Twists over étale groupoids and twisted vector bundles

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Motivation

For a groupoid $\mathcal{G}$ with $\mathcal{G}^{(0)} = M$ and a twist $\mathcal{R}$ over $\mathcal{G}$,

$$K_0(C_r^*(\mathcal{G};\mathcal{R}))$$

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When do twisted vector bundles exist?

Theorem (Farsi-G, [FG16])

Let $\mathcal{G}$ be an étale groupoid and let $\mathcal{R}$ be a twist over $\mathcal{G}$, of order n in $H^2(\mathcal{G}, \mathcal{S})$. Suppose that the classifying space $B\mathcal{G}$ is a compact CW complex, and that the principal $PU(n)$-bundle over $\mathcal{G}^{(0)}$ induced by $\mathcal{R}$ lifts to a $U(n)$ principal bundle. Then $\mathcal{R}$ admits a twisted vector bundle.

Outline

- Definitions

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- Proof of main Theorem

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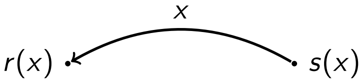

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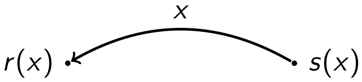
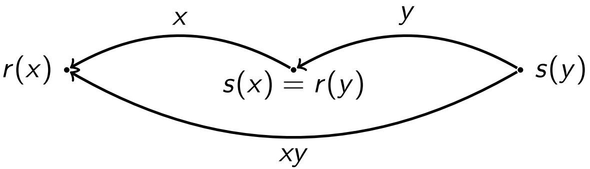


Then, the product xy is defined iff $s(x) = r(y)$:



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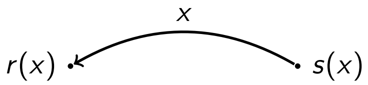

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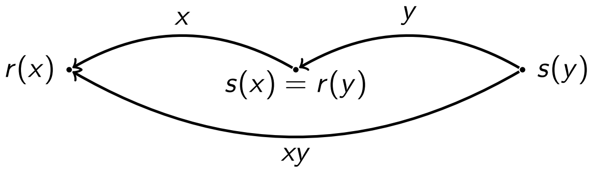

We write $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ for the set of composable arrows.

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- An equivalence relation $\sim$ on X gives rise to a groupoid R:

$$R = \{(x,y) \subseteq X \times X : x \sim y\}$$

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- A vector bundle is NOT étale.

Twists over groupoids

A twist over a (topological) groupoid $\mathcal{G}$ is a principal $\mathbb{T}$-bundle $p : \mathcal{R} \to \mathcal{G}$, such that $\mathcal{R}$ is also a groupoid in a compatible way:

$$r(\gamma) = r(p(\gamma)), \quad s(\gamma) = s(p(\gamma)), \quad p(\gamma\eta) = p(\gamma)p(\eta) \, \forall \, (\gamma, \eta) \in \mathcal{R}^{(2)}$$

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Example

If $c : \mathcal{G}^{(2)} \to \mathbb{T}$ is a continuous 2-cocycle, then $\mathcal{G} \times \mathbb{T}$ is a twist over $\mathcal{G}$:

$$(g, z)(h, w) := (gh, c(g, h)zw).$$

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Product of twists is given by Baer sum:

$$\mathcal{R}_1 * \mathcal{R}_2 = (\mathcal{R}_1 \times_{\mathcal{G}} \mathcal{R}_2)/\sim, \quad (\gamma_1, z\gamma_2) \sim (z\gamma_1, \gamma_2)\ \forall\ z \in \mathbb{T}$$

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If $\mathcal{G}$ is étale, each fiber $\mathcal{G}^u$ is discrete, so we take λ^u to be counting measure.

Reduced groupoid C^*-algebras

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$$C_r^*(\mathcal{G}) = \overline{C_c(\mathcal{G})} \subseteq B(L^2(\mathcal{G}, \nu^{-1})).$$

Twisted groupoid C^*-algebras

Given a twist $p : \mathcal{R} \to \mathcal{G}$,

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If $\mathcal{R}$ arises from a 2-cocycle c, then $C_r^*(\mathcal{G}; \mathcal{R}) \cong \overline{C_c(\mathcal{G}, c)}$:

$$f *_c g(\gamma) = \int_{\mathcal{G}} f(\gamma\eta) g(\eta^{-1}) c(\gamma\eta, \eta^{-1}) \, d\lambda^{s(\gamma)}(\eta).$$

Twisted vector bundles

Let $\mathcal{R}$ be a twist over $\mathcal{G}$ (principal $\mathbb{T}$-bundle over $\mathcal{G}$). A twisted vector bundle is a vector bundle

$$\pi : E \to \left(\mathcal{G}^{(0)} = \mathcal{R}^{(0)}\right)$$

which admits an action of $\mathcal{R}$ such that, for all $z \in \mathbb{T},\ \gamma \in \mathcal{R},\ e \in E$ with $\pi(e) = s(\gamma)$,

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Proposition (TXLG)

If $(\mathcal{G}, \mathcal{R})$ admits a twisted vector bundle of rank n, then $\mathcal{R}$ represents a class of order n in $H^2(\mathcal{G}, \mathcal{S})$.

Proof:

Classifying space

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Think of groupoid k-tuples as labeling k-simplices: the various possible partial products label the faces.

Classifying space

Note:

- For all k, the map $\phi_k : \mathcal{G}^{(k)} \to B\mathcal{G}$ given by

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- [Wil08] If $\mathcal{G} = M \rtimes G$, then $B\mathcal{G} \cong M \times_G \mathcal{E}G$, where $\mathcal{E}G$ is any contractible space with a free action of G.

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When do these hypotheses hold?

An example

$$\mathbb{R}P^2 = \{(\rho, \theta) : 0 \leq \rho \leq 1, 0 \leq \theta < 2\pi\}/\sim \text{ where } (1, \theta) \sim (1, \theta + \pi).$$

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Fix $x \in \mathbb{R} \backslash \mathbb{Q}$. Set $\mathcal{G} := M \rtimes_\alpha \mathbb{Z}$, where

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Künneth Theorem calculations tell us $\mathbb{Z}/2\mathbb{Z} \subseteq Tw(\mathcal{G}) \cong H^2(M, \mathbb{Z})$; moreover, for any n, the obstruction to lifting a $PU(n)$-bundle over M to a $U(n)$-bundle lives in

$$H^2(M, \mathbb{T}) \cong H^3(M, \mathbb{Z}) \cong H^3(\mathbb{R}P^2, \mathbb{Z}) \otimes H^0(S^4, \mathbb{Z}) = 0.$$

Proof sketch

- [Moe98] For any étale groupoid $\mathcal{G}$ and any abelian $\mathcal{G}$-sheaf $\mathcal{A}$, $H^2(\mathcal{G}, \mathcal{A}) \cong H^2(B\mathcal{G}, A)$;

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- If you have a rank-n twisted vector bundle E – a map $\mathcal{R} \to GL_n(E)$ – composing with the determinant & normalizing gives a map $\psi : \mathcal{R} \to \mathbb{T}$ such that $\psi(z \cdot \gamma) = z^n \psi(\gamma)$, for any $\gamma \in \mathcal{R}$.

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