When Ext is a Batalin-Vilkovisky algebra

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Motivation I: Cyclic operators on Hochschild cohomology

Let A be an associative algebra over a commutative ring k.

- Apparent asymmetry in defining a cyclic operator t on the Hochschild homology complex $C_\bullet(A, A)$ and a cocyclic one τ on the Hochschild cohomology complex $C^\bullet(A, A)$: whereas on the first, t is just cyclic permnutation, it is not so clear how to do that on the second, and one rather uses $C^\bullet(A, A^*)$ for $A^* = \mathrm{Hom}_k(A, k)$.
- You might want to add or object that for you this is not really a problem as Hochschild cohomology $H^\bullet(A, A)$ is **not** functorial in A, whereas $H^\bullet(A, A^*)$ is so, so: so what?
- Let me, however, remind you that the groups $H^\bullet(A, A)$ are interesting objects to study as they are related to deformation theory, so one might want to find a cyclic operator for $C^\bullet(A, A)$ as well.

Motivation II: BV on Hochschild cohomology

- The Hochschild cohomology $H^*(A, A)$ (which is $\mathrm{Ext}_{A^\mathrm{e}}(A, A)$ if A is k-projective) is always a Gerstenhaber algebra (Gerstenhaber 1963), that is, it has a (graded commutative) product $\smile$ and a (graded) Lie bracket $\{.,.\}$ plus compatibilities like a Leibniz rule.
- On the other hand, $\mathrm{Ext}_{A^\mathrm{e}}(A, A)$ is not always a Batalin-Vilkovisky algebra (**misleading** remark on `ncatlab.org`), that is, a Gerstenhaber algebra with a degree -1 differential B that fails to be a derivation of the cup product precisely by the Lie bracket.

It is so for

- **symmetric algebras** (Tradler, Menichi), $A \simeq A^*$ (as A-bimodules);
- **Frobenius algebras** with semisimple Nakayama automorphism (Lambre, Zhou, Zimmermann), $A_\sigma \simeq A^*$ (as A-bimodules);
- **Calabi-Yau algebras** (Ginzburg), $A \simeq \mathrm{Ext}^d_{A^\mathrm{e}}(A, A^\mathrm{e})$ (as A-bimodules);
- **twisted Calabi-Yau algebras** (K.-Krähmer), $A_\sigma \simeq \mathrm{Ext}^d_{A^\mathrm{e}}(A, A^\mathrm{e})$ (as A-bimodules). Examples here come from quantum groups, homogeneous spaces, the Podleś quantum sphere.

A unifying result

Goal: to unify these scattered results under the roof of the following result:

Theorem (K., 2016)

If M is a stable anti Yetter-Drinfel'd **contramodule** *over a left Hopf algebroid U, then $C^\bullet(U, M) := \mathrm{Hom}_{A^{\mathrm{op}}}(U^{\otimes_{A^{\mathrm{op}}}\bullet}, M)$ is a* **cocyclic module**. *If $M = A$, then $C^\bullet(U, A)$ is a* **cyclic operad** *with multiplication, which implies that the cohomology groups $H^\bullet(U, M)$ (resp. $\mathrm{Ext}^\bullet_U(A, M)$) form a* **BV algebra**.

Corollary

If A is a right A^{e}-contramodule with **contraaction** $\gamma : \mathrm{Hom}_k(A, A) \to A$,

$$(\tau f)(a_1, \ldots, a_n) := \gamma\big(a_1 f(a_2, \ldots, a_n, -)\big)$$

defines a cocyclic operator on the complex $C^\bullet(A, A) = \mathrm{Hom}_k(A^{\otimes\bullet}, A)$, and its Hochschild cohomology becomes a **BV algebra**.

This might (or might not) define a **new class of algebras**.

Gerstenhaber algebras

Definition

Let k be a commutative ring. A **Gerstenhaber algebra** over k is a graded commutative k-algebra $(V, \smile)$

$$V = \bigoplus_{p \in \mathbb{N}} V^p, \quad \alpha \smile \beta = (-1)^{pq} \beta \smile \alpha \in V^{p+q}, \quad \alpha \in V^p, \beta \in V^q$$

with a graded Lie bracket $\{\cdot,\cdot\} : V^{p+1} \otimes_k V^{q+1} \to V^{p+q+1}$ on the *desuspension*

$$V[1] := \bigoplus_{p \in \mathbb{Z}} V^{p+1}$$

for which all operators $\{\gamma, \cdot\}$ satisfy the graded Leibniz rule

$$\{\gamma, \alpha \smile \beta\} = \{\gamma, \alpha\} \smile \beta + (-1)^{pq} \alpha \smile \{\gamma, \beta\}, \quad \gamma \in V^{p+1}, \alpha \in V^q.$$

Batalin-Vilkovisky algebras

Definition

A Gerstenhaber algebra is **Batalin-Vilkovisky** if there exists

$$\mathsf{B} : V^p \to V^{p-1}, \quad \mathsf{BB} = 0$$

such that for $\alpha \in V^p, \beta \in V$ we have

$$\{\alpha, \beta\} = (-1)^p \big(\mathsf{B}(\alpha \smile \beta) - (\mathsf{B}\alpha) \smile \beta - (-1)^p \alpha \smile \mathsf{B}\beta\big).$$

- Examples appear naturally in (mathematical) physics in various field theories.
- A Batalin-Vilkovisky algebra is also called an **exact** Gerstenhaber algebra and the differential B is said to generate the Gerstenhaber bracket. It is not by pure coincidence that we call it B, see later.

