in \widehat{\mathscr{A}}, \tag{1.3}$$

where $\widehat{\boldsymbol{h}}_{\mathrm{L}}$ is the linear functional on $\widehat{\mathscr{A}}$ defined as

$$\widehat{\boldsymbol{h}}_{\mathrm{L}}(x) = \sum_{\beta \in \operatorname{Irr}(\mathbb{G})} \operatorname{Tr}(F_\beta) \operatorname{Tr}(F_\beta^{-1} x_\beta), \qquad x \in \widehat{\mathscr{A}}$$

with x_β denoting the component of x in the block $\mathrm{B}(\mathscr{H}_\beta) \subset \widehat{\mathscr{A}}$ (in particular the above sum is finite).

The proof that (1.2) and (1.3) are mutually inverse to one another involves orthogonality relations for matrix elements of irreducible representations $\{U^\alpha\}_{\alpha\in\mathrm{Irr}(\mathbb{G})}$ (see [2, Section 2]).

2. Main theorem

Theorem 2.1 ([2, Theorem 2.1])**.** *Let* B *be a* C^**-algebra and* $\mathrm{V}\in\mathrm{M}\big(\mathsf{B}\otimes\mathrm{C}(\mathbb{G})\big)$ *a unitary such that*

$$(\mathrm{id}\otimes\Delta_{\mathbb{G}})\mathrm{V}=\mathrm{V}_{12}\mathrm{V}_{13}.$$

Then there exists a unique $\Phi\in\mathrm{Mor}(\widehat{\mathsf{A}},\mathsf{B})$ *such that*

$$(\Phi\otimes\mathrm{id})\mathrm{W}=\mathrm{V}.$$

The formulation of Theorem 2.1 uses the notion of a morphism of C^*-algebras which had not been introduced before. We will not go into the technical difficulties related to morphisms between C^*-algebras and refer the reader to [4, 5].

Proof of Theorem 2.1. First let us address the question of uniqueness of Φ. Assume we have $\Phi\in\mathrm{Mor}(\widehat{\mathsf{A}},\mathsf{B})$ such that $(\Phi\otimes\mathrm{id})\mathrm{W}=\mathrm{V}$. Then for any $a\in\mathrm{Pol}(\mathbb{G})$

$$\begin{aligned}\Phi\big(\mathscr{F}(a)\big)&=\Phi\big((\mathrm{id}\otimes\boldsymbol{h})\big((\mathbb{1}\otimes a)\mathrm{W}^*\big)\big)\\&=(\mathrm{id}\otimes\boldsymbol{h})\big((\Phi\otimes\mathrm{id})\big((\mathbb{1}\otimes a)\mathrm{W}^*\big)\big)\\&=(\mathrm{id}\otimes\boldsymbol{h})\big((\mathbb{1}\otimes a)\mathrm{V}^*\big)\end{aligned}$$

Writing $a=\mathscr{F}^{-1}(x)$ with $x\in\widehat{\mathscr{A}}$ we obtain

$$\Phi(x)=(\mathrm{id}\otimes\boldsymbol{h})\big(\big(\mathbb{1}\otimes\mathscr{F}^{-1}(x)\big)\mathrm{V}^*\big),\qquad x\in\widehat{\mathscr{A}},$$

so Φ is determined uniquely on $\widehat{\mathscr{A}}$ which is dense in $\widehat{\mathsf{A}}$.

For the proof of existence of Φ let us denote the map

$$\mathrm{Pol}(\mathbb{G})\ni a\longmapsto(\mathrm{id}\otimes\boldsymbol{h})\big((\mathbb{1}\otimes a)\mathrm{V}^*\big)\in\mathrm{M}(\mathsf{B})$$

by $\mathscr{F}_{\mathrm{V}}$ and define a linear map $\Phi\colon\widehat{\mathscr{A}}\to\mathrm{M}(\mathsf{B})$ as the composition

$$\Phi=\mathscr{F}_{\mathrm{V}}\circ\mathscr{F}^{-1}.$$

Take $a\in\mathrm{Pol}(\mathbb{G})$. We will compute the expression

$$\mathcal{X}=(\mathrm{id}\otimes\mathrm{id}\otimes\boldsymbol{h})\big((\mathrm{id}\otimes\Delta_{\mathbb{G}})\big((\mathbb{1}\otimes a)\mathrm{V}^*\big)\big)$$

