The classification of unital graph C^*-algebras

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Content

Cuntz-Krieger 1980

4. Flow Equivalence

Topological Markov chains are said to be flow equivalent if their suspension flows act on spaces that are homeomorphic under homeomorphisms that respect the orientation of the orbits [11]. Equivalently they are flow equivalent if they induce isomorphic chains on some closed open subset, that is, if they are Kakutani equivalent. Parry and Sullivan have given a description of flow equivalence in terms of a matrix operation [11]. This description leads to a sort of instant computational proof of the invariance of the pair $(\bar{\mathcal{O}}_T, \bar{\mathcal{D}})$ under flow equivalence. We want to give this proof here. We point out, however, that a conceptual proof of this fact is also possible if one exploits the circumstance that $\bar{\mathcal{O}}_T$ arises as a crossed product.

4.1. **Theorem.** *If T_1 and T_2 are flow equivalent then*

$$(\bar{\mathcal{O}}_{T_1}, \bar{\mathcal{D}}) \sim (\bar{\mathcal{O}}_{T_2}, \bar{\mathcal{D}}).$$

Proof. From the transition matrix $A=(a_{ij})_{1 \leqq i,j \leqq n}$, form the transition matrix

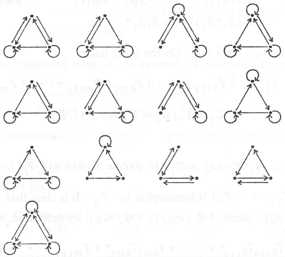
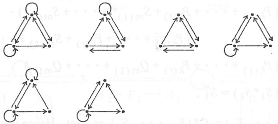
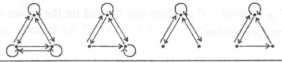
$$\tilde{A} = \begin{pmatrix} 0 & a_{11} & \dots & a_{1n} \\ 1 & 0 & \dots & 0 \\ 0 & a_{21} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

According to Parry and Sullivan, to prove the theorem it is enough to prove that

$$(\bar{\mathcal{O}}_{\bar{\sigma}_A}, \bar{\mathcal{D}}) \sim (\bar{\mathcal{O}}_{\bar{\sigma}_{\tilde{A}}}, \bar{\mathcal{D}}).$$

Enomoto-Fujii-Watatani 1981

The classification table of $\mathcal{O}_A$ for 3×3 irreducible matrices.

$K_0(\mathcal{O}_A)$	marker	digraph	representative
0	$\bar{0}$		$\mathcal{O}_2$
$\mathbb{Z}_2$	$\bar{0}$		$\mathcal{O}_3 \otimes M_2$
	$\bar{1}$		$\mathcal{O}_3$

Rørdam 1995

The class of all simple Cuntz–Krieger algebras is classified by K-theory. This is proved using a theorem of Cuntz, see the appendix, and the two Cuntz–Krieger algebras $\mathcal{O}_2$ and $\mathcal{O}_{2_-}$, where $\mathcal{O}_2$ corresponds to the 1×1 matrix (2) – or the 2×2 matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ – and

$$2_- = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Notice that $\det(1 - 2) = -1$ and $\det(1 - 2_-) = 1$.

LEMMA 6.4. *$\mathcal{O}_2$ is isomorphic to $\mathcal{O}_{2_-}$.*

Proof. Both C^*-algebras have trivial K-theory (both K_0 and K_1 are trivial), and so they are isomorphic by Theorem 6.2. □

The second part of the theorem below is due to Joachim Cuntz.

