Nodal sets of Laplace Eigenfunctions: pursuing the conjectures by Yau and Nadirashvili.

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Harmonic Analysis, Complex Analysis, Spectral Theory and all that.

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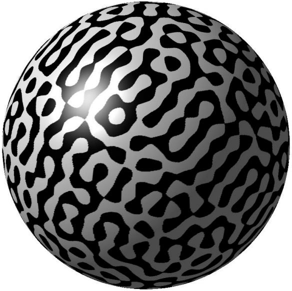
- Rescaled version in $\mathbb{R}^n$:

$$\text{Area}(\{u=0\} \cap B_R(0)) \geq cR^{n-1}.$$

Motivation: Yau's conjecture

- Let (M, g) be a compact C^∞-smooth Riemannian manifold (without boundary) of dimension n and let φ_λ be an eigenfunction of the Laplacian on M: $\Delta\varphi = -\lambda\varphi$.



Sign of a spherical harmonic of degree 40. Picture by Alex Barnett.

The nodal set $Z_{\varphi_\lambda} = \{\varphi = 0\}$.

Yau's conjecture:
$c\sqrt{\lambda} \le H^{n-1}(Z_{\varphi_\lambda}) \le C\sqrt{\lambda}$

for some c, C depending on (M, g) only and independent of λ

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- Thm(Donnelly & Fefferman). True for real analytic metrics.
- In dimension $n = 2$ the upper bound is open.
  $n = 2$: $c\sqrt{\lambda} \leq H^1(Z_{\varphi_\lambda}) \leq C\lambda^{3/4}$,
  The lower bound (n=2): Brunning + Yau.
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- Dong's inequality for $F = |\nabla\varphi|^2 + \frac{\lambda}{2}\varphi^2$:

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- Thm(A.L, Eu.Malinnikova): $H^1(Z_{\varphi_\lambda}) \leq C\lambda^{3/4-\varepsilon}, \varepsilon > 1/10^{10}$.

Bounds for Yau's conjecture, $n \geq 3$.

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- Let φ satisfy $\Delta \varphi + \lambda \varphi = 0$ in $\mathbb{R}^n$.
  Standard trick: define a harmonic function u in $\mathbb{R}^{n+1}$ by

  $$u(x,t) = \varphi(x) \exp(\sqrt{\lambda} t), \quad Z_u = Z_\varphi \times \mathbb{R}.$$

  Why $H^{n-1}(Z_\varphi \cap \{|x| < 1\}) \geq c\sqrt{\lambda}$ for $\lambda > \lambda_0$?

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- Projecting Nadirashvili's conjecture:
  If $\varphi(x) = 0$, then $H^{n-1}(Z_\varphi \cap B_1(x)) \geq c$ (not enough).
  Rescaling and projecting Nadirashvili's conjecture:
  $$H^{n-1}(Z_\varphi \cap B_{1/\sqrt{\lambda}}(x)) \geq c \left(\frac{1}{\sqrt{\lambda}}\right)^{n-1}.$$

From Nadirshvili's conjecture to Yau's conjecture

$$u(x,t) = \varphi(x)\exp(\sqrt{\lambda}t), \quad Z_u = Z_\varphi \times \mathbb{R}$$

Harnack's inequlaity: Z_u is $\frac{C}{\sqrt{\lambda}}$ dense in $\mathbb{R}^{n+1}$.

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Z_φ is also $\frac{C}{\sqrt{\lambda}}$ dense in $\mathbb{R}^n$.

One can find $\sim \lambda^{n/2}$ disjoint balls $B(x_i, \frac{1}{\sqrt{\lambda}})$ in B_1 such that $\varphi(x_i) = 0$. Rescaling (and projecting) Nadirashvili's conjectrure:

$$H^{n-1}(Z_\varphi \cap B_{1/\sqrt{\lambda}}(x_i)) \geq c\left(\frac{1}{\sqrt{\lambda}}\right)^{n-1}.$$

Thus $H^{n-1}(Z_\varphi \cap \{|x| < 1\}) \geq c\sqrt{\lambda}$.