Higher structures arising from operads

Question

Natural question: how to encode these higher structures as compactly as possible and how to construct them explicitly.

Theorem

- *Any* **operad with multiplication** *defines a cosimplicial k-module the cohomology of which carries the structure of a Gerstenhaber algebra.*
- *Any* **cyclic operad with multiplication** *defines a cocyclic k-module the cohomology of which carries the structure of a Batalin-Vilkovisky algebra (of which the associated Connes-Rinehart boundary B is the generator).*

- The first part is probably due to many people (for example, Gerstenhaber-Schack or McClure-Smith), the second to Menichi.

Operads: the fundamental example

Consider the space $\mathcal{O} := \mathrm{Hom}_k(A^{\otimes_k \bullet}, A)$ of Hochschild cochains with values in A.

- For $\phi \in \mathcal{O}^p$ and $\psi \in \mathcal{O}^q$, define the operation

$$\phi \circ_i \psi(a_1, \ldots, a_{p+q-1}) := \phi(a_1, \ldots, a_{i-1}, \psi(a_i, \ldots, a_{i+q-1}), a_{i+q}, \ldots, a_{p+q-1}),$$

  that is, insertion of ψ into ϕ at the ith slot.
- This apparently works replacing A by any vector space V, but in the Hochschild case there is a **distinguished element** $\mu \in \mathcal{O}^2$ given by

$$\mu(a, b) := ab, \quad a, b \in A,$$

  the multiplication in A, which will serve in a moment.
- The composition operation for operads is not (strictly speaking) associative, but sort of in a more general sense:

Associativity behaviour

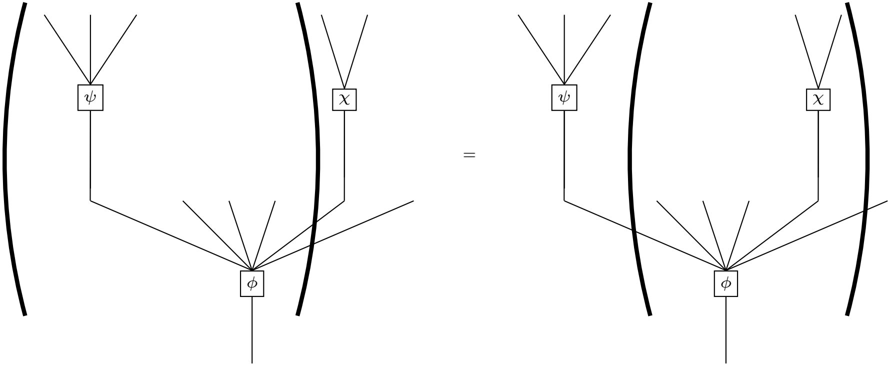

Figure: Parallel composition axiom

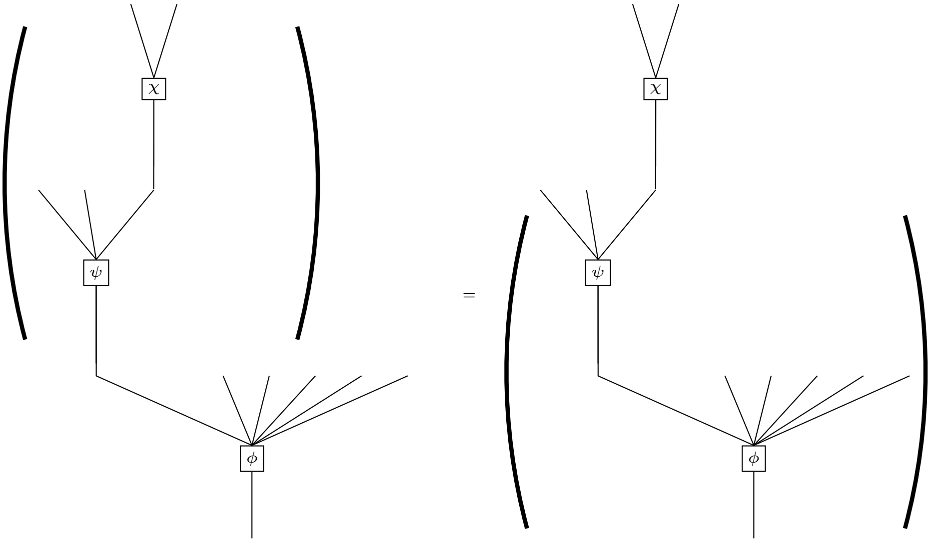

Figure: Sequential composition axiom

Operads: formal definition

Definition ("Partial" definition)

A (non-Σ, unital) **operad** (in k-**Mod**) is a sequence $\{\mathcal{O}^n\}_{n\geq 0}$ of k-modules with identity element $\mathbb{1} \in \mathcal{O}^1$ and k-bilinear operations $\circ_i : \mathcal{O}^p \otimes \mathcal{O}^q \to \mathcal{O}^{p+q-1}$ such that for $\varphi \in \mathcal{O}^p,\ \psi \in \mathcal{O}^q$, $\chi \in \mathcal{O}^r$:

$$
\begin{aligned}
\varphi \circ_i \psi &= 0 & & \text{if } p < i \quad \text{or} \quad p = 0,\\
(\varphi \circ_i \psi) \circ_j \chi &= \begin{cases} (\varphi \circ_j \chi) \circ_{i+r-1} \psi & \text{if } j < i,\\ \varphi \circ_i (\psi \circ_{j-i+1} \chi) & \text{if } i \leq j < q+i,\\ (\varphi \circ_{j-q+1} \chi) \circ_i \psi & \text{if } j \geq q+i, \end{cases}\\
\varphi \circ_i \mathbb{1} &= \mathbb{1} \circ_i \varphi = \varphi & & \text{for } i \leq p
\end{aligned}
$$

