Differentiable functions on modules and the equation $grad(v) = Mgrad(w)$

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Harmonic analysis, complex analysis, spectral theory and all that
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Basic definition

A - a finite dimensional commutative algebra over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$
B - an A-module
$U \subset B$ - an open set

A-differentiability

$f\colon U \to A$ is *A-differentiable* if it is differentiable and the derivative is A-linear: $Df(x)(ay) = aDf(x)(y)$ for all $x \in U$, $y \in B$ and $a \in A$.

A-analyticity

$f\colon U \to A$ is A-*analytic* if for every $b_0 \in U$ there exist an open neighbourhood $V \subset U$ of b_0, such that for $b \in V$

$$f(b) = \sum_{i=0}^{\infty} L_i(b - b_0, \dots, b - b_0),$$

for some symmetric A-multilinear $L_i\colon B^i \to A$ such that for $b \in V$

$$\sum_{i=0}^{\infty} \| L_i(b - b_0, \dots, b - b_0) \| < \infty.$$

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A-analyticity & $\mathbb{F}$-analyticity

$f\colon U \to A$ is A-differentiable and $\mathbb{F}$-analytic if and only if it is A-analytic.

Splitting

Assume that $A = \bigoplus_{i=1}^{m} A_i$, $e = \sum_{i=1}^{n} e_i$. Let $U \subset B$ be convex and open.

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Let $f \colon U \to A$ be A-differentiable. Then

$$f = \sum_{i=1}^{m} f_i \circ \pi_{B_i}$$

for some A_i-differentiable functions $f_i \colon \pi_{B_i}(U) \to A_i$.

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for some A_i-differentiable functions $f_i \colon \pi_{B_i}(U) \to A_i$.
Conversely, any function of this form is A-differentiable.
Thus, we can concentrate only on *local* algebras (i.e. those which have only one maximal ideal).

Banach algebras

$C^k_A(\overline{U}, A)$ - the set of all A-differentiable functions on U of class C^k, with all derivatives, of order up to k, continuous up to boundary.

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We have a Banach algebra norm

$$\|f\|_k = \sup_{x\in\overline{U}}\|f(x)\| + \sum_{i=1}^{k}\frac{1}{i!}\sup_{x\in\overline{U}}\sup_{\|y\|=1}\|D^i f(x)(y,\dots,y)\|.$$

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Any A-differentiable function in $f \in C^p_A(\overline{U}, A)$ may be written in the form

$$f = \sum_{k=0}^{l-1} \left(\rho_{(e^k)} f_k(\pi_{D_k}(u)) + \sum_{j=1}^{l-1-k} \frac{1}{j!} G^j_{(e^k)} D^j f_k(\pi_{D_k}(u))((u - \rho_{D_k}\pi_{D_k}(u))^j) \right).$$

for some functions

$$(f_k)_{k=0,1\ldots,l-1} \in \bigoplus_{k=0}^{l-1} C_{\mathbb{F}}^{p+l-1-k}(\overline{\pi_{D_k}(U)}, \mathbb{F}).$$

Conversely, any such function belongs to $C^p_A(\overline{U}, A)$. This assignment is an isomorphism of Banach spaces.

Generalised Laplace equations

A commutative algebra A is a *Frobenius algebra* if there is a linear functional $\phi\colon A \to \mathbb{F}$ such that the biliniear form $(x, y) \mapsto \phi(xy)$ is nondegenerate.

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Correspondence to A-differentiable functions

Suppose that A is a Frobenius algebra and that $U \subset B$ is open and simply connected. Let $t \in \mathbb{N}$. Let $v \colon U \to \mathbb{F}$ be a C^{2+t} function such that

$$D^2 v(b)(ax, y) = D^2 v(b)(x, ay).$$

Then $v = \phi(f)$ for some A-differentiable function f of class C^{2+t}. Such f is uniquely determined by v, up to a constant.

The equation $grad(w) = Mgrad(v)$

Let $t \in \mathbb{N}$. Let $U \subset \mathbb{F}^n$ be an open, simply connected set. Let A be an algebra generated by matrix $M^\intercal \in M_{n\times n}(\mathbb{F})$ and let $B = \mathbb{F}^n$ with the natural structure of A-module. Let $v, w \colon U \to \mathbb{F}$.

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(i) v, w are C^{2+t} functions satisfying $grad(w) = Mgrad(v)$,

(ii) $v = \phi(f)$ is a component function of an A-differentiable function $f \colon U \to A$ of class C^{2+t}, and $w = \phi(M^\intercal f) + c$, where c is a constant.

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Analyticity

The following conditions are equivalent:

(i) any functions v, w of class C^2 which satisfy $grad(w) = Mgrad(v)$ are analytic and are components of A-analytic functions,

(ii) M has no real eigenvalues.

Example

Consider the matrix

$$M = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix}.$$

The minimal polynomial of $M^\intercal$ is $P(x) = (x-\lambda)^2$. Algebra A generated by $M^\intercal$ is isomorphic to $\mathbb{R}[x]/(x-\lambda)^2$. A-module $B = \mathbb{R}^3$ has the decomposition

$$B = \mathbb{R}[x]/(x-\lambda) \oplus \mathbb{R}[x]/(x-\lambda)^2.$$

A has a basis $1, e$, where $e = x - \lambda$. B has a basis (e_1, e_2, e_3), such that $ee_1 = 0$, $ee_2 = 0$, $ee_3 = e_2$.