THEOREM 6.5. *Two simple Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_{A'}$ are stably isomorphic if and only if $K_0(\mathcal{O}_A)$ is isomorphic to $K_0(\mathcal{O}_{A'})$, and $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_{A'}$ if and only if $(K_0(\mathcal{O}_A), [1])$ and $(K_0(\mathcal{O}_{A'}), [1])$ are isomorphic (i.e. if there is a group isomorphism $K_0(\mathcal{O}_A) \to K_0(\mathcal{O}_{A'})$ that carries the class of the unit of $\mathcal{O}_A$ onto the class of the unit of $\mathcal{O}_{A'}$).*

Timeline

Classification results

- 1995: Simple Cuntz-Krieger algebras [Rørdam]
- 1997: Cuntz-Krieger algebras with a unique ideal [Rørdam]
- 2006: Cuntz-Krieger algebras with finitely many ideals [Restorff]

Timeline

Classification results

- 1995: Simple Cuntz-Krieger algebras [Rørdam]
- 1997: Cuntz-Krieger algebras with a unique ideal [Rørdam]
- 2006: Cuntz-Krieger algebras with real rank zero [Restorff]

Outline

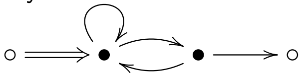
Definition

A graph is a tuple (E^0, E^1, r, s) with

$$r, s : E^1 \to E^0$$

and E^0 and E^1 countable sets.

We think of $e \in E^1$ as an edge from $s(e)$ to $r(e)$ and often represent graphs visually



or by an adjacency matrix

$$\mathsf{A}_E = \begin{bmatrix} 0 & \infty & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Singular and regular vertices

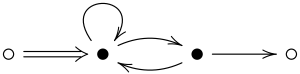
Definitions

Let E be a graph and $v \in E^0$.

- v is a *sink* if $|s^{-1}(\{v\})| = 0$
- v is an *infinite emitter* if $|s^{-1}(\{v\})| = \infty$

Definition

v is *singular* [$\circ$] if v is a sink or an infinite emitter. v is *regular* [$\bullet$] if it is not singular.



Definition

The *graph C^*-algebra* $C^*(E)$ is given as the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges subject to the Cuntz-Krieger relations

1. $s_e^* s_e = p_{r(e)}$
2. $s_e s_e^* \leq p_{s(e)}$
3. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every regular v

$C^*(E)$ is unital precisely when E has finitely many vertices.

Example

$\mathbb{C}$, $M_2(\mathbb{C})$, $\mathbb{K}$, $\mathcal{O}_2$, $\mathcal{E}_2$, $\mathcal{O}_\infty$, $\mathcal{T}$, $M_{2^\infty} \otimes \mathbb{K}$,$\mathbb{K}^\sim$,...

Observation

$$\gamma_z(p_v) = p_v \qquad \gamma_z(s_e) = zs_e$$

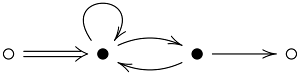
induces a **gauge action** $\mathbb{T} \mapsto \mathrm{Aut}(C^*(E))$

Theorem

Gauge invariant ideals are induced by **hereditary** *and* **saturated** *sets of vertices V:*

- $s(e) \in V \implies r(e) \in V$
- $r(s^{-1}(v)) \subseteq V \implies$ *[$v \in V$ or v is singular]*

and when there are no **breaking vertices**, *all such ideals arise this way.*



The gauge simple trichotomy

Theorem

If a graph C^*-algebra has no non-trivial gauge invariant ideals, it is either

- ■ *a simple AF algebra;*
- ■ *a Kirchberg algebra; or*
- □ $C(\mathbb{T}) \otimes \mathbb{K}(H)$ *for some Hilbert space* H.

It is easy to tell from the graph which case occurs: The first case occurs when the graph has no cycles; the second when one vertex supports several cycles.

Lemma

A graph C^*-algebra has real rank zero precisely when no gauge simple subquotient is □

The gauge simple trichotomy

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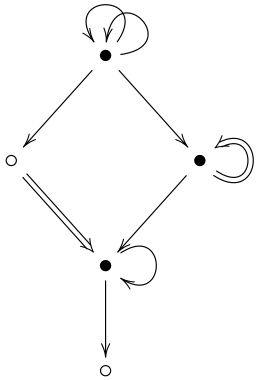
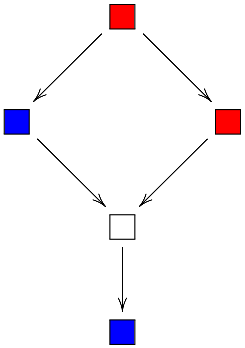