The operad $\mathcal{O}$ is called **operad with multiplication** if there exists a **multiplication** $\mu \in \mathcal{O}^2$ and a **unit** $e \in \mathcal{O}^0$ such that

$$\mu \circ_1 \mu = \mu \circ_2 \mu, \quad \text{and} \quad \mu \circ_1 e = \mu \circ_2 e \ = \ \mathbb{1}.$$

Sketch of the proof of the first part of the theorem

The explicit structure maps read as follows: for $\varphi \in \mathcal{O}^p, \psi \in \mathcal{O}^q$, set

$$\varphi \bar{\circ} \psi := \textstyle\sum_{i=1}^{p} (-1)^{|q||i|} \varphi \circ_i \psi \in \mathcal{O}^{p+q-1}, \qquad |n| := n-1,$$

and define their **Gerstenhaber bracket** by

$$\{\varphi, \psi\} := \varphi \bar{\circ} \psi - (-1)^{|p||q|} \psi \bar{\circ} \varphi.$$

The graded commutative product is given by the **cup product**

$$\varphi \smile \psi := (\mu \circ_2 \varphi) \circ_1 \psi \in \mathcal{O}^{p+q}.$$

Finally, the coboundary of the cosimplicial k-module results as

$$\delta \varphi = \{\mu, \varphi\},$$

and then the triple $(\mathcal{O}, \delta, \smile)$ **forms a DG algebra**.

Cyclic operads

Definition

A **cyclic** operad is a (non-Σ) operad O equipped with k-linear maps $\tau_n : \mathcal{O}^n \to \mathcal{O}^n$ subject to

$$\begin{aligned} \tau(\varphi \circ_1 \psi) &= \tau\psi \circ_q \tau\varphi, \quad \text{if } 1 \leq p, q, \\ \tau(\varphi \circ_i \psi) &= \tau\varphi \circ_{i-1} \psi, \quad \text{if } 0 \leq q \text{ and } 2 \leq i \leq p, \\ \tau^{n+1} &= \mathrm{id}_{\mathcal{O}^n}, \\ \tau_1 \mathbb{1} &= \mathbb{1} \end{aligned}$$

A **cyclic operad with multiplication** is simultaneously a cyclic operad and an operad with multiplication μ such that $\tau\mu = \mu$.

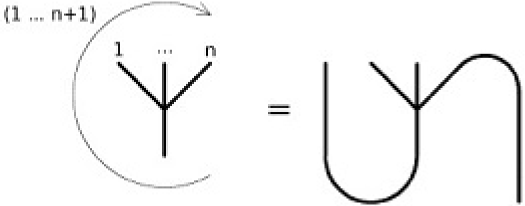

Illustration (of the opposite convention)

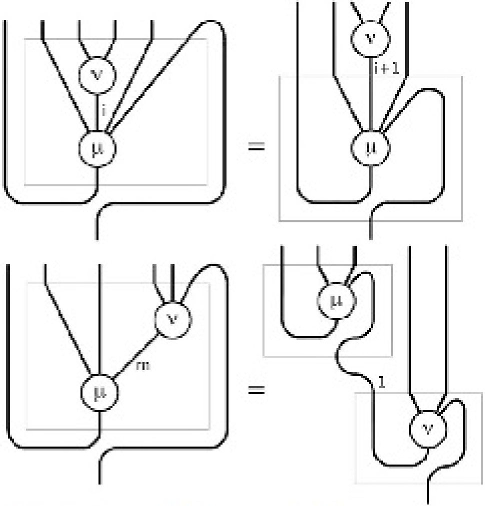

Fig. 13.2 $\tau(\mu \circ_i \nu) = \tau(\mu) \circ_{i+1} \nu$ and $\tau(\mu \circ_m \nu) = \tau(\nu) \circ_1 \tau(\mu)$

An alternative proof of Menichi's theorem

- In case of a cyclic operad w. mult., Menichi found an explicit homotopy formula for the Gerstenhaber bracket (that arises from the operad w. mult.) in terms of higher order operations (that arise from cyclicity).
- Slightly more fundamental approach:

Theorem (K., 2016)

If $(\mathcal{O}, \mu, \tau)$ is a cyclic operad with multiplication, then $\big(\mathcal{O}, \mathrm{Hom}_k(\mathcal{O}, k)\big)$ yields a (homotopy) noncommutative differential calculus. In particular,

$$\{f, g\} = \mathcal{L}_f g + (-1)^p \mathcal{L}_g f - B(f \smile g)$$

for $f, g \in \mathcal{O}$, where

$$\mathcal{L}_f := [B, f \smile \cdot] + [\delta, S_f] - S_{\delta f},$$

where δ is the cosimplicial differential induced by $(\mathcal{O}, \mu)$, and S_f a degree -2 operator. Hence, descending on cohomology gives the BV-equation.

