Operator-valued (L^p, L^q) Fourier multipliers

Jan Rozendaal, joint work with Mark Veraar (TU Delft)

August 1, 2016

Table of contents

Notation

Fix $d \in \mathbb{N}$ and Banach spaces X and Y. Let $m : \mathbb{R}^d \to \mathcal{L}(X, Y)$ be strongly measurable.

The Fourier multiplier operator $T_m : \mathcal{S}(\mathbb{R}^d; X) \to \mathcal{S}'(\mathbb{R}^d; Y)$ associated with m is given by

$$T_m(f) := \mathcal{F}^{-1}(m\mathcal{F}(f)) \qquad (f \in \mathcal{S}(\mathbb{R}^d; X)).$$

Under which conditions is $T_m : L^p(\mathbb{R}^d; X) \to L^q(\mathbb{R}^d; Y)$ bounded for $1 \leq p < q \leq \infty$?

Notation

Fix $d \in \mathbb{N}$ and Banach spaces X and Y. Let $m : \mathbb{R}^d \to \mathcal{L}(X, Y)$ be strongly measurable.
The Fourier multiplier operator $T_m : \mathcal{S}(\mathbb{R}^d; X) \to \mathcal{S}'(\mathbb{R}^d; Y)$ associated with m is given by

$$T_m(f) := \mathcal{F}^{-1}(m\mathcal{F}(f)) \qquad (f \in \mathcal{S}(\mathbb{R}^d; X)).$$

Under which conditions is $T_m : L^p(\mathbb{R}^d; X) \to L^q(\mathbb{R}^d; Y)$ bounded for $1 \leq p < q \leq \infty$?

Scalar case

Let $X = Y = \mathbb{C}$ and $p \leq 2 \leq q$. Suppose that

$$\sup\{|\xi|^{d(\frac{1}{p}-\frac{1}{q})}|m(\xi)| \mid \xi \in \mathbb{R}^d\} < \infty.$$

Scalar case

Let

$$
\begin{aligned}
m_1(\xi) &:= |\xi|^{-d(\frac{1}{p}-\frac{1}{2})}, \\
m_2(\xi) &:= |\xi|^{d(\frac{1}{p}-\frac{1}{q})} m(\xi), \\
m_3(\xi) &:= |\xi|^{-d(\frac{1}{2}-\frac{1}{q})}.
\end{aligned}
$$

Then, using Sobolev embeddings,

$$
\begin{array}{ccccc}
L^p(\mathbb{R}^d) & \xrightarrow{T_{m_1}} & \dot{H}_p^{d(\frac{1}{p}-\frac{1}{2})}(\mathbb{R}^d) & \longrightarrow & L^2(\mathbb{R}^d) \\
\Big\downarrow T_m & & & & \Big\downarrow T_{m_2} \\
L^q(\mathbb{R}^d) & \xleftarrow[T_{m_3}]{} & \dot{H}_q^{d(\frac{1}{2}-\frac{1}{q})}(\mathbb{R}^d) & \longleftarrow & L^2(\mathbb{R}^d)
\end{array}
$$

Scalar case

Let

$$m_1(\xi) := |\xi|^{-d(\frac{1}{p}-\frac{1}{2})},$$
$$m_2(\xi) := |\xi|^{d(\frac{1}{p}-\frac{1}{q})} m(\xi),$$
$$m_3(\xi) := |\xi|^{-d(\frac{1}{2}-\frac{1}{q})}.$$

Then, using Sobolev embeddings,

$$\begin{array}{ccccc} L^p(\mathbb{R}^d) & \xrightarrow{T_{m_1}} & \dot{H}_p^{d(\frac{1}{p}-\frac{1}{2})}(\mathbb{R}^d) & \longrightarrow & L^2(\mathbb{R}^d) \\ \Big\downarrow T_m & & & & \Big\downarrow T_{m_2} \\ L^q(\mathbb{R}^d) & \xleftarrow[T_{m_3}]{} & \dot{H}_q^{d(\frac{1}{2}-\frac{1}{q})}(\mathbb{R}^d) & \longleftarrow & L^2(\mathbb{R}^d) \end{array}$$

Banach space case

For general Banach spaces, this approach requires restrictions on X, Y and m.

X and Y should be UMD spaces. That is, $\text{sign}(\cdot)$ should be a bounded L^2-multiplier.

