From Gál's Theorem to Extreme Values of the Riemann Zeta Function

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Classical result: Gál's theorem (1949)

On Erdős's initiative, Wiskundig Genootschap te Amsterdam posed the following prize problem: Compute

$$\Gamma_1(N) := \sup_{n_1<\cdots<n_N} \frac{1}{N} \sum_{j,k=1}^{N} \frac{(n_j, n_k)}{[n_j, n_k]} = \sup_{n_1<\cdots<n_N} \frac{1}{N} \sum_{j,k=1}^{N} \frac{(n_j, n_k)^2}{n_j n_k}.$$

Example of $n_1, ..., n_N$: Suppose $N = 2^\ell$ and take the square-free numbers generated by the first ℓ primes. Then, by the special structure of the set and one of Mertens's theorems,

$$\frac{1}{N} \sum_{j,k=1}^{N} \frac{(n_j, n_k)}{[n_j, n_k]} = \prod_{j=1}^{\ell} (1 + p_j^{-1}) \sim c \log \ell \sim c \log\log N.$$

Gál's solution

$$\Gamma_1(N) \asymp (\log\log N)^2.$$

Here "Gál" is István Sándor Gál or Steven Gaal.

Origin of the problem

The problem posed by Erdős is about the matrix

$$\langle n_j, n_k \rangle := \frac{(n_j, n_k)}{[n_j, n_k]} = \prod_p p^{-|\alpha_j(p) - \alpha_k(p)|},$$

where

$$n_j = \prod_p p^{\alpha_j(p)}.$$

This matrix can appear in a number of ways; Gál mentions that

$$\int_0^1 \big(\{n_j x\} - 1/2\big) \cdot \big(\{n_k x\} - 1/2\big) dx = \frac{1}{12} \langle n_j, n_k \rangle,$$

which apparently was first noted by E. Landau (1927); Erdős, inspired by Koksma's work in the 1930s, had the idea of relating such GCD sums to the study of the distribution of the sequence $(\{n_j x\})$ for almost all x.

Corollary to Gál's theorem

Gál obtained the following corollary to his estimate (solving a problem attributed to Hardy and Littlewood).

Corollary

For every strictly increasing sequence (n_k) of positive integers

$$\sum_{k=1}^{N} (\{n_k x\} - 1/2) = o(\sqrt{N} \log^{2+\varepsilon} N)$$

for almost every x.

The exponent 2 was apparently improved to 3/2 in later joint work with Koksma, by the same method.

Other powers of $\langle n_j, n_k \rangle$ and spectral norms

In work of Mikolás (1957), in connection with a problem involving the Hurwitz zeta function, proposed to estimate

$$\Gamma_\sigma(N) := \sup_{n_1<\cdots<n_N} \frac{1}{N} \sum_{j,k=1}^{N} \langle n_j, n_k \rangle^\sigma$$

for $1/2 < \sigma < 1$. (He says: "I hope to deal with it in another paper.")The slightly more general problem of estimating

$$\Lambda_\sigma(N) := \sup_{n_1<\cdots<n_N} \max_{\|c\|_{\ell^2}=1} \sum_{j,k=1}^{N} c_j \overline{c_k} \langle n_j, n_k \rangle^\sigma$$

in the range $1/2 \le \sigma < 1$ was later considered by Dyer and Harman (1986), who obtained the first nontrivial estimates and used them to obtain certain results in metric diophantine approximation.

Dyer and Harman's estimates (1986) and beyond

Dyer and Harman proved that

$$\Lambda_\sigma(N) \ll \exp\Big((\log N)^{(4-4\sigma)/(3-2\sigma)}\Big),\ 1/2 < \sigma < 1$$

$$\Lambda_{1/2}(N) \ll \exp\Big(\frac{c \log N}{\log\log N}\Big).$$

Motivated by applications in the study of systems of dilated functions $f(n_j x)$ (originating in the work of Wintner (1944)) as well as diophantine approximation, Aistleitner and Berkes started a collaboration with me a few years ago, in an attempt to determine the precise asymptotics of $\Gamma_\sigma(N)$ and $\Lambda_\sigma(N)$. Our starting point was the observation that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{|\zeta(\sigma+it)|^2}{\zeta(2\sigma)}\Big|\sum_j c_j n_j^{-it}\Big|^2 dt = \sum_{j,k} c_j \overline{c_k} \langle n_j, n_k \rangle^\sigma,$$

where $\zeta(s)$ is the Riemann zeta function and $\sigma > 1$.