- Observe a certain resemblance to Koszul's formula in Poisson geometry
$$[\omega, \eta]_\pi = \mathcal{L}_{\pi^\#(\eta)}\omega - \mathcal{L}_{\pi^\#(\omega)}\eta - d\iota_\pi(\omega \wedge \eta).$$

Fundamental question

- How can one make the endomorphism operad $\mathrm{Hom}_k(A^{\otimes \bullet}, A)$ consisting of Hochschild cochains into a a cyclic operad with multiplication so as to obtain a BV-algebra structure on cohomology?
- For any **Hopf algebra** H over k with antipode S the cochains computing $\mathrm{Ext}_H(k, k)$ do form a cyclic operad with multiplication provided that $S^2(h) = ghg^{-1}$ for a grouplike element $g \in H$.
- Only that this does not help as A^{e} is not a Hopf algebra (over A or k).
- The two cases above can be included in the concept of a **bialgebroid** (roughly a bialgebra over a noncomuutative ring):

Theorem (K.-Krähmer, 2012)

Let (U, A) be a (left) **bialgebroid**. *Then $C^\bullet(U, A)$ forms an operad with multiplication. The cohomology groups $H^\bullet(U, A)$ (or $\mathrm{Ext}^\bullet_U(A, A)$) therefore yield a Gerstenhaber algebra.*

- Holds for more general coefficients (braided commutative YD-algebras).
- How can we possibly add a BV structure on these cohomology groups?

Left and right bialgebroids

Definition (Takeuchi, Lu, Xu)

A **(left)** A-**bialgebroid** is a k-module U that is simultaneously an A^{e}-ring (U, s^ℓ, t^ℓ) (a monoid in $(A^{\mathrm{e}} \otimes_k A^{\mathrm{e}})$-**Mod**) and an A-coring $(U, \Delta_\ell, \epsilon)$ (a comonoid in A^{e}-**Mod**) subject to:

- The bimodule structure in the coring is related to the A^{e}-ring by $a \triangleright u \triangleleft b := s^\ell(a)t^\ell(b)u, \quad a, b \in A, \ u \in U$;
- Δ_ℓ is a unital k-algebra morphism taking values in the Takeuchi subspace $U \times_A U$;
- $\epsilon(a \triangleright u \triangleleft b) = a\epsilon(u)b \quad \text{and} \quad \epsilon(uv) = \epsilon(us^\ell(\epsilon v)) = \epsilon(ut^\ell(\epsilon v))$.

Starting with the *right* A^{e}-module structure in an A^{e}-ring U,

$$a \blacktriangleright u \blacktriangleleft b := us(b)t(a)$$

denoted ${}_{\blacktriangleright}U_{\blacktriangleleft}$, leads to the notion of **right** bialgebroids (Kadison - Szlachányi), which is structurally the opposite & coopposite of the above.

Left Hopf algebroids

- For a left bialgebroid (U, A), define the **(Hopf-)Galois map**

$$\beta : U \otimes_{A^{\mathrm{op}}} U \to U \otimes_A U, \quad u \otimes_{A^{\mathrm{op}}} v \mapsto u_{(1)} \otimes_A u_{(2)} v.$$

- For bialgebras over fields, β is bijective if and only if U is a Hopf algebra, and

$$\beta^{-1}(u \otimes_k v) := u_{(1)} \otimes_k S(u_{(2)}) v,$$

where S is the antipode of U.
This motivates:

Definition (Schauenburg)

A left A-bialgebroid U is called a **left Hopf algebroid** (or $\times_A$-*Hopf algebra*) if β is a bijection.

- Denote the so-called *translation map* $\beta^{-1}(\cdot \otimes_A 1) : U \to U \otimes_{A^{\mathrm{op}}} U$. by

$$u_+ \otimes_{A^{\mathrm{op}}} u_- := \beta^{-1}(u \otimes_A 1).$$

Examples of left Hopf algebroids

- For $A = k$, left Hopf algebroids are simply Hopf algebras.
- The enveloping algebra $A^{\mathrm{e}} = A \otimes_k A^{\mathrm{op}}$ of an arbitrary (unital) k-algebra A is a left bialgebroid over A by means of

$$\begin{aligned} s(a) &:= a \otimes_k 1, & t(b) &:= 1 \otimes_k b, \\ \Delta(a \otimes_k b) &:= (a \otimes_k 1) \otimes_A (1 \otimes_k b), & \epsilon(a \otimes_k b) &:= ab, \end{aligned}$$

  and a left Hopf algebroid by

$$(a \otimes_k b)_+ \otimes_{A^{\mathrm{op}}} (a \otimes_k b)_- := (a \otimes_k 1) \otimes_{A^{\mathrm{op}}} (b \otimes_k 1).$$

- The universal enveloping algebra of a (geometric or algebraic) Lie algebroid is a left Hopf algebroid.

Observe: there is no notion of antipode for left Hopf algebroids.

The explicit construction

- We already know that $\mathrm{Cotor}^\bullet_U(A, M)$ (the right derived functor of the cotensor product) is a BV algebra if U is a (left or right) Hopf algebroid and M a (stable) anti Yetter-Drinfel'd **module** (a U-module and a U-comodule plus compatibility).
- The idea of how to obtain the structure of a cocyclic module on $C^\bullet(U, M)$ is as follows: if U is f.g.p. A^{op}-projective, the (right) **dual** $U^* := \mathrm{Hom}_{A^{\mathrm{op}}}(U, A)$ is a (right) bialgebroid again. The isomorphism

$$\xi : C^\bullet_{\mathrm{co}}(U^*, M) := M \otimes_{A^{\mathrm{op}}} U^* \otimes_{A^{\mathrm{op}}} \cdots \otimes_{A^{\mathrm{op}}} U^* \xrightarrow{\simeq} C^\bullet(U, M)$$

  is one of cosimplicial k-modules, where $M \in \mathbf{Comod}\text{-}U^* \simeq U\text{-}\mathbf{Mod}$.
- To declare this to be an isomorphism of cocyclic k-modules and hence to deduce the cocyclic operator τ on $C^\bullet(U, M)$, the dual U^* has to carry a Hopf structure of some kind, which was not known.
- Once obtained τ this way, it should **make sense** even if U is **not** f.g.p.

