Ratios of harmonic functions

Eugenia Malinnikova
NTNU (Trondheim, Norway)

joint work with A. Logunov (St.Petersburg/Tel Aviv)

Harmonic Analysis, Complex Analysis, Spectral Theory and all that
Bedlewo, August 2, 2016

Ratios of harmonic functions and Harnack's inequalities

Let $\Omega \subset \mathbb{R}^n$, $u, v : \Omega \to \mathbb{R}$, $\Delta u = \Delta v = 0$. Suppose that the zero sets of u and v coincide: $Z(u) = Z(v) = Z$. We consider $f = u/v$.

Claim: f is a real-analytic function which satisfies

$$\sup_K |f| \leq C_1 \inf_K |f|,$$

$$\sup_K |\nabla f| \leq C_2 \inf_K |f|,$$

where $K \subset\subset \Omega$ and the constants C_1, C_2 depend on K and the nodal set Z only.

The result was proved by Dan Mangoubi (Jerusalem) for $n = 2$ in 2014. Dan also conjectured that it holds in higher dimensions and that in dimension two the constants depend only on the number of nodal domains.

Examples in dimension two

Let $f : \Omega \to \mathbb{D}$ be a bounded analytic function and $h : \mathbb{D} \to \mathbb{R}$ is a harmonic function such that $h(x + iy) > 0$ if and only if $x > 0$. then

$$u = \Re(f), \quad v = h(f)$$

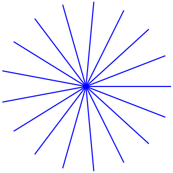
is a pair of harmonic functions with a common zero set.

If we fix a harmonic function u in Ω and let $Z = Z(u)$, we can find a large family of functions with this zero set. It is a local property of functions on a bounded domain Ω.

Globally, if three harmonic functions in $\mathbb{R}^2$ have the same zero set then they are linearly dependent.

More specific examples in dimension two

Let $u = \Im z^k$ and $\Omega = \mathbb{D}$, Z(u) is



Suppose that $c_j \in \mathbb{R}$ are small ($\sum_j |c_j| < 1$) and choose

$$v = u + \Im \left(\sum_{j=2}^{\infty} c_j \Im z^{jk} \right).$$

Then $Z(v) = Z(u)$ and

$$f = v/u = 1 + \sum_{j=2}^{\infty} c_j \sum_{m=0}^{[(j-1)/2]} (-1)^m u^{2m} \tilde{u}^{j-2m-1}, \quad z^k = \tilde{u} + iu.$$

Compactness of the family of functions with bounded number of nodal domains

Lemma (N. Nadirashvili, 1991)

Let u_n be a sequence of harmonic functions in $\mathbb{D}$ and let $N \in \mathbb{N}$. Suppose that the number of nodal domains of each u_n is less than N. Then there exist a subsequence u_{n_k}, a sequence α_{n_k} of real numbers and a non-zero function u such that $\alpha_{n_k} u_{n_k}$ converge to u uniformly on compact subsets of $\mathbb{D}$. Clearly, u is harmonic in $\mathbb{D}$.

It follows from the fact that given a sequence of continuous functions on $[0, 1]$ with uniformly bounded number of zero points, one can choose a subsequence that after a renormalization converges to a non-zero distribution.

Structure results

Lemma

Let $\{u_n\}$ and $\{v_n\}$ be sequences of harmonic functions in $\mathbb{D}$ such that $Z(u_n) = Z(v_n), u_n = f_n v_n, f_n > 0$ and $u_n \rightrightarrows u$, $v_n \rightrightarrows v$ in $\mathbb{D}$, where u and v are non-zero functions. Then $Z(u) = Z(v)$.

Proposition

Let U and V be analytic functions in $\mathbb{D}$ such that $Z(\Im U) = Z(\Im V)$. Assume also that $\Omega = V^{-1}\{r_1 < |z| < r_2\}$ is connected for some $r_1 < r_2$ and there exists integer k such that $V|_\Omega$ is a k-cover of $\{r_1 < |z| < r_2\}$. Then we have $U(z) = g \circ V(z)$ when $z \in \Omega$, where g is an analytic function on $\{|z| < r_2\}$ with real coefficients.