in two ways. First, using the fact that $\boldsymbol{h}$ is the Haar measure we find that

$$\mathcal{X}=\mathscr{F}_{\mathrm{V}}(a)\otimes\mathbb{1}.$$

On the other hand, using the fact that $\Delta_{\mathbb{G}}$ is a $*$-homomorphism and formula $(\mathrm{id}\otimes\Delta_{\mathbb{G}})\mathrm{V}=\mathrm{V}_{12}\mathrm{V}_{13}$, we obtain

$$\begin{aligned}\mathcal{X}&=(\mathrm{id}\otimes\mathrm{id}\otimes\boldsymbol{h})\big(\big(\mathbb{1}\otimes\Delta_{\mathbb{G}}(a)\big)(\mathrm{id}\otimes\Delta_{\mathbb{G}})\mathrm{V}^*\big)\\&=(\mathrm{id}\otimes\mathrm{id}\otimes\boldsymbol{h})\big(\big(\mathbb{1}\otimes\Delta_{\mathbb{G}}(a)\big)\mathrm{V}_{13}^*\mathrm{V}_{12}^*\big)\\&=\big[(\mathrm{id}\otimes\boldsymbol{h})\big(\Delta_{\mathbb{G}}(a)_{23}\mathrm{V}_{13}^*\big)\big]\mathrm{V}^*.\end{aligned}$$

Since V is unitary, we immediately see that this implies

$$(\mathrm{id}\otimes\boldsymbol{h})\big(\Delta_{\mathbb{G}}(a)_{23}\mathrm{V}_{13}^*\big)=\big(\mathscr{F}_{\mathrm{V}}(a)\otimes\mathbb{1}\big)\mathrm{V}.\tag{2.1}$$

Now we can multiply both sides of (2.1) from the right by $\mathbb{1}\otimes b^*$ (with $b\in\mathrm{Pol}(\mathbb{G})$) and apply $(\mathrm{id}\otimes\boldsymbol{h})$ to obtain

$$(\mathrm{id}\otimes\boldsymbol{h})\big[\big((\mathrm{id}\otimes\boldsymbol{h})\big(\Delta_{\mathbb{G}}(a)_{23}\mathrm{V}_{13}^*\big)\big)(\mathbb{1}\otimes b^*)\big]=(\mathrm{id}\otimes\boldsymbol{h})\big(\mathscr{F}(a)_{12}\mathrm{V}(\mathbb{1}\otimes b^*)\big)$$

which can be rewritten as

$$\begin{aligned}\mathscr{F}_{\mathrm{V}}\big((\boldsymbol{h}\otimes\mathrm{id})\big(\Delta(a)(b^*\otimes\mathbb{1})\big)\big)&=\mathscr{F}_{\mathrm{V}}(a)\cdot(\mathrm{id}\otimes\boldsymbol{h})\big(\mathrm{V}(\mathbb{1}\otimes b^*)\big)\\&=\mathscr{F}_{\mathrm{V}}(a)\big[(\mathrm{id}\otimes\boldsymbol{h})\big((\mathbb{1}\otimes b)\mathrm{V}^*\big)\big]^*\\&=\mathscr{F}_{\mathrm{V}}(a)\mathscr{F}_{\mathrm{V}}(b)^*\end{aligned}\tag{2.2}$$

Exactly the same calculation with V replaced by W yields

$$\mathscr{F}\big((\boldsymbol{h}\otimes\mathrm{id})\big(\Delta(a)(b^*\otimes\mathbb{1})\big)\big)=\mathscr{F}(a)\mathscr{F}(b)^* \tag{2.3}$$

for all $a,b\in\mathrm{Pol}(\mathbb{G})$. It follows that if $x=\mathscr{F}(a)$ and $y=\mathscr{F}(b)$ then