Duals of bialgebroids

- Simply **guessing** τ it is not so easy: for Hopf algebras, τ in degree 1 is $\tau(f) = f \circ S$; for Hopf algebroids this does not make sense and even if (as for (A^{e}, A), where an S exists), it is not a map $U^* \to U^*$.
- So, we need to discuss the explicit formula for the Hopf structure on the dual first.
- If $U_{\triangleleft}$ is f.g. A-projective, the (right) dual $U^* := \mathrm{Hom}_{A^{\mathrm{op}}}(U_{\triangleleft}, A_A)$ is a right bialgebroid (Kadison-Szlachányi); for example, the coproduct is

$$\langle \phi^{(1)}, \langle \phi^{(2)}, v \rangle \blacktriangleright u \rangle = (v \rightharpoonup \phi)(u)$$

for the left action

$$(v \rightharpoonup \phi)(u) := \phi(uv).$$

- Using a dual basis $\{e_i\}_{1 \leq i \leq n} \in U, \ \{e^i\}_{1 \leq i \leq n} \in U^*$, this can be written as

$$\Delta_r : U^* \to U^*_{\blacktriangleleft} \otimes_A {}_{\blacktriangleright}U^*, \quad \phi \mapsto \textstyle\sum_i (e_i \rightharpoonup \phi) \otimes_A e^i.$$

The Hopf structure on the dual

The following answers a long-standing open question what kind of structure the dual of a Hopf algebroid carries:

Proposition (K., 2016)

Let (U, A) be a left Hopf algebroid, and $U_\triangleleft$ f.g.p. Then (U^*, A) is a right Hopf algebroid: the map $\beta^{-1}(- \otimes 1) : U^* \to U^* \otimes_{A^{\mathrm{op}}} U^*$ given by

$$(\beta^{-1}(\phi \otimes 1))(u, v) = (u \rhd \phi)(v) = \varepsilon(\phi(u_- v) \blacktriangleright u_+)$$

yields a translation map which defines an inverse to the Hopf-Galois map

$$\beta : \phi \otimes_{A^{\mathrm{op}}} \psi \mapsto \phi\psi^{(1)} \otimes_A \psi^{(2)}.$$

Explicitly,

$$\phi^- \otimes_{A^{\mathrm{op}}} \phi^+ := \textstyle\sum_i e^i \otimes_{A^{\mathrm{op}}} (e_i \rhd \phi).$$

This statement also explicitly illustrates the corresponding result obtained by Schauenburg a couple of months ago.

Cyclic modules for right bialgebroids

For any right bialgebroid, say, (V, B),

$$\delta'_i(m \otimes_B w) = \begin{cases} m \otimes_B v_1 \otimes_B \cdots \otimes_B v_n \otimes_B 1 & \text{if } i = 0, \\ m \otimes_B v_1 \otimes_B \cdots \otimes_B \Delta_r(v_{n-i+1}) \otimes_B \cdots \otimes_B v_n & \text{if } 1 \leq i \leq n, \\ m^{(0)} \otimes_B m^{(1)} \otimes_B v_1 \otimes_B \cdots \otimes_B v_n & \text{if } i = n+1, \end{cases}$$
$$\sigma'_j(m \otimes_B w) = m \otimes_B v_1 \otimes_B \cdots \otimes_B \partial(v_{n-j}) \otimes_B \cdots \otimes_B v_n \qquad 0 \leq j \leq n-1,$$
$$\tau'(m \otimes_B w) = v_n^+ m^{(0)} \otimes_B m^{(1)} v_n^{-(1)} \otimes_B v_1 v_n^{-(2)} \otimes_B \cdots \otimes_B v_{n-1} v_n^{-(n)}$$

(where $w := v^1 \otimes_B \cdots \otimes_B v^n$) defines a cocyclic k-module structure on the complex $C^\bullet_{\mathrm{co}}(V, M)$.

Here, M is a **(stable) anti Yetter-Drinfel'd module**: a **left** V**-module** and **right** V**-comodule** where one knows what happens if **coaction is followed by action** and **vice versa**.

If V is B-flat, the simplicial part of $C^\bullet_{\mathrm{co}}(V, M)$ computes $\mathrm{Cotor}^\bullet_V(M, B)$.

Theorem (K., 2013)

$C^\bullet_{\mathrm{co}}(V, M)$ is a cyclic operad with multiplication and hence $\mathrm{Cotor}^\bullet_V(M, B)$ a BV-algebra.

BV structures on Cotor

Hence, $C^\bullet(U^*, M)$ is a cyclic operad with multiplication. For the cocyclic structure on $C^\bullet(U, M) := \mathrm{Hom}_{A^{\mathrm{op}}}(U^{\otimes_{A^{\mathrm{op}}}\bullet}, M)$, computing Ext, define

$$\delta_i := \xi \circ \delta'_i \circ \xi^{-1}, \quad \sigma_j := \xi \circ \sigma'_j \circ \xi^{-1}, \quad \tau := \xi \circ \tau' \circ \xi^{-1},$$

where $\xi : C^n_{\mathrm{co}}(U^*, M) \to C^n(U, M)$ is the canonical isomorphism as before. Once obtained the operators, they will make sense even without f.g.p. assumptions.

- What happens to the coefficients M when passing from U^* to U (and then forgetting about finiteness assumptions)?
- M for the moment is a left U^*-module and right U^*-comodule plus compatibility conditions. One might be tempted to think that the left U^*-module structure corresponds to some U-comodule structure, but one rather needs the notion of U-**contramodules** which were introduced by Eilenberg-Moore more years ago than you can remember but later somehow forgotten.