Boundedness on L^2 uses Mikhlin's Theorem and requires smoothness of m_2:

$$\sup_{\xi\in\mathbb{R}^d} |\xi|^{|\alpha|} \| m_2^{(\alpha)}(\xi) \| < \infty.$$

Today's talk

For $p < q$ other conditions on X and Y can be used, and no smoothness is required of m.

Why might this interest you?

Applications to:

- Stability theory for semigroups;
- functional calculus theory;
- Schur multipliers;
- and more.

Gaussian sequence

Let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence of independent complex Gaussian random variables on some probability space $(\Omega, \mathbb{P})$.

Type and cotype

X has *type* $p \in [1,2]$ if

$$\Big(\mathbb{E}\Big\|\sum_{k=1}^{n}\gamma_k x_k\Big\|_X^p\Big)^{1/p} \lesssim \Big(\sum_{k=1}^{n}\|x_k\|_X^p\Big)^{1/p}$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$.

X has *cotype* $q \in [2,\infty)$ if

$$\Big(\sum_{k=1}^{n}\|x_k\|_X^q\Big)^{1/q} \lesssim \Big(\mathbb{E}\Big\|\sum_{k=1}^{n}\gamma_k x_k\Big\|_X^q\Big)^{1/q}$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$.

Type and cotype

X has *type* $p \in [1,2]$ if

$$\Big(\mathbb{E}\Big\|\sum_{k=1}^{n}\gamma_k x_k\Big\|_X^p\Big)^{1/p} \lesssim \Big(\sum_{k=1}^{n}\|x_k\|_X^p\Big)^{1/p}$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$.
X has *cotype* $q \in [2,\infty)$ if

$$\Big(\sum_{k=1}^{n}\|x_k\|_X^q\Big)^{1/q} \lesssim \Big(\mathbb{E}\Big\|\sum_{k=1}^{n}\gamma_k x_k\Big\|_X^q\Big)^{1/q}$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$.

Type and cotype

Each Banach space has type 1 and cotype ∞. X has type 2 and cotype 2 if and only if X is isomorphic to a Hilbert space.
For $r \in [1, \infty)$, $L^r(\Omega)$ has type $\min(r, 2)$ and cotype $\max(r, 2)$.

γ-boundedness

A collection $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ is γ-bounded if

$$\Big(\mathbb{E}\Big\|\sum_{k=1}^{n}\gamma_k T_k x_k\Big\|_Y^2\Big)^{1/2} \lesssim \Big(\mathbb{E}\Big\|\sum_{k=1}^{n}\gamma_k x_k\Big\|_X^2\Big)^{1/2}$$

for all $n \in \mathbb{N}$, $T_1, \ldots, T_n \in \mathcal{L}(X, Y)$ and all $x_1, \ldots, x_n \in X$.
Each γ-bounded collection is uniformly bounded.
Each uniformly bounded collection $\mathcal{T} \subseteq \mathbb{C} \subseteq \mathcal{L}(X)$ is γ-bounded.

γ-boundedness

A collection $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ is γ-bounded if

$$\Big(\mathbb{E}\Big\|\sum_{k=1}^{n}\gamma_k T_k x_k\Big\|_Y^2\Big)^{1/2} \lesssim \Big(\mathbb{E}\Big\|\sum_{k=1}^{n}\gamma_k x_k\Big\|_X^2\Big)^{1/2}$$

for all $n \in \mathbb{N}$, $T_1, \ldots, T_n \in \mathcal{L}(X, Y)$ and all $x_1, \ldots, x_n \in X$.
Each γ-bounded collection is uniformly bounded.
Each uniformly bounded collection $\mathcal{T} \subseteq \mathbb{C} \subseteq \mathcal{L}(X)$ is γ-bounded.

Theorem

Suppose that X has type $p_0 \in (1,2]$ and Y cotype $q_0 \in [2,\infty)$, and let $p \in (1,p_0)$, $q \in (q_0,\infty)$. Suppose that

$$\{|\xi|^{d(\frac{1}{p}-\frac{1}{q})} m(\xi) \mid \xi \in \mathbb{R}^d \setminus \{0\}\} \subseteq \mathcal{L}(X,Y)$$

is γ-bounded.
Then $T_m : L^p(\mathbb{R}^d;X) \to L^q(\mathbb{R}^d;Y)$ is bounded.

Other results

Sharp results hold for Banach lattices and in the Besov scale.

For smoother multipliers the boundedness results extrapolate to all (p, q) with $\frac{1}{p} - \frac{1}{q}$ constant.

Using a transference principle, one obtains results for multipliers on the torus $\mathbb{T}^d$.