Refinement of the result for $n = 2$

Theorem (L-M,2016)

Let u and v be harmonic functions in the unit disc $\mathbb{D} \subset \mathbb{R}^2$ such that $Z(u) = Z(v)$ and suppose the number of nodal domains of u (and v) is less than a fixed number N. Let f be the ratio of u and v, then for any compact set $K \subset \mathbb{D}$ there exist constants $C_1 = C_1(K, N)$ and $C_2 = C_2(K, N)$ depending on K and N only such that the Harnack inequality

$$\sup_K |f| \leq C_1 \inf_K |f|; \tag{1}$$

and the following gradient estimate

$$\sup_K |\nabla f| \leq C_2 \inf_K |f|, \tag{2}$$

hold.

Harmonic polynomials and Brelot-Choquet theorem

Harmonic functions are better than real analytic!

Lemma (Division Lemma)

Suppose Q is a homogeneous harmonic polynomial and P is a polynomial such that $Z(Q) \subset Z(P)$. Then $P = QR$ for some $R \in \mathbb{R}[x_1, x_2, \ldots, x_n]$

Let u be real analytic and v be harmonic

$$u = \sum_{i=k}^{\infty} u_i, \quad v = \sum_{i=l}^{\infty} v_i,$$

where u_i and v_i denote homogeneous polynomials of degree i; u_k and v_l are non-zero polynomials.

Lemma

If $Z(v) \subset Z(u)$, then $Z(v_l) \subset Z(u_k)$.

Division by power series in higher dimensions

Theorem (L-M, 2015)

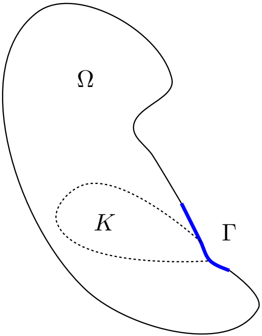
Suppose that v is harmonic, u is real analytic and $Z(v) \subset Z(u)$. If $Z(v) \subset Z(u)$, then there exist a real-analytic function f in Ω such that $u = vf$.
If u is also harmonic then f satisfies the maximum and minimum principles.

The proof also implies the estimates

$$\sup_K D^\alpha f \leq C_{\alpha,K} \sup_K f$$

Thus the Harnack principle for the ratios would imply the gradient estimate.

Boundary Harnack Principle



Let Ω be a *good* domain in $\mathbb{R}^n$ and $\Gamma \subset \partial\Omega$. If u, v are positive harmonic in Ω, $u, v \in C(\Omega \cup \Gamma$ and $u|_\Gamma = v|_\Gamma = 0$, then

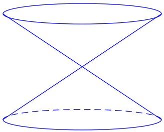
$$\inf_K |u/v| \geq C(K, \Omega) \sup_K |u/v|$$

when $K \subset \Omega \cup \Gamma$.

Kemper, Ancona, Wu, Aikawa, Bass, Burdzy, Bañuelos, Popovici, Volberg

Examples in dimension three: bad news

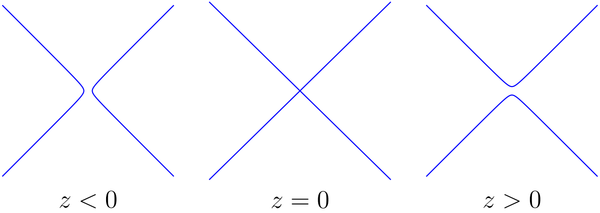
Example 1



$Z_1 = Z(u) = \{x^2 + y^2 - 2z^2 = 0\}$, does there exist a harmonic function v in the unit ball such that $v \neq cu$ and $Z(v) = Z(u)$?