Contramodules I

Definition (Böhm-Brzeziński-Wisbauer)

A *right contramodule* over a left bialgebroid (U, A) is a right A-module M together with a right A-module map $\gamma : \mathrm{Hom}_{A^{\mathrm{op}}}(U_{\triangleleft}, M) \to M$ subject to

$$\begin{tikzcd}
\mathrm{Hom}_{A^{\mathrm{op}}}(U, \mathrm{Hom}_{A^{\mathrm{op}}}(U, M)) \arrow[rr, "\mathrm{Hom}_{A^{\mathrm{op}}}(U,\gamma)"] \arrow[d, "\simeq"'] & & \mathrm{Hom}_{A^{\mathrm{op}}}(U, M) \arrow[d, "\gamma"] \\
\mathrm{Hom}_{A^{\mathrm{op}}}(U_{\triangleleft} \otimes_{A} {}_{\triangleright}U, M) \arrow[r, "\mathrm{Hom}_{A^{\mathrm{op}}}(\Delta_\ell, M)"'] & \mathrm{Hom}_{A^{\mathrm{op}}}(U, M) \arrow[r, "\gamma"'] & M \\
& & \mathrm{Hom}_{A^{\mathrm{op}}}(A, M) \arrow[ul, "\mathrm{Hom}_{A^{\mathrm{op}}}(\varepsilon, M)"] \arrow[u, "\simeq"']
\end{tikzcd}$$

The category **Contramod**-U of right contramodules is not known to be monoidal.

Contramodules II

Lemma

- *There exists a functor*

$$U\text{-}\mathbf{Comod} \to \mathbf{Mod}\text{-}U^*.$$

If $U_{\triangleleft}$ is finitely generated A-projective, this functor has a quasi-inverse $\mathbf{Mod}\text{-}U^* \to U\text{-}\mathbf{Comod}$ *such that there is an equivalence of categories.*

- *Let (U, A) be a left bialgebroid. There is a functor*

$$\mathbf{Contramod}\text{-}U \to U^*\text{-}\mathbf{Mod},$$

which if $U_{\triangleleft}$ is finitely generated A-projective has a quasi-inverse $U^*\text{-}\mathbf{Mod} \to \mathbf{Contramod}\text{-}U$ *giving an equivalence of categories.*

The first part might seem banal to you, but observe that for the left dual U_* one has (if ${}_{\triangleright}U$ is f.g. A-projective), somewhat unexpectedly,

$$\mathbf{Comod}\text{-}U \simeq \mathbf{Mod}\text{-}U_*.$$

Anti Yetter-Drinfel'd contramodules

We need the following bialgebroid version of a notion of Brzeziński:

Definition (K., 2016)

An **anti Yetter-Drinfel'd contramodule** M over a left Hopf algebroid (U, A) is a left U-module and a right U-contramodule s. th. both underlying A-bimodule structures coincide, and s. th. action and contraaction are compatible in the sense that

$$u(\gamma(f)) = \gamma\big(u_{+(2)} f(u_-(-)u_{+(1)})\big), \qquad \forall u \in U, \; f \in \mathrm{Hom}_{A^{\mathrm{op}}}(U, M).$$

An anti Yetter-Drinfel'd contramodule is called **stable** if

$$\gamma((-)m) = m, \quad \forall m \in M.$$

Observe that the above condition **is** (somehow miraculously) well-defined, although in general the adjoint action $(u, m) \mapsto u_- m u_+$ on a U-bimodule M **is not**.

Categorical equivalences

So, what happens to the stable anti Yetter-Drinfel'd modules over U^* under the isomorphism ξ?

Lemma

Let U be additionally a left Hopf algebroid and $U_\triangleleft$ be f.g.p. over A. One then has an equivalence of categories

$$ {}_{U^*}\mathbf{aYD}^{U^*} \simeq {}_{U}\mathbf{aYD}^{\mathrm{contra}-U} $$

between the categories of (stable) aYD modules over U^* and (stable) aYD contramodules over U.

None of the above categories is monoidal.

BV structures on Ext

Computing then the operators $\delta_i := \xi \circ \delta'_i \circ \xi^{-1}$, $\sigma_j := \xi \circ \sigma'_j \circ \xi^{-1}$, $\tau := \xi \circ \tau' \circ \xi^{-1}$, leads to

$$(\delta_i f)(u^1, \ldots, u^{n+1}) = \begin{cases} f(u^1, \ldots, \varepsilon(u^{n+1}) \blacktriangleright u^n) & \text{if } i = 0, \\ f(u^1, \ldots, u^{n-i+1} u^{n-i+2}, \ldots, u^{n+1}) & \text{if } 1 \leq i \leq n, \\ u^1 f(u^2, \ldots, u^{n+1}) & \text{if } i = n+1, \end{cases}$$

$$(\sigma_i f)(u^1, \ldots, u^{n-1}) = f(u^1, \ldots, u^{n-i}, 1, u^{n-i+1}, \ldots, u^n) \qquad 0 \leq i \leq n-1,$$

$$(\tau f)(u^1, \ldots, u^n) = \gamma\big(u^1_+ f(u^2_+, \ldots, u^n_+, u^n_- \cdots u^1_-(-))\big)$$

where M is now a left U-module right U-contramodule, and now these operators make sense even if U is **not** f.g.p. anymore.