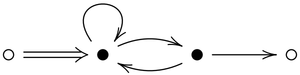


Outline

Timeline

Classification results

- 1998: Simple graph C^*-algebras [Kumjian-Pask-Raeburn]
- 2010: Graph C^*-algebras with a unique ideal [E-Tomforde]
- 2013: Graph C^*-algebras with restricted mixing [E-Restorff-Ruiz]

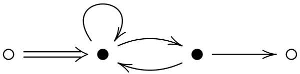


The K-theory of $C^*(E)$ is easily computed by the *regular adjacency matrix* A_E° obtained by deleting all rows at singular vertices:

Theorem

$K_0(C^*(E)) = \operatorname{coker}(\mathsf{A}_E^\circ - I^\circ)^T$ *and* $K_1(C^*(E)) = \ker(\mathsf{A}_E^\circ - I^\circ)^T$, *where* $K_0(C^*(E))_+$ *is precisely the elements having a non-negative representative.*

$$\left(\begin{bmatrix} 0 & \infty & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)^T$$



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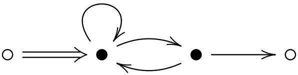

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$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The gauge simple trichotomy

Theorem (Kumjian-Pask-Raeburn 1998)

For $C^*(E)$ with no non-trivial gauge invariant ideals we have one of

- *a simple AF algebra;*
- *a Kirchberg algebra; or*
- $C(\mathbb{T}) \otimes \mathbb{K}(H)$ *for some Hilbert space H.*

The gauge simple trichotomy

Theorem (Kumjian-Pask-Raeburn 1998)

For $C^*(E)$ with no non-trivial gauge invariant ideals we have one of

- $K_0(C^*(E))_+ \neq K_0(C^*(E))$, $K_1(C^*(E)) = 0$;
- *a Kirchberg algebra; or*
- $C(\mathbb{T}) \otimes \mathbb{K}(H)$ *for some Hilbert space* H.

The gauge simple trichotomy

Theorem (Kumjian-Pask-Raeburn 1998)

For $C^*(E)$ with no non-trivial gauge invariant ideals we have one of

- $K_0(C^*(E))_+ \neq K_0(C^*(E))$, $K_1(C^*(E)) = 0$;
- $K_0(C^*(E))_+ = K_0(C^*(E))$; *or*
- $C(\mathbb{T}) \otimes \mathbb{K}(H)$ *for some Hilbert space* H.

The gauge simple trichotomy

Theorem (Kumjian-Pask-Raeburn 1998)

For $C^*(E)$ with no non-trivial gauge invariant ideals we have one of

- ■ (blue) $K_0(C^*(E))_+ \neq K_0(C^*(E))$, $K_1(C^*(E)) = 0$;
- ■ (red) $K_0(C^*(E))_+ = K_0(C^*(E))$; *or*
- □ $K_0(C^*(E))_+ \neq K_0(C^*(E))$, $K_1(C^*(E)) \neq 0$.

The gauge simple trichotomy

Theorem (Kumjian-Pask-Raeburn 1998)

For $C^*(E)$ with no non-trivial gauge invariant ideals we have one of

- $K_0(C^*(E))_+ \neq K_0(C^*(E))$, $K_1(C^*(E)) = 0$;
- $K_0(C^*(E))_+ = K_0(C^*(E))$; *or*
- $K_0(C^*(E))_+ \neq K_0(C^*(E))$, $K_1(C^*(E)) \neq 0$.

Thus, we can appeal to either Elliott or Kirchberg-Phillips to complete the classification of gauge simple graph C^*-algebras.