Growth of Laplace eigenfunctions on compact manifolds

$$\Delta\varphi + \lambda\varphi = 0$$

Donnelly-Fefferman growth estimate for Laplace eigenfunctions on compact Riemannian manifolds:
For any geodesic ball $B_r(x)$

$$\frac{\sup_{B_{2r}(x)} |\varphi|}{\sup_{B_r(x)} |\varphi|} \leq 2^{C\sqrt{\lambda}}.$$

Old question for elliptic PDE:
Estimates for the zero set in terms of growth.

Zeroes and growth of harmonic functions on the plane.

Let u be a harmonic function in $\mathbb{R}^2$.
Doubling index:

$$N(B_r) = \log \frac{\max_{B_{2r}} |u|}{\max_{B_r} |u|}$$

Theorem(Nadirashvili, Robertson, Gelfond)

$$cN(B_{1/2}) - C \leq H^1(Z_u \cap B_1) \leq CN(B_2) + C$$

Nadirashvili's conjecture(true): Zero set of a non-constant harmonic function in $\mathbb{R}^3$ has infinite area.

Frequency

- Let $H(x,r) = \frac{\int_{\partial B_r(x)} u^2}{|\partial B_r|}$ be the meanvalue of u^2 over $\partial B_r(x)$.
- Frequency of a harmonic function:

$$N(B_r(x)) = \frac{rH'(x,r)}{H(x,r)}.$$

- Frequency is comparable to doubling index.
- Growth of harmonic functions:

$$\frac{H(1)}{H(1/2)} \leq 2^{N(1)} \leq \frac{H(2)}{H(1)} \leq 2^{N(2)}.$$

- Q. Estimates of $H^{n-1}(Z_u)$ in terms of the frequency.

Frequency is monotonic

$$N(B_r) = \frac{rH'(r)}{H(r)}, H(r) = \frac{\int_{\partial B_r} u^2}{|\partial B_r|}.$$

- Frequency is a monotonic function of r.
- If u is a homogeneous harmonic polynomial, then N is exactly twice the degree of u.
- $\lim_{r \to +0} N(B_r) =$ twice the vanishing order of u at 0.
  If $u(x) = 0$, then $N(B_r(x)) \geq 2$.

Is the frequency "additive" in some sense?

Goal:To show that if the $H^{n-1}(Z_u \cap B)$ is big, then $N(2B)$ is also big.

- Model question: Given many disjoint balls B_1, B_2,. . . , B_k with $N(B_i) \geq 1$ for each i, how large must be the frequency for a giant ball B, which contains all small balls B_i?
  Monotonicity of the frequency: if $r \leq 1$, then

$$N(rB) \leq N(B).$$

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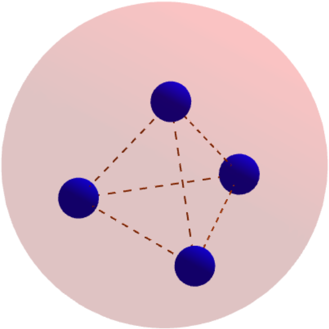
- Suppose that $N(B_1) = N(B_2) = N$.
  It is not true that if a ball B contains $B_1 \cup B_2$, then $N(B) > N$.
  For instance, if $u(x, y, z) = \Re(x + iy)^n$ and a point p lies on the z-axis, then $N(B_r(p)) = 2n$ for any $r > 0$.

4 balls are enough in $\mathbb{R}^3$



Blue balls B_i are disjoint balls with the same radii and centers at the vertices of the equilateral simplex.

Simplex Lemma:

If for each blue ball $N(B_i) \geq A > 1000$, $n = 1, 2, 3, 4$, then the frequency of the giant red ball $N(B) > A(1 + c)$, $c > 0$.

Hint.

