Compensated Compactness
Interpolatory Estimates
Riesz Transforms, Wavelet- and Haar Projections

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Content

Primary Sources

- J. Lee, S. Müller and PFXM, Compensated Compactness, and interpolatory estimates between Riesz transforms and Haar projections.
  Comm. P. D. E. 36; (2011) 547–601.

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- J. Lee, S. Müller and PFXM, Compensated Compactness, and interpolatory estimates between Riesz transforms and Haar projections.
  Comm. P. D. E. 36; (2011) 547–601.
- S. Müller and PFXM, Interpolatory Estimates, Riesz Transforms and Wavelet Projection. **Revista Math. Iberoam.** (2016).
- A. Kamont and PFXM, Directional wavelet Projections–revisited. Preprint 2016

Fourier- and Riesz Transforms

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Fourier Transformation

$$\mathcal{F}(u)(\xi) = \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi} dx$$

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The Riesz transformation as Fourier multiplier

$$\mathcal{F}(R_j(u))(\xi) = -\sqrt{-1}\frac{\xi_j}{|\xi|}\mathcal{F}(u)(\xi) \quad \text{with} \quad 1 \leq i \leq n, \quad \xi = (\xi_1, \ldots, \xi_n).$$

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R_j is $L^p(\mathbb{R}^n)-$bounded

$$\|R_j\|_p \leq Cp^2/(p-1), \quad 1 < p < \infty.$$

Lower Semi-continuity

Theorem (J. Lee, S. Müller, P.F.X.M.)

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then we have lower semi-continuity: $\forall \varphi \geq 0$,

$$\int_{\mathbb{R}^n} f(v(x))\varphi(x)dx \leq \liminf_{r \to \infty} \int_{\mathbb{R}^n} f(v_r(x))\varphi(x)dx,$$

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provided that

$$\forall i \neq j \qquad \|R_i(v_r^{(j)})\|_{L^p(\mathbb{R}^n)} \to 0.$$

Semi continuity and convexity.

Let $1 < p < \infty$, $f : \mathbb{R}^n \to \mathbb{R}^+$, $0 < f(x) \leq C(1 + |x|^p)$.

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A) Fatou on lower semi continuity

If f is continuous and $w_j \to w$ in L^p norm convergent, then

$$\int_{[0,1]^n} f(w(x))dx \leq \liminf_{j\to\infty} \int_{[0,1]^n} f(w_j(x))dx.$$

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B) Hahn-Banach on lower semi continuity

If f is convex and $w_j \rightharpoonup w$ in L^p weakly convergent then again

$$\int_{[0,1]^n} f(w(x))dx \leq \liminf_{j\to\infty} \int_{[0,1]^n} f(w_j(x))dx,$$

and conversely.

Weak lower semi continuity implies convexity

Let $\mathrm{h} : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{n}}$ be bounded and $[0,1]^{\mathrm{n}}$ periodic. We prove

$$\mathrm{f}\left(\int_{[0,1]^{\mathrm{n}}} \mathrm{h}(\mathrm{x})\mathrm{dx}\right) \leq \int_{[0,1]^{\mathrm{n}}} \mathrm{f}(\mathrm{h}(\mathrm{x}))\mathrm{dx}.$$

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$$f(\int_{[0,1]^n} h(x)dx) \leq \int_{[0,1]^n} f(h(x))dx.$$

Form highly oscillatory and hence weakly convergent, $w_j(x) = h(jx)$,

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Weak lower semi continuity implies Jensen's Inequality

$$f\left(\int_{[0,1]^n} h(x)dx\right) \le \liminf_{j\to\infty} \int_{[0,1]^n} f(w_j(x))dx = \int_{[0,1]^n} f(h(x))dx.$$

Interpretation

Comparison of A and B:

Extending the class of admissible testing sequences from norm-converging to weakly converging **is compensated by restricting** the class of admissible integrands f from continuous to convex.

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The weakly convergent sequence satisfies **Jensen's inequality** with respect to f.

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Extending the class of admissible testing sequences from norm-converging to weakly converging **is compensated by restricting** the class of admissible integrands f from continuous to convex.

Emphasize

The weakly convergent sequence satisfies **Jensen's inequality** with respect to f.