Filtered K-theory

Definition

Let $\mathfrak{A}$ be a C^*-algebra with only finitely many gauge invariant ideals. The collection of all sequences

$$\begin{array}{ccccc} K_0(\mathfrak{J}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{J}) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{K}/\mathfrak{J}) & \longleftarrow & K_1(\mathfrak{K}/\mathfrak{I}) & \longleftarrow & K_1(\mathfrak{J}/\mathfrak{I}) \end{array}$$

with gauge invariant $\mathfrak{I} \lhd \mathfrak{J} \lhd \mathfrak{K} \lhd \mathfrak{A}$ is called the *filtered K-theory* of $\mathfrak{A}$ and denoted $\mathsf{FK}^{\gamma}(\mathfrak{A})$. Equipping all K_0-groups with order we arrive at the *ordered, filtered K-theory* $\mathsf{FK}^{\gamma,+}(\mathfrak{A})$.

Working conjecture [E-Restorff-Ruiz 2010]

$\mathsf{FK}^{\gamma,+}(-)$ is a complete invariant, up to stable isomorphism, for graph C^*-algebras of real rank zero and finitely many (gauge invariant) ideals.

- Confirmed in the non-mixed cases: ■ by Elliott 1976 and ■ by Bentmann-Meyer 2014 amended by Restorff-Ruiz.
- First open cases have 3 ideals and K-groups not finitely generated.
- No counterexamples are known, even allowing for □ subquotients. The non-mixed □ case is in fact confirmed by the work presented today.

Outline

Move **(S)**

Remove a regular source, as

Move **(R)**

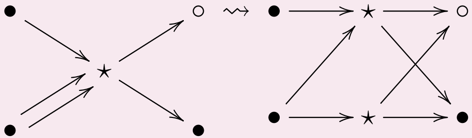
Reduce a configuration with a transitional regular vertex, as

or

Moves

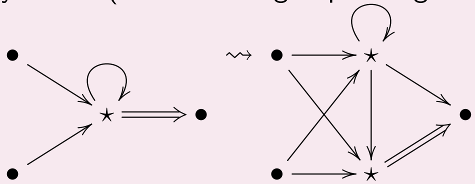
Move (I)

Insplit at regular vertex



Move (O)

Outsplit at any vertex (at most one group of edges infinite)



Definition

$E \sim_{\mathrm{ME}} F$ when there is a finite sequence of moves of type

(S),**(R)**,**(O)**,**(I)**,

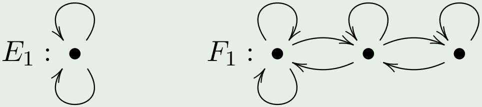
and their inverses, leading from E to F.

Theorem (Cuntz-Krieger, Bates-Pask)

$$E \sim_{\mathrm{ME}} F \Longrightarrow C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$$

Example (Cuntz 1986, Rørdam 1995)

When



we get that

$$C^*(E_1) \otimes \mathbb{K} \simeq C^*(F_1) \otimes \mathbb{K},$$

yet $E_1 \not\sim_{\mathrm{ME}} F_1$.

Move **(C)**

"Cuntz splice" on a vertex supporting two cycles

Definition

$E \sim_{\mathrm{CE}} F$ when there is a finite sequence of moves of type

(S),(R),(O),(I),(C)

and their inverses, leading from E to F.

Theorem (E-Restorff-Ruiz-Sørensen 2015)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras with **real rank zero**. *Then the following are equivalent*

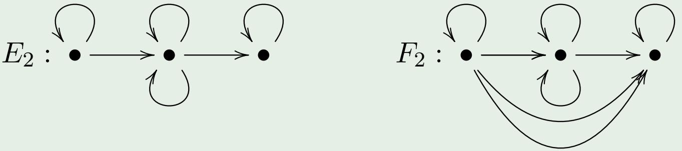
(i) $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

(ii) $E \sim_{\mathrm{CE}} F$

(iii) $\mathsf{FK}^{\gamma,+}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}(C^*(F))$

Example (E-Restorff-Ruiz-Sørensen 2015)

When



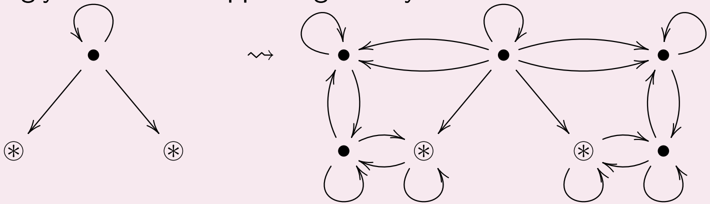
we get that

$$C^*(E_2) \otimes \mathbb{K} \simeq C^*(F_2) \otimes \mathbb{K},$$

yet $E_2 \not\sim_{\mathrm{CE}} F_2$.