$$\begin{aligned}\Phi(xy^*)&=\mathscr{F}_{\mathrm{V}}\big(\mathscr{F}^{-1}(xy^*)\big)\\&=\mathscr{F}_{\mathrm{V}}\big(\mathscr{F}^{-1}\big(\mathscr{F}(a)\mathscr{F}(b)^*\big)\big)\\&\overset{(2.3)}{=}\mathscr{F}_{\mathrm{V}}\big((\boldsymbol{h}\otimes\mathrm{id})\big(\Delta(a)(b^*\otimes\mathbb{1})\big)\big)\\&\overset{(2.2)}{=}\mathscr{F}_{\mathrm{V}}(a)\mathscr{F}_{\mathrm{V}}(b)^*\\&=\mathscr{F}_{\mathrm{V}}\big(\mathscr{F}^{-1}(x)\big)\mathscr{F}_{\mathrm{V}}\big(\mathscr{F}^{-1}(y)\big)^*=\Phi(x)\Phi(y)^*.\end{aligned} \tag{2.4}$$

Now any $z\in\widehat{\mathscr{A}}$ can be written in the form xy^* with $x,y\in\widehat{\mathscr{A}}$ and then $z^*=yx^*$, so

$$\Phi(z^*)=\Phi(yx^*)=\Phi(y)\Phi(x)^*=\big(\Phi(x)\Phi(y)^*\big)^*=\Phi(z)^*,$$

i.e. Φ is a $*$-map. Using this and (2.4) we obtain also multiplicativity of Φ:

$$\Phi(xy)=\Phi\big(x(y^*)^*\big)=\Phi(x)\Phi(y^*)^*=\Phi(x)\Phi(y),\qquad x,y\in\widehat{\mathscr{A}}.$$

There are two points whose proof we will skip, namely:

- ► the map $\Phi\colon\widehat{\mathscr{A}}\to\mathrm{M}(\mathsf{B})$ is continuous and consequently it extends to a $*$-homomorphism of C*-algebras $\widehat{\mathsf{A}}\to\mathrm{M}(\mathsf{B})$,
- ► Φ is non-degenerate, i.e. the span of elements of the form $\Phi(y)b$ with all possible $x\in\widehat{\mathsf{A}}$ and $b\in\mathsf{B}$ is dense in B.

Both are treated in detail in [2].

Let us finish the proof by sketching an argument showing that Φ defined above does indeed satisfy

$$(\Phi\otimes\mathrm{id})\mathrm{W}=\mathrm{V}.$$

To that end let $\widetilde{\mathrm{V}}=(\Phi\otimes\mathrm{id})\mathrm{W}$. For any $a\in\mathrm{Pol}(\mathbb{G})$ we have

$$\begin{aligned}(\mathrm{id}\otimes\boldsymbol{h})\big((\mathbb{1}\otimes a)\widetilde{\mathrm{V}}^*\big)&=(\mathrm{id}\otimes\boldsymbol{h})\big((\mathbb{1}\otimes a)(\Phi\otimes\mathrm{id})\mathrm{W}^*\big)\\&=\Phi\big((\mathrm{id}\otimes\boldsymbol{h})\big((\mathbb{1}\otimes a)\mathrm{W}^*\big)\big)\\&=\Phi\big(\mathscr{F}(a)\big)\\&=\mathscr{F}_{\mathrm{V}}\big(\mathscr{F}^{-1}\big(\mathscr{F}(a)\big)\big)\\&=\mathscr{F}_{\mathrm{V}}(a)=(\mathrm{id}\otimes\boldsymbol{h})\big((\mathbb{1}\otimes a)\mathrm{V}^*\big).\end{aligned}$$

This result is sufficient to conclude that $\widetilde{\mathrm{V}}=\mathrm{V}$, but one has to use decomposition of (infinite dimensional) unitary representations into irreducible ones and orthogonality relations or the argument given on [2, Page 397]. In any case we do get $(\Phi\otimes\mathrm{id})\mathrm{W}=\widetilde{\mathrm{V}}=\mathrm{V}$ which ends the proof. □