γ-radonifying operators (Kalton, Weis (2002))

Let H be a Hilbert space. Then $\gamma(H,X)$ is the closure of $H\otimes X\subseteq\mathcal{L}(H,X)$ in the norm

$$\Big\|\sum_{k=1}^{n}h_k\otimes x_k\Big\|_{\gamma(H,X)}:=\Big(\mathbb{E}\Big\|\sum_{k=1}^{n}\gamma_k x_k\Big\|_X^2\Big)^{1/2}$$

for $h_1,\ldots,h_n\in H$ orthonormal.

Ideal property

If $R \in \mathcal{L}(X)$, $S \in \gamma(H, X)$ and $T \in \mathcal{L}(H)$ then $RST \in \gamma(H, X)$ and

$$\|RST\|_{\gamma(H,X)} \leq \|R\|_{\mathcal{L}(X)} \|S\|_{\gamma(H,X)} \|T\|_{\mathcal{L}(H)}.$$

Square functions

Let $\gamma(\mathbb{R}^d; X)$ consist of all strongly measurable, weakly L^2 functions $f : \mathbb{R}^d \to X$ such that

$$g \mapsto \int_{\mathbb{R}^d} g(s)f(s)\mathrm{d}s$$

is an element of $\gamma(L^2(\mathbb{R}^d), X)$.
If $f \in \gamma(\mathbb{R}^d; X)$ then $\mathcal{F}f \in \gamma(\mathbb{R}^d; X)$ and

$$\|\mathcal{F}f\|_\gamma = \|f\|_\gamma.$$

Square functions

Let $\gamma(\mathbb{R}^d; X)$ consist of all strongly measurable, weakly L^2 functions $f : \mathbb{R}^d \to X$ such that

$$g \mapsto \int_{\mathbb{R}^d} g(s) f(s) \mathrm{d}s$$

is an element of $\gamma(L^2(\mathbb{R}^d), X)$.
If $f \in \gamma(\mathbb{R}^d; X)$ then $\mathcal{F}f \in \gamma(\mathbb{R}^d; X)$ and

$$\|\mathcal{F}f\|_\gamma = \|f\|_\gamma.$$

γ-multiplier theorem

If $\{m(s) \mid s \in \mathbb{R}^d\} \subseteq \mathcal{L}(X, Y)$ is γ-bounded and $f \in \gamma(\mathbb{R}^d; X)$ then $mf \in \gamma(\mathbb{R}^d; Y)$ and

$$\|mf\|_{\gamma(\mathbb{R}^d;Y)} \leq \gamma(\{m(s) \mid s \in \mathbb{R}^d\})\|f\|_{\gamma(\mathbb{R}^d;X)}.$$

Fourier multipliers on $\gamma(\mathbb{R}^d; X)$

If $\{m(s) \mid s \in \mathbb{R}^d\} \subseteq \mathcal{L}(X, Y)$ is γ-bounded and $f \in \gamma(\mathbb{R}^d; X)$ then $\mathcal{F}^{-1}(m\mathcal{F}f) \in \gamma(\mathbb{R}^d; Y)$ and

$$\begin{aligned}\|\mathcal{F}^{-1}(m\mathcal{F}f)\|_{\gamma(\mathbb{R}^d;Y)} &= \|m\mathcal{F}f\|_{\gamma(\mathbb{R}^d;Y)} \\ &\leq \gamma(\{m(s) \mid s \in \mathbb{R}^d\})\|\mathcal{F}f\|_{\gamma(\mathbb{R}^d;X)} \\ &= \gamma(\{m(s) \mid s \in \mathbb{R}^d\})\|f\|_{\gamma(\mathbb{R}^d;X)}.\end{aligned}$$

Embeddings

For $s \in \mathbb{R}$ and $p \in [1, \infty]$ let $\dot{H}^s_p(\mathbb{R}^d; X)$ consist of all $f \in \mathcal{S}'(\mathbb{R}^d; X)$ such that

$$\mathcal{F}^{-1}(|\xi|^s \mathcal{F} f(\xi)) \in L^p(\mathbb{R}^d; X).$$

Then

$$\dot{H}_p^{d(\frac{1}{p}-\frac{1}{2})}(\mathbb{R}^d; X) \subseteq \gamma(\mathbb{R}^d; X)$$

and

$$\gamma(\mathbb{R}^d; Y) \subseteq \dot{H}_q^{d(\frac{1}{q}-\frac{1}{2})}(\mathbb{R}^d; Y).$$