Move **(P)**

"Butterfly move" on a vertex supporting a single cycle emitting only singly to vertices supporting two cycles



Definition

$E \sim_{\mathrm{PE}} F$ when there is a finite sequence of moves of type

(S),**(R)**,**(O)**,**(I)**,**(C)**,**(P)**

and their inverses, leading from E to F.

Theorem (E-Restorff-Ruiz-Sørensen)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras. Then the following are equivalent

(i) $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

(ii) $E \sim_{\mathrm{PE}} F$

(iii) $\mathsf{FK}^{\gamma,+}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}(C^*(F))$

Outline

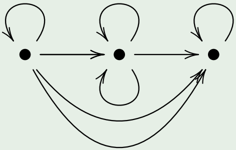
(iii)$\Longrightarrow$(ii)

Lemma

For any pair of graphs (E, F) with $\mathsf{FK}^{\gamma,+}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}(C^*(F))$ there is a pair of graphs (E', F') so that the regular adjacency matrices have identically, suitably sized upper triangular block matrix forms, and so that $E \sim_{\mathrm{ME}} E'$ and $F \sim_{\mathrm{ME}} F'$. We say that (E', F') is in **standard form**.

Example

F_2 :



$\rightsquigarrow$

$$\left[\begin{array}{c|cccc} 1 & 1 & 0 & 0 & 2 \\ \hline 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

(iii)$\Longrightarrow$ (ii)

Amended from symbolic dynamics (Boyle-Huang):

Proposition

When (E, F) is in standard form, we have

- If $\mathsf{FK}^{\gamma,+}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}(C^*(F))$, then there exist $U, V \in \mathsf{GL}^{\boxplus}(\mathbb{Z})$ so that
$$V(\mathsf{A}_E^\circ - I^\circ)^T = (\mathsf{A}_F^\circ - I^\circ)^T U$$
with $\det V\{i\} = 1$ at all ■ or □ blocks.
- If there exist $U, V \in \mathsf{SL}^{\boxplus}(\mathbb{Z})$ so that
$$V(\mathsf{A}_E^\circ - I^\circ)^T = (\mathsf{A}_F^\circ - I^\circ)^T U,$$
then $E \sim_{\mathrm{ME}} F$.

(iii) $\Longrightarrow$ (ii)

Restorff showed how to use **(C)** to arrange $\det U\{i\} = \det V\{i\} = 1$ at all ■ blocks, and the ■ case is automatic, leaving the □ case. Comparing the essential parts of the K-theory of E_2, F_2:

$$\begin{array}{ccccccc} & & (\mathbb{Z}, \mathbb{Z}^+) & & & & \\ & & \big\downarrow{\scriptstyle 1} & & & & \\ \mathbb{Z} & \xrightarrow[\pm 1]{} & (\mathbb{Z}, \mathbb{Z}) & \xrightarrow[0]{} & (\mathbb{Z}, \mathbb{Z}) & \longrightarrow & (\mathbb{Z}, \mathbb{Z}^+) \end{array}$$

one sees that in this case, $\det U\{i\} = -1$ is inevitable.

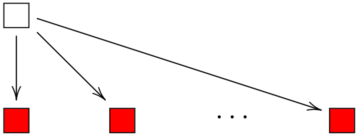
(iii)$\Longrightarrow$ (ii)

The **(P)** move exactly changes the sign of the map $K_1 \to K_0$ at the appropriate location. To arrange to apply it, we use an old idea:

Definition

We say that E satisfies *Condition (H)* if for any regular vertex v supporting a unique cycle, either this path has no exit, or there is a vertex $w \neq v$ which is singular or supports a unique cycle so that there is a path from v to w, and so that any path from v to w passes through vertices not supporting two distinct cycles.