3. The dual quantum group

3.1. **Comultiplication.** Let $\mathsf{B}=\widehat{\mathsf{A}}\otimes\widehat{\mathsf{A}}$ and define $\mathrm{V}\in\mathrm{M}\big(\mathsf{B}\otimes\mathrm{C}(\mathbb{G})\big)$ as

$$\mathrm{V}=\mathrm{W}_{23}\mathrm{W}_{13}.$$

We have

$$\begin{aligned}(\mathrm{id}\otimes\Delta_{\mathbb{G}})\mathrm{V}&=\big[(\mathrm{id}\otimes\Delta_{\mathbb{G}})\mathrm{W}\big]_{234}\big[(\mathrm{id}\otimes\Delta_{\mathbb{G}})\mathrm{W}\big]_{134}\\&=\mathrm{W}_{23}\mathrm{W}_{24}\mathrm{W}_{13}\mathrm{W}_{14}\\&=\mathrm{W}_{23}\mathrm{W}_{13}\mathrm{W}_{24}\mathrm{W}_{14}\\&=\mathrm{V}_{12}\mathrm{V}_{13},\end{aligned}$$

where in the first three lines the leg numbers refer to $\widehat{\mathsf{A}}\otimes\widehat{\mathsf{A}}\otimes\mathrm{C}(\mathbb{G})\otimes\mathrm{C}(\mathbb{G})$ and in the last line they refer to $(\widehat{\mathsf{A}}\otimes\widehat{\mathsf{A}})\otimes\mathrm{C}(\mathbb{G})\otimes\mathrm{C}(\mathbb{G})$ (the tensor product $\widehat{\mathsf{A}}\otimes\widehat{\mathsf{A}}$ is treated as one leg). The element V is unitary, so by Theorem 2.1 there exists a unique $\widehat{\Delta}\in\mathrm{Mor}(\widehat{\mathsf{A}},\widehat{\mathsf{A}}\otimes\widehat{\mathsf{A}})$ such that

$$(\widehat{\Delta}\otimes\mathrm{id})\mathrm{W}=\mathrm{W}_{23}\mathrm{W}_{13}.$$

The morphism $\widehat{\Delta}$ is coassociative. Indeed, we have

$$\begin{aligned}\big((\widehat{\Delta}\otimes\mathrm{id})\circ\widehat{\Delta}\otimes\mathrm{id}\big)\mathrm{W}&=(\widehat{\Delta}\otimes\mathrm{id}\otimes\mathrm{id})\mathrm{W}_{23}\mathrm{W}_{13}\\&=(\widehat{\Delta}\otimes\mathrm{id}\otimes\mathrm{id})(\mathrm{W}_{23})\cdot(\widehat{\Delta}\otimes\mathrm{id}\otimes\mathrm{id})(\mathrm{W}_{13})\\&=\mathrm{W}_{34}\cdot\mathrm{W}_{24}\mathrm{W}_{14}\end{aligned}$$

and

$$\begin{aligned}\big((\mathrm{id}\otimes\widehat{\Delta})\circ\widehat{\Delta}\otimes\mathrm{id}\big)\mathrm{W}&=(\mathrm{id}\otimes\widehat{\Delta}\otimes\mathrm{id})\mathrm{W}_{23}\mathrm{W}_{13}\\&=(\mathrm{id}\otimes\widehat{\Delta}\otimes\mathrm{id})(\mathrm{W}_{23})\cdot(\mathrm{id}\otimes\widehat{\Delta}\otimes\mathrm{id})(\mathrm{W}_{13})\\&=\mathrm{W}_{34}\mathrm{W}_{24}\cdot\mathrm{W}_{14}.\end{aligned}$$

In particular

$$\big((\widehat{\Delta}\otimes\mathrm{id})\circ\widehat{\Delta}\otimes\mathrm{id}\big)\mathrm{W}^*=\big((\mathrm{id}\otimes\widehat{\Delta})\circ\widehat{\Delta}\otimes\mathrm{id}\big)\mathrm{W}^*.$$

Multiplying both sides from the left by $(\mathbb{1}\otimes\mathbb{1}\otimes a)$ (with $a\in\mathrm{Pol}(\mathbb{G})$) and applying $(\mathrm{id}\otimes\mathrm{id}\otimes\boldsymbol{h})$ we see that

$$(\widehat{\Delta}\otimes\mathrm{id})\circ\widehat{\Delta}=(\mathrm{id}\otimes\widehat{\Delta})\circ\widehat{\Delta}$$

on $\widehat{\mathscr{A}}$ and since this algebra is dense in $\widehat{\mathsf{A}}$, we obtain coassociativity of $\widehat{\Delta}$.