Compensated Compactness obtains

weak lower semicontinuity of non convex Lagrangians.This is possible only when weakly converging testing functions satisfy additional constrains.

Quasi- Convexity

Gradients are curl-free

$\mathrm{w} : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$.

$$\mathrm{curl w} = (\partial_{\mathrm{i}} \mathrm{w}^{(\mathrm{m,j})} - \partial_{\mathrm{j}} \mathrm{w}^{(\mathrm{m,i})})_{\mathrm{i,j}=1}^{\mathrm{n}}, \qquad \mathrm{m} \leq \mathrm{n}.$$

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Jensen's inequality for gradients = quasi convex.

If $L : \mathbb{R}^{n\times n} \to \mathbb{R}^+, \quad L(x) \le (1 + |x|^p)$ satisfies Jensen's inequality for gradients,

$$\int_{[0,1]^n} L(a + w) \ge L(a), \quad \int_{[0,1]^n} w = 0, \quad \mathrm{curl} w = 0.$$

then $L : \mathbb{R}^{n\times n} \to \mathbb{R}^+$ is called quasi convex.

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Assume quasi convexity, $\mathrm{L} : \mathbb{R}^{n \times n} \to \mathbb{R}^{+}, \quad \mathrm{L}(x) \leq (1 + |x|^{p})$ and

$$\int_{[0,1]^{n}} \mathrm{L}(a + w) \geq \mathrm{L}(a), \quad \int_{[0,1]^{n}} w = 0, \quad \mathrm{curl} w = 0.$$

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If

$$\begin{cases} w_r \rightharpoonup v \quad \text{weakly in} \quad L^p(\mathbb{R}^n, \mathbb{R}^{n\times n}), \\ \operatorname{curl}(w_r) = 0, \end{cases}$$

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then

$$\int L(w)dx \le \liminf \int L(w_k)dx,$$

and conversely.

Morrey's Theorem extended by Murat and Tartar.

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$$\int \mathrm{L}(w)dx \leq \liminf \int \mathrm{L}(w_k)dx.$$

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then

$$\int L(w)dx \le \liminf \int L(w_k)dx.$$

Note : $w_k(x)$ are weakly converging $n \times n$ matrices.

Recall Decomposition Principle for curl.

For sequences $(\mathrm{v_r} : \mathbb{R}^n \to \mathbb{R}^n)$ satisfying

$$\mathrm{v_r} \rightharpoonup 0 \quad \text{weakly in} \quad \mathrm{L^p}, \quad \mathrm{curl v_k} \to 0 \quad \text{in} \quad \mathrm{W^{-1,p}}$$

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there exists a decomposition $\mathrm{v_r} = \mathrm{u_r} + \mathrm{w_r}$, so that :

$$\mathrm{curl w_k} = 0, \quad \left(\Longrightarrow \int_{[0,1]^n} \mathrm{L(a + w_k)} \geq \mathrm{L(a)}, \right)$$

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$$\text{curl} w_k = 0, \quad \left(\Longrightarrow \int_{[0,1]^n} L(a + w_k) \geq L(a), \right)$$

and

$$\|u_k\|_{L^p} \to 0.$$

The Ball-Murat specialization

The system

$$\mathcal{A}_0(\mathrm{v}) = \mathrm{grad}(\mathrm{v}) - \mathrm{diag}(\partial_1 \mathrm{v}_1, \dots \partial_\mathrm{n} \mathrm{v}_\mathrm{n}), \qquad \mathrm{v} = (\mathrm{v}_1, \dots, \mathrm{v}_\mathrm{n}), \mathrm{v}_\mathrm{i} : \mathbb{R}^\mathrm{n} \to \mathbb{R}.$$

satisfies

$$\mathcal{A}_0(\mathrm{v}) = 0 \Longrightarrow \mathrm{v}_\mathrm{i}(\mathrm{x}) = \mathrm{v}_\mathrm{i}(\mathrm{x}_\mathrm{i}).$$

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If f is separately convex and $\mathcal{A}_0(\mathrm{v}) = 0$ then Jensen's inequality holds

$$\int_{[0,1]^n} \mathrm{f}(\mathrm{a} + \mathrm{v}) \geq \mathrm{f}(\mathrm{a}), \quad \int_{[0,1]^n} \mathrm{v} = 0.$$