Embeddings

For $s \in \mathbb{R}$ and $p \in [1, \infty]$ let $\dot{H}^s_p(\mathbb{R}^d; X)$ consist of all $f \in \mathcal{S}'(\mathbb{R}^d; X)$ such that

$$\mathcal{F}^{-1}(|\xi|^s \mathcal{F}f(\xi)) \in L^p(\mathbb{R}^d; X).$$

Then

$$\dot{H}_p^{d(\frac{1}{p}-\frac{1}{2})}(\mathbb{R}^d; X) \subseteq \gamma(\mathbb{R}^d; X)$$

and

$$\gamma(\mathbb{R}^d; Y) \subseteq \dot{H}_q^{d(\frac{1}{q}-\frac{1}{2})}(\mathbb{R}^d; Y).$$

Proof

Let

$$m_1(\xi) := |\xi|^{-d(\frac{1}{p}-\frac{1}{2})},$$
$$m_2(\xi) := |\xi|^{d(\frac{1}{p}-\frac{1}{q})} m(\xi),$$
$$m_3(\xi) := |\xi|^{-d(\frac{1}{2}-\frac{1}{q})}.$$

Then, using γ-embeddings,

$$\begin{array}{ccccc} L^p(\mathbb{R}^d;X) & \xrightarrow{T_{m_1}} & \dot{H}_p^{d(\frac{1}{p}-\frac{1}{2})}(\mathbb{R}^d;X) & \longrightarrow & \gamma(\mathbb{R}^d;X) \\ \downarrow T_m & & & & \downarrow T_{m_2} \\ L^q(\mathbb{R}^d;Y) & \xleftarrow[T_{m_3}]{} & \dot{H}_q^{d(\frac{1}{2}-\frac{1}{q})}(\mathbb{R}^d;Y) & \longleftarrow & \gamma(\mathbb{R}^d;Y) \end{array}$$

Proof

Let

$$m_1(\xi) := |\xi|^{-d(\frac{1}{p}-\frac{1}{2})},$$
$$m_2(\xi) := |\xi|^{d(\frac{1}{p}-\frac{1}{q})} m(\xi),$$
$$m_3(\xi) := |\xi|^{-d(\frac{1}{2}-\frac{1}{q})}.$$

Then, using γ-embeddings,

$$\begin{array}{ccccc}
L^p(\mathbb{R}^d;X) & \xrightarrow{T_{m_1}} & \dot{H}_p^{d(\frac{1}{p}-\frac{1}{2})}(\mathbb{R}^d;X) & \longrightarrow & \gamma(\mathbb{R}^d;X) \\
\Big\downarrow T_m & & & & \Big\downarrow T_{m_2} \\
L^q(\mathbb{R}^d;Y) & \xleftarrow[T_{m_3}]{} & \dot{H}_q^{d(\frac{1}{2}-\frac{1}{q})}(\mathbb{R}^d;Y) & \longleftarrow & \gamma(\mathbb{R}^d;Y)
\end{array}$$

Theorem

Suppose that X has type $p_0 \in (1,2]$ and Y cotype $q_0 \in [2,\infty)$, and let $p \in (1,p_0)$, $q \in (q_0,\infty)$. Suppose that

$$\{|\xi|^{d(\frac{1}{p}-\frac{1}{q})} m(\xi) \mid \xi \in \mathbb{R}^d \setminus \{0\}\} \subseteq \mathcal{L}(X,Y)$$

is γ-bounded.
Then $T_m : L^p(\mathbb{R}^d;X) \to L^q(\mathbb{R}^d;Y)$ is bounded and

$$\|T_m\| \lesssim \gamma(\{|\xi|^{d(\frac{1}{p}-\frac{1}{q})} m(\xi) \mid \xi \in \mathbb{R}^d \setminus \{0\}\}).$$

Proof of embeddings

- First prove inclusions for f with compact Fourier support;
- Obtain from this embeddings between vector-valued Besov spaces and γ-spaces (Kalton, van Neerven, Veraar, Weis (2008));
- Use embeddings between vector-valued Bessel spaces and Besov spaces.

Stability for semigroups

Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on X. Suppose

$$\sigma(A) \subseteq \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq -\epsilon\}$$

for some $\epsilon > 0$, with appropriate resolvent bounds.