This rules out the configuration



(iii)$\Longrightarrow$ (ii)

Theorem (E-Restorff-Ruiz-Sørensen 2015)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras with **real rank zero**. *Then the following are equivalent*

(i) $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

(ii) $E \sim_{\mathrm{CE}} F$

(iii) $\mathsf{FK}^{\gamma,+}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}(C^*(F))$

(iii)$\Longrightarrow$ (ii)

Theorem (E-Restorff-Ruiz-Sørensen 2015)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras with **Condition (H)**. *Then the following are equivalent*

(i) $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

(ii) $E \sim_{\mathrm{CE}} F$

(iii) $\mathsf{FK}^{\gamma,+}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}(C^*(F))$

(ii)$\Longrightarrow$(i)

General invariance of moves

(S)	Obvious	
(I)	Bates-Pask	2004
(O)	Bates-Pask	2004
(R)	Bates-Pask	2004
(C)	Bentmann, ERRS	2016

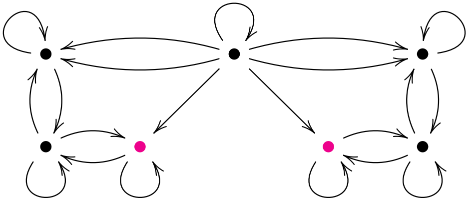
We only know that **(P)** is invariant when applied to unital graph C^*-algebras.

(ii)⟹(i)

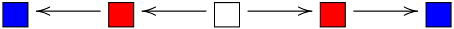
With a legal **(P)** move indicated by "#", the key steps are:

1. $C^*(E) \otimes \mathbb{K} \simeq C^*(E_{\#,\#}) \otimes \mathbb{K}$
2. $E \sim_{\mathrm{ME}} E_{\#,\#}$
3. $C^*(F_\#, S_\#) \simeq C^*(F_{\#,\#}, S_{\#,\#})$

where the C^*-algebras in (iii) are *relative* graph algebras given by specific graphs such as



with structure



which is just within reach of classification.

Outline

Decidability

Theorem (E-Restorff-Ruiz-Sørensen)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras in standard form. Then the following are equivalent

(i) $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

(ii) $E \sim_{\mathrm{PE}} F$

(iii) *There exist* $U, V \in \mathsf{GL}^{\boxplus}(\mathbb{Z})$ *so that*

$$V(\mathsf{A}_E^\circ - I^\circ)^T = (\mathsf{A}_F^\circ - I^\circ)^T U$$

with $\det V\{i\} = 1$ *at all* ■ *or* □ *blocks.*

Arranging for standard form is algorithmic, and if there are no ■ blocks, the condition in (iii) is linear. But in general, the invertibility condition takes us into the realm of Hilbert's tenth problem.

Decidability

Solution as amended by Boyle and Steinberg: Apply the Grunewald–Segal theorem!

Theorem (Boyle-Steinberg, E-Restorff-Ruiz-Sørensen)

Stable isomorphism among unital graph C^*-algebras is decidable.

Open problem

Find a concrete implementation!

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Exact classification

In general, exact ("on-the-nose") classification can be obtained by stable classificaion when the result is **strong**!

Observation [Arklint-Restorff-Ruiz]

$\mathrm{FK}^{\gamma,+}(-)$ fails to give strong classification already for Cuntz-Krieger algebras of real rank zero.

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Exact classification

Theorem (E-Restorff-Ruiz-Sørensen)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras. Then the following are equivalent

(i) $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

(ii) $E \sim_{\mathrm{PE}} F$

(iii) $\mathsf{FK}^{\gamma,+}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}(C^*(F))$

(iv) $\mathsf{FK}^{\gamma,+}_{\mathsf{red}}(C^*(E)) \simeq \mathsf{FK}^{\gamma,+}_{\mathsf{red}}(C^*(F))$

and any given isomorphism on $\mathsf{FK}^{\gamma,+}_{\mathsf{red}}(-)$ lifts to a $*$-isomorphism.