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Consider the generalised Laplace equations

$$D^2 v(\cdot)(z, M^\intercal y) = D^2 v(\cdot)(M^\intercal z, y).$$

Equivalently

$$\frac{\partial^2 v}{\partial x_1 \partial x_2} = 0, \qquad \frac{\partial^2 v}{\partial x_2^2} = 0.$$

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Any such v is given by $v = \phi(f)$ for some A-differentiable f of class $C^2(\overline{U})$. Further, any such f is uniquely determined by two functions - f_0 in $C^3_{\mathbb{R}}(\overline{\pi_{D_0}(U)}, \mathbb{R})$, and f_1 in $C^2_{\mathbb{R}}(\overline{\pi_{D_1}(U)}, \mathbb{R})$, where

$$D_0 = \mathbb{R}[x]/(x-\lambda)^2$$
$$D_1 = \mathbb{R}[x]/(x-\lambda) \oplus \mathbb{R}[x]/(x-\lambda)^2,$$

and

$$\pi_{D_0} \colon B \to D_0/(x-\lambda)D_0, \quad \pi_{D_0}(x_1 e_1 + x_2 e_2 + x_3 e_3) = x_2[e_2],$$
$$\pi_{D_1} \colon B \to D_1/(x-\lambda)D_1, \quad \pi_{D_1}(x_1 e_1 + x_2 e_2 + x_3 e_3) = x_1[e_1] + x_2[e_2].$$

Let

$$\rho_{D_0} \colon D_0/(x-\lambda)D_0 \to B, \quad \rho_{D_0}(x_2[e_2]) = x_2 e_2,$$
$$\rho_{D_1} \colon D_1/(x-\lambda)D_1 \to B, \quad \rho_{D_1}(x_1[e_1] + x_2[e_2]) = x_1 e_1 + x_2 e_2,$$
$$\rho_{(e^0)} \colon \mathbb{R} \to A, \quad \rho_{(e^0)}(x) = x1,$$
$$\rho_{(e^1)} \colon \mathbb{R} \to A, \quad \rho_{(e^1)}(x) = xe.$$

The extension is given by

$$f(u) = \rho_{(e^0)} f_0(\pi_{D_0}(u)) + G^1_{(e^0)}(Df_0(\pi_{D_0}(u))(u - \rho_{D_0}\pi_{D_0} u) + \rho_{(e^1)} f_1(\pi_{D_1}(u))$$
$$= f_0(u_2[e_2])1 + u_3 \frac{\partial f_0}{\partial x_2}(u_2[e_2])e + f_1(u_1[e_1] + u_2[e_2])e.$$

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Therefore any solution to the equation $grad(w) = Mgrad(v)$ is of the form

$$v(u) = \phi(f(u)) = u_3 \frac{\partial f_0}{\partial x_2}(u_2[e_2]) + f_1(u_1[e_1] + u_2[e_2]),$$
$$w(u) = \phi(M^T f(u)) = f_0(u_2[e_2]) + \lambda u_3 \frac{\partial f_0}{\partial x_2}(u_2[e_2]) + \lambda f_1(u_1[e_1] + u_2[e_2]).$$

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Therefore any solution to the equation $grad(w) = Mgrad(v)$ is of the form

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$$w(u) = \phi(M^T f(u)) = f_0(u_2[e_2]) + \lambda u_3 \frac{\partial f_0}{\partial x_2}(u_2[e_2]) + \lambda f_1(u_1[e_1] + u_2[e_2]).$$

We see that there is a unique solution v of generalized Laplace equations such that it has fixed values on $\overline{\rho_{D_1}\pi_{D_1}(U)}$ and such that $w - \lambda v$ has fixed, up to a constant, values on $\overline{\rho_{D_0}\pi_{D_0}(U)}$.

Boundary value problem

$U \subset \mathbb{R}^n$ - convex, open and bounded set, $t \geq 2$. Then for any functions

$$f_i \in C_{\mathbb{F}}^{t+l-1-i}(\overline{\pi_{D_i}(U)}, \mathbb{F}), i = 0, \ldots, l-1,$$

there exists a unique $v \in C^t(\overline{U})$ such that

$$D^2 v(\cdot)(M^\intercal x, y) = D^2 v(\cdot)(x, M^\intercal y),$$
$$v|_{\overline{\rho_{D_{l-1}} \pi_{D_{l-1}}(U)}} = \mu_{\mathbb{F}} f_{l-1} \circ \pi_{D_{l-1}},$$
$$Dv(\cdot)((M^\intercal - \lambda I)^{l-1-i} x)|_{\overline{\rho_{D_i} \pi_{D_i}(U)}} = \mu_{\mathbb{F}} D(f_i \circ \pi_{D_i})(\cdot)(x),$$
$$x \in \rho_{D_i} \pi_{D_i}(B), i = 0, \ldots, l-2.$$

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&Dv(\cdot)((M^\intercal - \lambda I)^{l-1-i}x)|_{\overline{\rho_{D_i}\pi_{D_i}(U)}} = \mu_{\mathbb{F}} D(f_i \circ \pi_{D_i})(\cdot)(x),\\
&x \in \rho_{D_i}\pi_{D_i}(B), i = 0, \ldots, l-2.
\end{aligned}$$

The unique solution is given by $v = \phi_{\mathbb{F}}(Tf)$, where

$$Tf = \sum_{k=0}^{l-1} \left(\rho_{(e^k)} f_k(\pi_{D_k}(u)) + \sum_{j=1}^{l-1-k} \frac{1}{j!} G^j_{(e^k)} D^j f_k(\pi_{D_k}(u))((u - \rho_{D_k}\pi_{D_k}(u))^j) \right)$$

Thank you for your attention!

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http://arxiv.org/abs/1607.05624