Let $\alpha \geq 0$. If $(-A)^{-\alpha} R(-\delta + \mathrm{i}\cdot, A)$ is a bounded $(L^p(\mathbb{R}; X), L^q(\mathbb{R}; X))$ Fourier multiplier for some $p, q \in [1, \infty]$ and all $\delta \in (0, \epsilon)$, then $T(t)x = O(\mathrm{e}^{-\delta t})$ for all $x \in D((-A)^{\alpha})$ and all $\delta \in (0, \epsilon)$.

Stability for semigroups

Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on X. Suppose

$$\sigma(A) \subseteq \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq -\epsilon\}$$

for some $\epsilon > 0$, with appropriate resolvent bounds.
Let $\alpha \geq 0$. If $(-A)^{-\alpha}R(-\delta + \mathrm{i}\cdot, A)$ is a bounded $(L^p(\mathbb{R};X), L^q(\mathbb{R};X))$ Fourier multiplier for some $p, q \in [1,\infty]$ and all $\delta \in (0,\epsilon)$, then $T(t)x = O(\mathrm{e}^{-\delta t})$ for all $x \in D((-A)^{\alpha})$ and all $\delta \in (0,\epsilon)$.

Stability for semigroups

Theorem (van Neerven (2009))

Suppose that X has type $p \in [1,2]$ and cotype $q \in [2,\infty)$ and that $\{R(z,A) \mid \mathrm{Re}(z) > \epsilon\} \subseteq \mathcal{L}(X)$ is γ-bounded. Then $T(t)x = O(\mathrm{e}^{-\delta t})$ for all $x \in D((-A)^{\frac{1}{p}-\frac{1}{q}})$ and all $\delta \in (0,\epsilon)$.

Functional calculus theory

Let A generate a C_0-group on X. For f a bounded holomorphic function on a large enough strip St_ω, define $f(A)$ on X by the holomorphic functional calculus.

Modulo technicalities:

$f(A)$ can be factorized via an $(L^p(\mathbb{R};X), L^q(\mathbb{R};X))$ Fourier multiplier with symbol $f(\pm\omega + \mathrm{i}\cdot)$, for $p, q \in [1,\infty]$.

Functional calculus theory

Let A generate a C_0-group on X. For f a bounded holomorphic function on a large enough strip St_ω, define $f(A)$ on X by the holomorphic functional calculus.

Modulo technicalities:

$f(A)$ can be factorized via an $(L^p(\mathbb{R};X), L^q(\mathbb{R};X))$ Fourier multiplier with symbol $f(\pm\omega + \mathrm{i}\cdot)$, for $p, q \in [1, \infty]$.

Functional calculus theory

Theorem

Suppose that X has type $p \in [1,2]$ and cotype $q \in [2,\infty)$. If $f(z) = O(|z|^{-\alpha})$ as $|z| \to \infty$ for $\alpha > \frac{1}{p} - \frac{1}{q}$, then $f(A) \in \mathcal{L}(X)$.

For $s \in (1,\infty)\setminus\{2\}$, let $\mathscr{C}^s$ be the Schatten s-class of matrices with s-summable singular values. Let $m:\mathbb{Z}\to\mathbb{C}$.

Theorem

Let $r \in [1,\infty)$ be such that $\frac{1}{r} < |\frac{1}{s}-\frac{1}{2}|$. Suppose that $\sup_{k\in\mathbb{Z}}(1+|k|^{1/r})|m_k| < \infty$. Then $(a_{ij})_{ij} \mapsto (m_{i-j}a_{ij})_{ij}$ is a bounded map on $\mathscr{C}^s$.

For $s \in (1,\infty) \setminus \{2\}$, let $\mathscr{C}^s$ be the Schatten s-class of matrices with s-summable singular values. Let $m : \mathbb{Z} \to \mathbb{C}$.

Theorem

Let $r \in [1,\infty)$ be such that $\frac{1}{r} < |\frac{1}{s} - \frac{1}{2}|$. Suppose that $\sup_{k\in\mathbb{Z}}(1 + |k|^{1/r})|m_k| < \infty$. Then $(a_{ij})_{ij} \mapsto (m_{i-j}a_{ij})_{ij}$ is a bounded map on $\mathscr{C}^s$.

References

- J. Rozendaal and M. Veraar. *Fourier multiplier theorems involving type and cotype*, http://arxiv.org/abs/1605.09340.
- J. Rozendaal and M. Veraar. *Fourier multiplier theorems on Besov spaces using type and cotype*, http://arxiv.org/abs/1606.03272.
- J. Rozendaal. *Functional calculus for C_0-groups using (co)type*, http://arxiv.org/abs/1508.02036.