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Exact classification

Corollary [E-Restorff-Ruiz-Sørensen]

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras. Then the following are equivalent

(i) $C^*(E) \simeq C^*(F)$

(ii) $(\mathsf{FK}^{\gamma,+}_{\mathsf{red}}(C^*(E)), [1]) \simeq (\mathsf{FK}^{\gamma,+}_{\mathsf{red}}(C^*(F)), [1])$

We (Arklint-E-Ruiz) have a complete list of exact moves, but not a very aestethically appealing one.

Application: Quantum lens spaces

Definition

The *Vaksman-Soibelman odd quantum sphere* $C(S_q^{2n-1})$ is the universal C^*-algebra for generators $z_1, \ldots, z_n$ subject to

$$\begin{aligned} z_j z_i &= q z_i z_j \qquad i < j \\ z_j^* z_i &= q z_i z_j^* \qquad i \neq j \\ z_i^* z_i &= z_i z_i^* + (1 - q^2) \sum_{j>i} z_j z_j^* \\ 1 &= \sum_{i=1}^{n} z_i z_i^* \end{aligned}$$

for $q \in (0, 1)$.

Application: Quantum lens spaces

Let n and r be given, set $\theta = e^{2\pi i/r}$ and note that

$$\Lambda_{\underline{m}}(z_i) = \theta^{m_i} z_i$$

with $\underline{m} = (m_1, \ldots, m_n)$ defines $\Lambda_{\underline{m}} \in \operatorname{Aut} C(S_q^{2n-1})$ when $(m_i, r) = 1$ for all i.

Definition [Hong-Szymanski 2002]

Given r, n, and $\underline{m} \in \mathbb{N}^n$. The **quantum lens space** $C(L_q^{2n-1}(r; \underline{m}))$ is the fixed point space

$$C(S_q^{2n-1})^{\Lambda_{\underline{m}}}$$

Theorem (Hong-Szymanski 2002)

$C(L_q^{2n-1}(r; \underline{m}))$ is a unital graph C^*-algebra, in fact a Cuntz-Krieger algebra, with all blocks of type $\square$.

Application: Quantum lens spaces

Let us say that $C(L_q^{2n-1}(r;\underline{m}))$ **depends on** $\underline{m}$ when for some $\underline{m}$ and $\underline{m}'$, we have

$$C(L_q^{2n-1}(r;\underline{m})) \not\simeq C(L_q^{2n-1}(r;\underline{m}'))$$

Theorem (E-Restorff-Ruiz-Sørensen, Jensen-Klausen-Rasmussen)

$C(L_q^{2n-1}(r;\underline{m}))$ *depends on* $\underline{m}$ *precisely when*

$$n \geq 2b, \qquad 2b > a > 2, \qquad a \mid r$$

2	3	4	5	6	7	8	9	10	11	12	13
∞	4	6	6	4	8	6	4	6	12	4	14

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Characterizing $\sim_{\mathrm{ME}}$:

$C^*(E)$ contains a canonical abelian subalgebra $\mathcal{D}_E$ which is Cartan under modest assumptions.

Conjecture

The following are equivalent

(i) $E \sim_{\mathrm{ME}} F$

(ii) $(C^*(E) \otimes \mathbb{K}, \mathcal{D}_E \otimes c_0) \simeq (C^*(F) \otimes \mathbb{K}, \mathcal{D}_F \otimes c_0)$

Evidence

- (i)$\Longrightarrow$ (ii) holds as noted by Cuntz-Krieger.
- Confirmed when $C^*(E)$ is simple (Matsumoto-Matui 2014, Sørensen 2013)
- Confirmed for Cuntz-Krieger algebras (Arklint-E-Ruiz, Carlsen-E-Restorff-Ruiz)
- $(C^*(E_2) \otimes \mathbb{K}, \mathcal{D}_{E_2} \otimes c_0) \not\simeq (C^*(F_2) \otimes \mathbb{K}, \mathcal{D}_{F_2} \otimes c_0)$

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Open problem

Confirm the conjecture!