The Ball-Murat specialization

Following Ball-Murat we specialize Morrey's theorem to **diagonal** matrices

$$\mathrm{w}(\mathrm{x}) = \sum_{\mathrm{m}=1}^{\mathrm{n}} \mathrm{v}^{(\mathrm{m})}(\mathrm{x}) \mathrm{e}_{\mathrm{m}} \otimes \mathrm{e}_{\mathrm{m}}, \quad \text{and} \quad \mathrm{L}(\mathrm{w}) = \mathrm{f}(\mathrm{v}^{(1)}, \ldots, \mathrm{v}^{(\mathrm{n})}).$$

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The **curl for diagonal** martices w : $\mathrm{curl}(w) \iff \mathcal{A}_0(v)$.

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Theorem of Tartar:

Weak lower semicontinuity of

$$\int \mathrm{f}(\mathrm{v}_\mathrm{r}(\mathrm{x}))\mathrm{dx}, \qquad \mathcal{A}_0(\mathrm{v}_\mathrm{r}) \text{ pre} - \text{compact in } \mathrm{W}^{-1,\mathrm{p}}$$

implies that f is separately convex.

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Problem of Tartar

Does separate convexity imply weak lower semicontinuity?

Theorem (J. Lee, S. Müller, P.F.X.M.)

Let $\mathrm{f} : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{+},$ separatly convex, with, $0 \leq \mathrm{f}(\mathrm{x}) \leq (1 + |\mathrm{x}|^{\mathrm{p}})$.

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Let $f : \mathbb{R}^n \to \mathbb{R}^+,$ separatly convex, with, $0 \leq f(x) \leq (1 + |x|^p)$.

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implies that for each non negative testing function φ,

$$\int_{\mathbb{R}^n} f(v(x))\varphi(x)dx \leq \liminf_{r\to\infty} \int_{\mathbb{R}^n} f(v_r(x))\varphi(x)dx.$$

Recall. $\mathcal{A}_0(u) = \text{grad}(v) - \text{diag}(\partial_1 v_1, \dots \partial_n v_n)$,
$v = (v_1, \dots, v_n), v_i : \mathbb{R}^n \to \mathbb{R}$.

The decomposition. (J. Lee, S. Müller, P.F.X.M.)

For sequences $(v_r : \mathbb{R}^n \to \mathbb{R}^n)$ with support in the unit cube satisfying

$$v_r \rightharpoonup 0 \quad \text{weakly in} \quad L^p, \quad \mathcal{A}_0(v_r) \to 0 \quad \text{in} \quad W^{-1,p}$$

there exists a decomposition $v_r = u_r + w_r$, so that :

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1. For each separately convex f we have Jensen's Inequality

$$\int_{[0,1]^n} f(a + u_r(x))dx \geq f(a).$$

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$$\int_{[0,1]^n} f(a + u_r(x))dx \geq f(a).$$

2. $$\|w_r\|_{L^p} \to 0.$$

The Haar System

Dyadic intervals $\mathcal{D}$ in $\mathbb{R}$ are $[\mathrm{k}2^{-\mathrm{n}}, (\mathrm{k}+1)2^{-\mathrm{n}}[$. A Haar function $\mathrm{h_I}$ is supported on $\mathrm{I} \in \mathcal{D}$ and $\mathrm{h_I} = 1$ on the left half of I and $\mathrm{h_I} = -1$ on the right half of I.

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Isotropic Haar basis in $L^2(\mathbb{R}^n)$:

$x=(x_1,\dots,x_n),\quad |I_1|=\cdots=|I_n|,\quad \mathcal{A}=\{0,1\}^n\setminus\{0\}.$

$$h^{(\varepsilon)}_{I_1\times\cdots\times I_n}(x)=h^{\varepsilon_1}_{I_1}(x_1)\cdots h^{\varepsilon_n}_{I_n}(x_n),\qquad \varepsilon=(\varepsilon_1,\dots,\varepsilon_n)\in\mathcal{A}.$$

Subsystems of the Haar system

The j−th unit vector: $e_j \in \mathbb{R}^n$.

$$\mathcal{H}^j = \{h^{(e_j)}_{I_1 \times \cdots \times I_n} : |I_1| = \cdots = |I_n|\}.$$

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$P^{(e_j)}$ is the orthogonal projection onto $\mathrm{span}\mathcal{H}^{(j)}$.