Example 2

$Z_2 = \{x^2 - y^2 + z^3 - 3x^2 z = 0\}$, some sections of Z_2:



$\Omega = \{x^2 - y^2 + z^3 - 3x^2 z > 0\}$ is a very bad domain (violates the Harnack chain condition)

Examples in higher dimensions: good news (no pictures)

Example 3 (Axially symmetric harmonic functions in $\mathbb{R}^4$)

$$u_k(x, x_4) = \frac{\Im(x_4 + i|x|)^{3k}}{|x|}, \quad x \in \mathbb{R}^3, \quad k \in \mathbb{N},$$

$Z(u_k) \supset Z(u_1)$, $u_1 = 3x_4^2 - |x|^2$, and we have local families of harmonic functions with the same zero set.

Example 4 (A family of harmonic functions in $\mathbb{R}^9$)

$$h_k(x_1, x_2, x_3) = \frac{\sin(3k|x_1|)\sin(4k|x_2|)\sinh(5k|x_3|)}{|x_1||x_2||x_3|}, \quad x_1, x_2, x_3 \in \mathbb{R}^3,$$

once again, $Z(h_k) \supset Z(h_1)$ for $k \in \mathbb{N}$.

Łojasiewicz exponent

For any function f, real analytic in B_1, with $Z(f) \neq \emptyset$ there exist constants $l, L, \gamma > 0$ depending on f such that

$$L \cdot d(x, Z(f)) \geq |f(x)| \geq l \cdot d(x, Z(f))^{\gamma}$$

for any $x \in B_{1/2}$, γ is called the Łojasiewicz exponent of f.

Main lemma 1

We fix a harmonic function v and assume that $\sup_{B_{1/2}} |v| = 1$, let u be another harmonic function (varying) with $Z(u) = Z(v) = Z$. We denote $d(x, Z) = \delta(x)$.

Lemma

There exists a constant $C = C(v) > 1$ such that for any $x \in B_{1/4}$ with $v(x) \neq 0$ there is $\tilde{x} \in B_{1/2}$ with $|x - \tilde{x}| \leq \frac{3}{4}\delta(x)$ and $|v(\tilde{x})| \geq C|v(x)|$.

Using this lemma and starting with some $x_1 \in B_{1/4}$ we construct a sequence $x_1, ..., x_m$ with $x_m \in B_{1/2} \setminus B_{1/4}$.

First chain

We have $1 \geq |v(x_m)| \geq C^m |v(x_1)|$. Using the Łojasiewiecz exponent we can also show that $\delta(x_m) > c = c(v)$.

Now if we consider the values of u along this sequence, we see by the usual Harnack inequality that

$$|u(x_m)| \geq q^m |u(x_1)|.$$

Then $|u(x_m)| \geq |u(x_1)||v(x_1)|^\alpha$.

Thus for every $x \in B_{1/4}$ there exists $y \in B_{1/2}$ such that

$$|u(y)| \geq |u(x)||v(x)|^\alpha, \ \delta(y) > c.$$

Main lemma 2 and the second chain

Lemma

There exists a constant $K > 1$ depending on the dimension n only such that for any function u harmonic in B_1 with $Z(u) \cap B_{1/2} \neq \emptyset$ and any $x \in B_{1/4}$ with $u(x) \neq 0$ and $\delta_u(x) < (4K)^{-1}$ there exists a point $\tilde{x}$ for which $|\tilde{x} - x| \leq K\delta(x)$ and $|u(\tilde{x})| \geq 2|u(x)|$.

Suppose that $\sup_{\delta(x)>c, x\in B_{1/2}} |u| = 1$ but $u(x_0) = M >> 1$ for some $x_0 \in B_{1/8}$. We construct points x_j applying the second lemma.

For each such x_j we can find y_j with $1 \geq |u(y_j)| \geq |u(x_j)||v(x_j)|^\alpha$ and $|u(x_j)| > 2^j M$.Therefore $|v(x_j)| \leq 2^{-j/\alpha} M^{-1/\alpha}$ and

$$\delta(x_j) \leq C_v M^{-\beta} 2^{-j\beta}$$

by the Łojasiewicz inequality! Contradiction.

Final step

We have the following inequality now

$$\sup_{B_{1/8}} |u| \leq C(v) \sup_{x \in B_{1/2}, \delta(x) > c} |u|.$$

The estimates for the ratio follow now from the usual Harnack inequality inside the nodal domains and a simple compactness argument.