3.2. **Counit.** Setting $\mathsf{B}=\mathbb{C}$ and $\mathrm{V}=1\otimes\mathbb{1}\in\mathbb{C}\otimes\mathrm{C}(\mathbb{G})$ and using Theorem 2.1 we get a unique character $\widehat{e}$ of $\widehat{\mathsf{A}}$ with the property that

$$(\widehat{e}\otimes\mathrm{id})\mathrm{W}=\mathbb{1}.$$

A similar trick as the one leading to coassociativity of $\widehat{\Delta}$ in Section 3.1 shows that

$$(\widehat{e}\otimes\mathrm{id})\circ\Delta=\mathrm{id}=(\mathrm{id}\otimes\widehat{e})\circ\Delta \tag{3.1}$$

Indeed, for the first formula we compute

$$\begin{aligned}\big(\big((\widehat{e}\otimes\mathrm{id})\circ\Delta\big)\otimes\mathrm{id}\big)\mathrm{W}&=(\widehat{e}\otimes\mathrm{id}\otimes\mathrm{id})(\mathrm{W}_{23}\mathrm{W}_{13})\\&=\mathrm{W}\cdot\big(\mathbb{1}\otimes\big((\widehat{e}\otimes\mathrm{id})\mathrm{W}\big)\big)=\mathrm{W}\cdot(\mathbb{1}\otimes\mathbb{1})=\mathrm{W}\end{aligned}$$

and thus $(\widehat{e}\otimes\mathrm{id})\circ\Delta=\mathrm{id}$ on $\widehat{\mathscr{A}}$ as before. The other formula of (3.1) is proved in the same way.

3.3. **Haar measures.** In Section 1.3 we introduced the functional $\widehat{\boldsymbol{h}}_{\mathrm{L}}$ on $\widehat{\mathscr{A}}$:

$$\widehat{\boldsymbol{h}}_{\mathrm{L}}(x)=\sum_{\beta\in\mathrm{Irr}(\mathbb{G})}\mathrm{Tr}(F_\beta)\,\mathrm{Tr}(F_\beta^{-1}x_\beta),\qquad x\in\widehat{\mathscr{A}}$$

(recall that for $x\in\widehat{\mathscr{A}}$ the symbol x_β denotes the component of x in the direct summand $\mathrm{B}(\mathscr{H}_\beta)$). Similarly let

$$\widehat{\boldsymbol{h}}_{\mathrm{R}}(x)=\sum_{\beta\in\mathrm{Irr}(\mathbb{G})}\mathrm{Tr}(F_\beta)\,\mathrm{Tr}(F_\beta x_\beta),\qquad x\in\widehat{\mathscr{A}}.$$

Then, although for $x\in\widehat{\mathscr{A}}$ the element $\widehat{\Delta}(x)$ does not (usually) belong to $\widehat{\mathscr{A}}\otimes_{\mathrm{alg}}\widehat{\mathscr{A}}$, one can make sense of the expressions

$$(\mathrm{id}\otimes\widehat{\boldsymbol{h}}_{\mathrm{L}})\widehat{\Delta}(x),\qquad(\widehat{\boldsymbol{h}}_{\mathrm{R}}\otimes\mathrm{id})\widehat{\Delta}(x)$$

and, moreover, show that we in fact have

$$\begin{aligned}(\mathrm{id}\otimes\widehat{\boldsymbol{h}}_{\mathrm{L}})\widehat{\Delta}(x)&=\widehat{\boldsymbol{h}}_{\mathrm{L}}(x)\mathbb{1}_{\mathrm{M}(\mathsf{B})},\\(\widehat{\boldsymbol{h}}_{\mathrm{R}}\otimes\mathrm{id})\widehat{\Delta}(x)&=\widehat{\boldsymbol{h}}_{\mathrm{R}}(x)\mathbb{1}_{\mathrm{M}(\mathsf{B})},\end{aligned}\qquad x\in\widehat{\mathscr{A}}.$$

Thus $\widehat{\boldsymbol{h}}_{\mathrm{L}}$ and $\widehat{\boldsymbol{h}}_{\mathrm{R}}$ are respectively left and right invariant functionals on $\widehat{\mathscr{A}}$. They extend to so called *weights* on the C*-algebra $\widehat{\mathscr{A}}$ which have many desirable properties (e.g. are *faithful* and *locally finite* as well as faithful when appropriately extended to $\mathrm{M}(\widehat{\mathsf{A}})$). In what follows we will refer to $\widehat{\boldsymbol{h}}_{\mathrm{L}}$ and $\widehat{\boldsymbol{h}}_{\mathrm{R}}$ as the left and right *Haar measure* on the dual of $\mathbb{G}$.