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$$P(v) = (P^{(e_1)}(v_1), \ldots, P^{(e_n)}(v_n)) \quad v = (v_1, \ldots, v_n)$$

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If f is separately convex then Jensen's inequality holds on the range of P,

$$\int_{[0,1]^n} f(P(v(x))dx \geq f\left(\int_{[0,1]^n} P(v)(x)dx\right).$$

Directional Haar projection:

For $\varepsilon \in \mathcal{A} = \{0,1\}^{\mathrm{n}} \setminus 0$ put $P^{(\varepsilon)} u = \sum_{Q} \langle u, h_Q^{(\varepsilon)} \rangle h_Q^{(\varepsilon)} |Q|^{-1}.$

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If $\varepsilon = (\varepsilon_1, \ldots \varepsilon_n) \in \mathcal{A}$ with $\varepsilon_{i_0} = 1$, then

$$||P^{(\varepsilon)}(u)||_p \leq A_p \|u\|_p^{1/2} \|R_{i_0}(u)\|_p^{1/2}, \qquad 2 \leq p < \infty.$$

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Exponents are sharp! For instance,

$$\sup_{w \in L^p} \frac{\|P^{(\varepsilon)} w\|_p}{\|R_{i_0} w\|_p^{1/2+\delta} \|w\|_p^{1/2-\delta}} = \infty, \qquad 2 \leq p < \infty.$$

Harvest: L^p Estimates for $(v - P(v))$.

$$\textit{If} \quad v_r \rightharpoonup 0 \quad \textit{weakly in} \quad L^p \qquad \mathcal{A}_0(v_r) \to 0 \quad \textit{in} \quad W^{-1,p}$$

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$$\|R_j w\|_p \le C \|\partial_j w\|_{W^{-1,p}} + \|Tw\|_p,$$

The decomposition made explicit

For sequences $(v_r : \mathbb{R}^n \to \mathbb{R}^n)$ with support in the unit cube satisfying

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Replace the Haar system by Hölder smooth wavelets. Conversely, do the interpolatory exponents contain then information about the smoothness of the wavelet system? (A. Kamont, S.Müller, PFXM).

Admissible wavelet systems (1).

$\mathcal{S}$ is the collection of all dyadic cubes in $\mathbb{R}^n$ and
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  where
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Haar projections are not limiting cases of the above. Check the exponents!

Dyadic decomposition of Haar Projections:

We resolve the discontinuities of the Haar system step by step. At each dyadic scale we relate the Riesz transforms to Haar projections.
$b \in C^{\infty}(\mathbb{R})$, with support in $[-1,1]$, so that

$$b(t) = b(-t),\ \mathrm{Lip}(b) \leq 8,\ \textit{and}\ \int_{-1}^{+1} b(t)dt = 1.$$

$$d_0(x) = 2^n b(2x_1) \cdot \dots \cdot b(2x_n) - b(x_1) \cdot \dots \cdot b(x_n).$$

$$u = \sum_{\ell=-\infty}^{\infty} u * d_\ell, \quad d_\ell(x) = d_0(2^\ell x) 2^{n\ell}.$$

Decomposing $P^{(\varepsilon)}(u)$

$$\mathcal{S}_j = \{Q \in \mathcal{S} : |Q| = 2^{-j}\}, \qquad \Delta_{j+\ell}(h_Q^{(\varepsilon)}) = h_Q^{(\varepsilon)} * d_\ell.$$

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$$T_\ell^{(\varepsilon)} u = \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{S}_j} \langle u, \Delta_{j+\ell}(h_Q^{(\varepsilon)})\rangle h_Q^{(\varepsilon)} |Q|^{-1}, \qquad P^{(\varepsilon)}(u) = \sum_{\ell=-\infty}^{\infty} T_\ell^{(\varepsilon)} u.$$