Theorem (K., 2016)

If M is an aYD contramodule over any left Hopf algebroid U, then $C^\bullet(U, M)$ by the above is a cocyclic module. If $M = A$, then $C^\bullet(U, A)$ is a cyclic operad with multiplication and hence the cohomology groups $H^\bullet(U, A)$ (resp. $\mathrm{Ext}^\bullet_U(A, M)$ if $U_\lhd$ is projective) form a BV algebra.

Example: Hopf algebras

- For a Hopf algebra H over k, the evaluation

$$\mathrm{Hom}_k(H, k) \to k, \quad \phi \mapsto \phi(g)$$

for a grouplike element $g \in H$ always gives a right H-contramodule structure γ on the ground ring k, which is why one does not "see" (or need) the contraaction in Hopf algebra theory.
The resulting cocyclic module (δ, σ, τ) is the well-known one from Menichi which is cyclic if $S^2(h) = ghg^{-1}$, and in this case $\mathrm{Ext}_H(k, k)$ is a BV-algebra.

Observe that evaluation in general is not allowed as it would not be right A-linear.

Examples: associative algebras and Hochschild theory

- For the bialgebroid (A^{e}, A), one obviously has

$$\mathrm{Hom}_{A^{\mathrm{op}}}(A^{\mathrm{e}}, A) \simeq \mathrm{Hom}_k(A, A).$$

- Hence, a right A^{e}-contramodule is a right A-module M along with a map

$$\gamma : \mathrm{Hom}_k(A, M) \to M$$

subject to

$$\begin{array}{rcll} \gamma\big(f(a(-))\big) & = & \gamma(f)a, & \forall\ f \in \mathrm{Hom}_k(A, M), \\ \dot{\gamma}\big(\ddot{\gamma}(g(\cdot \otimes \cdot\cdot))\big) & = & \gamma\big(g(\cdot \otimes 1_A)\big), & \forall\ g \in \mathrm{Hom}_k(A \otimes A, M), \\ \gamma(m\mathrm{id}_A(-)) & = & m, & \forall\ m \in M, \end{array}$$

- Any right A^{e}-contramodule is **automatically** a stable anti Yetter-Drinfel'd contramodule (easy check).

Example: Symmetric algebras

- In case of a **symmetric algebra**, that is, $A \simeq A^* := \mathrm{Hom}_k(A, k)$ as A-bimodules, the contraaction $\gamma : \mathrm{Hom}_k(A, A) \to A$ from the slide before for $M = A$ reduces to giving a map

$$\begin{tikzcd}
\mathrm{Hom}_k(A, A) \arrow[rr, "\gamma"] \arrow[d, "\simeq"'] & & A \\
\mathrm{Hom}_k(A, A^*) \arrow[r, "\simeq"'] & \mathrm{Hom}_k(A \otimes A, k) \arrow[r, "\tilde{\gamma}"'] & A^* \arrow[u, "\simeq"']
\end{tikzcd}$$

  subject to the above three conditions, and it is simple to see that

$$\tilde{\gamma} : g \mapsto g(- \otimes 1_A)$$

  yields such a map.
- Hence, in this case, $H^\bullet(A, A)$ carries the structure of a Batalin-Vilkovisky algebra.

Example: Frobenius algebras

Let k be a(n algebraically closed) field.

- A *Frobenius algebra* is a f.d. k-algebra A with a nondegenerate bilinear form $\mathfrak{b} : A \otimes A \to k$ subject to

  $$\mathfrak{b}(ab, c) = \mathfrak{b}(a, bc),$$

  for $a, b, c \in A$. There exists a unique $\sigma(a) \in A$ such that $\mathfrak{b}(a, -) = \mathfrak{b}(-, \sigma(a))$, and this determines the **Nakayama** automorphism $\sigma \in \mathrm{Aut}_k(A)$. In particular,

  $$\mathfrak{b}(ab, c) = \mathfrak{b}(b, c\sigma(a)),$$

  for all $a, b, c \in A$.
- The map $a \mapsto \mathfrak{b}(a, -)$ establishes an A-bimodule isomorphism $A \simeq {}_{\sigma}A^*$, inducing a nondegenerate bilinear form $\mathfrak{b}^* : A^* \otimes A^* \to k$ on A^*.

Example: Frobenius algebras

- One then proceeds by generalising the case of symmetric algebras:

$$\begin{tikzcd} \mathrm{Hom}_k(A, A) \arrow[rr, "\gamma"] \arrow[d, "\mathfrak{b}^{\#}", "\simeq"'] & & A \\ \mathrm{Hom}_k(A, {}_\sigma A^*) \arrow[r, "\simeq"'] & \mathrm{Hom}_k(A \otimes A_\sigma, k) \arrow[r, "\tilde{\gamma}"'] & A^*, \arrow[u, "{\mathfrak{b}^*_{\sigma^{-1}}}^{\#}"', "\simeq"] \end{tikzcd}$$

  where $\tilde{\gamma} : g \mapsto g(- \otimes 1_A)$ is the same evaluation as before.
- Explicitly, for a dual basis $\{e_i\}_{1 \leq i \leq n} \in A$, $\{e^i\}_{1 \leq i \leq n} \in A^*$, this map reads as

$$\gamma : \mathrm{Hom}_k(A, A) \to A, \quad f \mapsto \textstyle\sum_{i,j} \mathfrak{b}\big(f(e_i), 1\big) \mathfrak{b}^*\big(e^i, (\sigma^{-1})^*(e^j)\big) e_j.$$