3.4. **Definition of a discrete quantum group.** We have by now established a number of properties of the objects

$$(\widehat{\mathsf{A}},\widehat{\Delta},\widehat{\boldsymbol{h}}_{\mathrm{L}},\widehat{\boldsymbol{h}}_{\mathrm{R}}).$$

On way of summarizing some of them is that they satisfy the axioms of a *reduced* C^**-algebraic locally compact quantum group* as defined by J. Kustermans and S. Vaes in [1, Definition 4.1]. This quantum group is usually denoted by the symbol $\widehat{\mathbb{G}}$. Accordingly we introduce new notation:

We write	For
$\mathrm{c}_0(\widehat{\mathbb{G}})$	$\widehat{\mathsf{A}}$
$\mathrm{c}_{00}(\widehat{\mathbb{G}})$	$\widehat{\mathscr{A}}$
$\Delta_{\widehat{\mathbb{G}}}$	$\widehat{\Delta}$

Locally compact quantum groups obtained from compact quantum groups via the procedure described above are called *discrete*. One can show that a locally compact quantum group $\mathbb{G}$ is discrete if and only if the corresponding C^*-algebra is a direct sum of finite dimensional algebras (and many other equivalent conditions for a locally compact quantum group to be discrete can be given).

3.5. **(Non-)Unimodularity.** The collection $(F_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}$ of operators introduced in proposition 1.1 is not necessarily bounded (in norm), so it is not an element of either $\mathrm{c}_0(\widehat{\mathbb{G}})$ nor of $\mathrm{M}\big(\mathrm{c}_0(\widehat{\mathbb{G}})\big)$. It is, however, an element *affiliated* with $\mathrm{c}_0(\widehat{\mathbb{G}})$ (see [4]) which means that it can be thought of as a possibly unbounded continuous function on the quantum space $\widehat{\mathbb{G}}$. Let us denote this element simply by F.

One can show that the following formulas hold for any $x\in\mathrm{c}_{00}(\widehat{\mathbb{G}})$:

$$\begin{aligned}(\widehat{\boldsymbol{h}}_{\mathrm{L}}\otimes\mathrm{id})\widehat{\Delta}(x)&=\widehat{\boldsymbol{h}}_{\mathrm{L}}(x)F^2,\\(\mathrm{id}\otimes\widehat{\boldsymbol{h}}_{\mathrm{R}})\widehat{\Delta}(x)&=\widehat{\boldsymbol{h}}_{\mathrm{R}}(x)F^{-2}.\end{aligned}$$

This means that the failure of $\widehat{\boldsymbol{h}}_{\mathrm{L}}$ to be right invariant (as well as failure of $\widehat{\boldsymbol{h}}_{\mathrm{R}}$ to be left invariant) is controlled by F^2 (F^{-2} respectively) in much the same way as the modular function of a locally compact group controls similar phenomena for a non-unimodular group. For that reason F is called the *modular element* for $\widehat{\mathbb{G}}$. Since there are compact quantum groups with non-trivial matrices $(F_\alpha)_{\alpha\in\mathrm{Irr}(\mathbb{G})}$, we see that discrete quantum groups may be non-unimodular.

Let us finish with the following theorem which can be found in [6, 2]:

Theorem 3.1. *Let $\mathbb{G}$ be a compact quantum group. Then the following statements are equivalent:*
1. *the Haar measure of $\mathbb{G}$ is a trace,*
2. *$\widehat{\boldsymbol{h}}_L=\widehat{\boldsymbol{h}}_R$ on $\widehat{\mathbb{G}}$, i.e. $\widehat{\mathbb{G}}$ is unimodular,*
3. *$F=\mathbb{1}$,*
4. *the antipode S of $\mathbb{G}$ satisfies $S^2=\mathrm{id}$,*
5. *the antipode S is bounded.*

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