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The inverse $R_{i_0}^{-1}$ has symbol $\dfrac{|\xi|}{\xi_{i_0}} = \dfrac{\xi_{i_0}}{|\xi|} + \sum_{i=1, i \neq i_0} \dfrac{\xi_i}{\xi_{i_0}} \cdot \dfrac{\xi_i}{|\xi|}.$

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$$T_\ell^{(\varepsilon)} R_{i_0}^{-1} u = T_\ell^{(\varepsilon)} R_{i_0} u + \sum_{i=1, i\neq i_0} T_\ell^{(\varepsilon)} \mathbb{E}_{i_0} \partial_i R_i.$$

Fix $p \geq 2$. We have the Norm-Estimates

$$\| T_\ell^{(\varepsilon)} \|_p \leq 2^{-\ell/2}, \quad \| T_\ell^{(\varepsilon)} R_{i_0}^{-1} \|_p \leq 2^{+\ell/2}; \quad \ell > 0,$$

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$$2^{M/2}\|R_{i_0} u\|_p + 2^{-M/2}\|u\|_p \le C\|R_i u\|_p^{1/2}\|u\|_p^{1/2}.$$

Decomposing $W^{(\varepsilon)}(u)$

A similar decomposition based on Calderon's reproducing formula gives a decomposition of the wavelet projection

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Norm estimates for $T_\ell^{(\varepsilon)}, \quad T_\ell^{(\varepsilon)} R_{i_0}^{-1}$. Reduction to permutation operators and to projections onto block bases of the Haar system:

$$T_\ell^{(\varepsilon)} u = \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{S}_j} \langle u, \, \Delta_{j+\ell}(h_Q^{(\varepsilon)}) \rangle h_Q^{(\varepsilon)} |Q|^{-1}.$$

$\ell > 0 \longrightarrow$ projections. $\quad \ell < 0 \longrightarrow$ rearrangements.
$Q \in \mathcal{S}_j, \quad |Q| = 2^{-nj}. \quad D^{(\varepsilon)}(Q)$ discontinuities of $h_Q^{(\varepsilon)}$.
$D_{j+\ell}^{(\varepsilon)}(Q) = \{z : d(z, D^{(\varepsilon)}(Q)) \leq 2^{-(j+\ell)}\}$ strips of width $2^{-(j+\ell)}$ around the discontinuities.
$\Delta_{j+\ell}(h_Q^{(\varepsilon)})$ *lives on* $D_{j+\ell}^{(\varepsilon)}(Q)$ and oscyllates at scale $\sim 2^{-\ell} diam Q$.

$$|\Delta_{j+\ell}(h_Q^{(\varepsilon)})| \le C, \quad Lip(\Delta_{j+\ell}(h_Q^{(\varepsilon)})) \le C2^{\ell}/diamQ.$$

Cover $Q \cap D_{j+\ell}^{(\varepsilon)}(Q)$ with pairwise disjoint cubes of diameter $2^{-\ell}diamQ$.
$\longrightarrow \{E_1(Q), \ldots, E_M(Q)\}, \quad M \le C2^{n(\ell-1)}$.

$$\text{Form} \quad d_Q = \sum_{i=1}^{M} h_{E_i(Q)} \quad \text{and} \quad G(u) = \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{S}_j} \langle u,\, h_Q^{(\varepsilon)}\rangle d_Q |Q|^{-1},$$

and compare with

$$T_\ell^{(\varepsilon)*}u = \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{S}_j} \langle u,\, h_Q^{(\varepsilon)}\rangle \Delta_{j+\ell}(h_Q^{(\varepsilon)})|Q|^{-1}.$$

We have

$$\|T_\ell^{(\varepsilon)*}u\|_q \le C\|G(u)\|_q,$$

and

$$\|(T_\ell^{(\varepsilon)}R_{i_0}^{-1})^*u\|_q \le C2^\ell\|G(u)\|_q,$$

Estimates for the projection itself,

$$\|G(u)\|_q \le 2^{-\ell/2}\|u\|_q \quad \text{for} \quad q \le 2.$$

$$\|G(u)\|_q \le 2^{-\ell/p}\|u\|_q \quad \text{for} \quad q \ge 2.$$