- If one tries to verify the above three conditions for γ, one notices that this seemingly only works if σ is diagonalisable, meaning that there is a decomposition of k-vector spaces

$$A = \textstyle\bigoplus_{\lambda \in \Sigma \subseteq k \setminus \{0\}} A_\lambda, \quad A_\lambda = \{a \in A \mid \sigma(a) = \lambda a\}.$$

Fancier approaches

- A Gerstenhaber algebra is an e_2-**algebra**, where e_2 is the Gerstenhaber operad which is isomorphic to the singular homology operad $H_\bullet(D_2, k)$ of the little disks operad D_2 (Cohen 1976).
- There are cases in which the bracket (which is of degree -1) disappears but in which a "higher" bracket (i.e., of degree -2 or $1 - n$) is still there, and which fulfils analogous Leibniz-kind of relations; this is then called an e_3-**algebra** (resp. e_n-**algebra**).
- Lurie and Kontsevich and maybe also everybody else calls this an n-**algebra**, where n is the number of compositions one has at hand; in the Gerstenhaber, that is, the 2-algebra case, this would be the **cup product** $\smile$ and Getzler's **brace operation**, which is "half" of the Gerstenhaber bracket (the pre-Lie structure), see below.
- For example, the cohomology groups $\mathrm{Ext}_{D(H)}(k, k)$ governing bialgebra deformations (that is, Gerstenhaber-Schack cohomology) has a vanishing Gerstenhaber bracket but a bracket of degree -2, hence is a 3-algebra (Shoikhet 2011).
- This probably generalises to the cohomology of any braided bialgebra.

— Additional material —

Noncommutative differential calculi

Definition

A **nc differential calculus** is a quintuple $(V, \Omega, B, \frown, \mathcal{L})$, where $V = (V, \smile, \{.,.\})$ is a Gerstenhaber algebra, and $(\Omega, \frown)$ both a graded module over $(V, \smile)$

$$\Omega = \bigoplus_{n \in \mathbb{Z}} \Omega_n, \quad \iota_\alpha x := \alpha \frown x \in \Omega_{n-p}, \quad \alpha \in V^p, x \in \Omega_n,$$

and graded Lie algebra module over $(V[1], \{\cdot, \cdot\})$

$$\mathcal{L} : V^{p+1} \otimes_k \Omega_n \to \Omega_{n-p}, \quad \alpha \otimes_k x \mapsto \mathcal{L}_\alpha(x)$$

which satisfies the mixed Leibniz rule

$$\beta \frown \mathcal{L}_\alpha(x) = \{\beta, \alpha\} \frown x + (-1)^{pq} \mathcal{L}_\alpha(\beta \frown x).$$

On top, there is a k-linear differential $\mathrm{B} : \Omega_n \to \Omega_{n+1}$, such that $\mathcal{L}_\alpha$ is for $\alpha \in V^p$ given by the **homotopy formula**

$$\mathcal{L}_\alpha(x) = \mathrm{B}(\alpha \frown x) - (-1)^p \alpha \frown \mathrm{B}(x).$$

- (**Classical geometric example**) X compact smooth manifold, $k := \mathbb{C}$.

$$V := \bigwedge_A \mathrm{Der}_{\mathbb{C}}(A), \quad \Omega = \Omega(X), \quad A := C^\infty(X, \mathbb{C})$$

  yields a ncd calculus, with the well-known operations of Lie derivative, contraction, B:=d, and the Cartan homotopy $\mathcal{L} = [\iota, \mathrm{d}]$.
- (**Classical algebraic example**) The pair

$$\big(\mathrm{Ext}^\bullet_{A^{\mathrm{e}}}(A, A), \mathrm{Tor}^{A^{\mathrm{e}}}_\bullet(A, A)\big)$$

  of Hochschild cohomology and homology forms a calculus (Rinehart 1963, Connes, Getzler, Goodwillie, Nest-Tsygan (80/90s)).
- (**Sort-of universal example**) For a (left) Hopf algebroid U and (somehow technically complicated) coefficient modules M, N,

$$\big(\mathrm{Ext}^\bullet_U(A, N), \mathrm{Tor}^U_\bullet(M, N)\big)$$

  yields a calculus (K.-Krähmer 2012, K. 2013).
- (**Even-more universal example**) Let $\mathcal{O}$ be an operad with multiplication, $\mathcal{M}$ a "cyclic comp module" over $\mathcal{O}$. Then

$$\big(H^\bullet(\mathcal{O}), H_\bullet(\mathcal{M})\big)$$

  forms a calculus (K. 2013).

Duality differential calculi

Definition (Lambre, 2009)

A ncd calculus $(V^\bullet, \Omega_\bullet, B, \frown, \mathcal{L})$ is called a **(Poincaré) duality** calculus if there is a $d \geq 0$ and $\omega \in \Omega_d$ such that $1_V \frown \omega = \omega$ and such that

$$\cdot \frown \omega : V^i \to \Omega_{d-i}$$

is an isomorphism for all $i \geq 0$. The element ω is called the **fundamental class** of the calculus.

Theorem (Lambre, 2009)

For a duality calculus $(V^\bullet, \Omega_\bullet, B, \frown, \mathcal{L}, \omega)$, the operator

$$\partial := (-1)^d DBD^{-1},$$

where $D := (\cdot \frown \omega)^{-1}$, generates the Gerstenhaber bracket of $V^\bullet$. Hence, $V^\bullet$ is a Batalin-Vilkovisky algebra.

Poincaré duality

- At least in the first three cases the requirements when such a Poincaré duality exists can be formulated (Van den Bergh 2002 for the second, K.-Krähmer 2008 for the third, but I spare you the details).
- The idea in the following is to find a more direct approach.