Euclidean geometry fact: with four unit balls in $\mathbb{R}^3$ one can cover a slightly bigger ball with radius $1 + \varepsilon$.

Tools: Three spheres theorem + monotonicity of the frequency

Is the frequency "additive" in some sense?

Let $x_1, x_2, \ldots, x_{n+1}$ be the vertices of a non-degenerate simplex S in $\mathbb{R}^n$. The relative width $w(S)$ is defined as $\frac{\text{width}(S)}{\text{diam}(S)}$. Let x_0 be the barycenter of S.

Simplex lemma.

Let $r \leq \text{diam}(S)$ and $B_i = B(x_i, r)$.
If $N(B_i) \geq N$, $i = 1, \ldots, n+1$, then

$$N(x_0, A\,\text{diam}(S)) \geq (1+c)N - C.$$

Where $c = c(w(S), n) > 0$, $A = A(w(S), n) > 1$ and $C = C(w(s), n) > 0$.

Is the frequency "additive" in some sense?

What happens if the balls are concentrated near the hyperplane?
Fix $n = 3$.

Hyperplane lemma

If $N(B_1(i,j,0)) \geq N$ for $i = -100, \ldots, 100$, $j = -100, \ldots, 100$, then

$$N(B_{100}) \geq 2N - C.$$

Hint.

The quantitative version of the Cauchy uniqueness theorem:
Let u be a harmonic function in the unit cube Q such that $|u| < 1$ in Q. Let F be one face of Q. If $|u|$ and $|\nabla u|$ are smaller than ε on F, then

$$|u| \leq \varepsilon^{\alpha}$$

in $\frac{1}{2}Q$, $\alpha = 1/100$.

For a given cube Q define $N(Q) = \sup_{B \subset 2Q} N(B)$.

Simplex lemma + hyperplane lemma imply

Lemma. There exist an integer A depending on n only such that the following holds. Let Q be a cube in $\mathbb{R}^n$, which is partitioned into A^n equal subcubes Q_i. If there are $\frac{1}{2}A^{n-1}$ subcubes Q_i with $N(Q_i) \geq N$, then $N(Q) \geq N(1+c) - C$.

Iterations of the lemma are used in the proof of the polynomial upper bound

$$H^{n-1}(Z_u \cap Q) \leq CN^C(Q) \quad \Longrightarrow \quad H^{n-1}(Z_{\varphi_\lambda}) \leq C\lambda^C$$

and in the proof of the lower bound:

$$H^{n-1}(Z_u \cap B_1) \geq c2^{\frac{c \log N}{\log \log N}} \quad (\text{for } N = N(B_{1/2}) > C).$$

Lower bounds for harmonic functions.

- If $N(B_{1/2}) = N > C$, then

$$H^{n-1}(Z_u \cap B_1) \geq c2^{\frac{c \log N}{\log \log N}}.$$

- In particular, if $u(0) = 0$, then

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- This estimate is sufficient for the lower bound in Yau's conjecture

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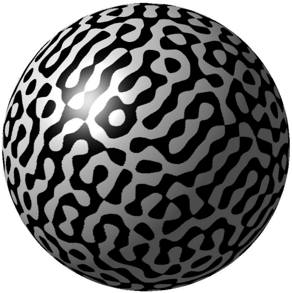
- Assuming the upper bound $H^{n-1}(Z_\varphi) \leq C\sqrt{\lambda}$:
- Most of the balls $B_{\frac{1}{\sqrt{\lambda}}}(x_i)$,covering the manifold, satisfy

$$N(B_{\frac{1}{\sqrt{\lambda}}}(x_i)) < C.$$

If Yau's conjecture is true:

$$c\sqrt{\lambda} \leq H^{n-1}(Z_{\varphi_\lambda}) \leq C\sqrt{\lambda}.$$



Then $\quad c \leq \dfrac{H_n(\varphi > 0)}{H_n(\varphi < 0)} \leq C.$

Happy Birthday Sasha!!