Asymptotics of $\Gamma_\sigma(N)$ & $\Lambda_\sigma(N)$

σ-range	Size of $\Gamma_\sigma(N)$ & $\Lambda_\sigma(N)$	By
$\sigma > 1$	$\to \zeta^2(\sigma)/\zeta(2\sigma)$	
$\sigma = 1$	$(\frac{6}{\pi^2} e^{2\gamma} + o(1))(\log_2 N)^2$	Gál (1949); Lewko–Radziwiłł (arXiv 2014)
$\frac{1}{2} < \sigma < 1$	$\exp\left(B\frac{(\log N)^{1-\sigma}}{(\log_2 N)^\sigma}\right)$	Aistleitner–Berkes–S (2015)
$\sigma = \frac{1}{2}$	$\exp\left(A\sqrt{\frac{\log N \log_3 N}{\log_2 N}}\right)$	Bondarenko–S (2015)
$0 < \sigma < \frac{1}{2}$	$N^{1-2\sigma}(\log N)^{c(\sigma)}$	Bondarenko–Hilberdink–S (2016)

Size of primes in (nearly) optimal sets

	Size of $\Gamma_\sigma(N)$ & $\Lambda_\sigma(N)$	Size of primes ("friability")
$\sigma > 1$	$\to \zeta^2(\sigma)/\zeta(2\sigma)$	
$\sigma = 1$	$(\frac{6}{\pi^2}e^\gamma + o(1))(\log_2 N)^2$	$p \ll \log N$
$\frac{1}{2} < \sigma < 1$	$\exp\left(B\frac{(\log N)^{1-\sigma}}{(\log_2 N)^\sigma}\right)$	$\ll \log N \log_2 N$
$\sigma = \frac{1}{2}$	$\exp\left(A\sqrt{\frac{\log N \log_3 N}{\log_2 N}}\right)$	$p \ll \log N \; e^{(\log_2 N)^\beta},\ 0 < \beta < 1$
$0 < \sigma < \frac{1}{2}$	$N^{1-2\sigma}(\log N)^{c(\sigma)}$	$p \ll N^\delta,\ 0 < \delta < 1$

Further remarks

- There is an interesting similarity with the anticipated extreme values of $t \mapsto |\zeta(\sigma + it)|$ on $[1, T]$ and the size of $\Lambda_\sigma(N)$ when $1/2 < \sigma \le 1$. This is explained heuristically by Lewko and Radziwiłł's proof: They bound the GCD sums in terms of the square of a certain random model of $\zeta(s)$.
- When $\sigma = 1$ or $0 < \sigma < 1/2$, it *makes a difference* whether you restrict to n_j that are square-free. For $\sigma = 1$, Gál proved for instance that the corresponding supremum in the square-free case is $\log\log N$; the power of $\log N$ may possibly be removed in the square-free case when $0 < \sigma < 1/2$.
- When $\sigma = 1/2$, it is *essential for our method* that we are able to restrict to the square-free case. (See next slide.)

Key ingredient in proof of upper bound when $\sigma = 1/2$

We restrict to sets $\mathscr{M}$ of N distinct square-free numbers $n_1, ..., n_N$. We are interested in computing

$$\Gamma_{1/2,\mathrm{sf}}(N) := \frac{1}{N} \sum_{j,k=1}^{N} \langle n_j, n_k \rangle^{1/2}.$$

Then:

- By a division algorithm of Gál, extremal sets exist and any such set may be assumed to be divisor closed.
- Divisor closed extremal sets $\mathscr{M}$ enjoy the following completeness property: If $n \in \mathscr{M}$, $p|n$, $p' < p$, then either $p'|n$ or $p'n/p \in \mathscr{M}$.

This combinatorial component enables us to use analytic arguments! (This insight was also essential for our construction of nearly optimal sets (sets “close enough” to being extremal).)

Main application of upper bound when $1/2 < \sigma < 1$

Applications to systems of dilated functions $f(n_j x)$ all rely on the following kind of Carleson–Hunt inequality.

Theorem (Aistleitner–Berkes–S ('15); Lewko–Radziwiłł('14))

Suppose f is a 1-periodic function of mean zero with Fourier coefficients satisfying $|\widehat{f}(k)| = O(k^{-1})$, and let (n_k) be a strictly increasing sequence of positive integers. Then

$$\int_0^1 \max_{1\le M\le N} \left| \sum_{k=1}^{M} c_k f(n_k x) \right|^2 dx \ll (\log_2 N)^2 \sum_{k=1}^{N} |c_k|^2.$$

Here the power of $\log_2 N$ is optimal.

Very roughly, to prove this, one makes a number theoretic ("friable") splitting of the Fourier series of f and use the classical Carleson–Hunt theorem for the "friable" part and GCD sums and the Rademacher–Menshov theorem for the rest.

One consequence of the Carleson–Hunt inequality

Theorem (Lewko–Radziwiłł (2014))

Suppose f is a 1-periodic function of bounded variation and mean 0, and let (n_k) be a strictly increasing sequence of positive integers. Then for almost every x

$$\sum_{k \le N} f(n_k x) \ll \sqrt{N \log N} (\log \log N)^{3/2+\varepsilon}.$$

This is an improvement of a theorem of Aistleitner–Berkes–S (2015) with $5/2$ instead of $3/2$; by a result of Berkes and Philipp (1994), the best exponent cannot be smaller than $1/2$ which is believed to be the right one. This result is the culmination of a long series of work that in part has been motivated by a desire to obtain almost sure bounds for the discrepancy of the sequence $(\{n_k x\})$ (cf. Gál's corollary).

Application of upper bound when $\sigma = 1/2$

There is a recent arXiv preprint of Aistleitner–Larcher-Lewko (including an appendix of Bourgain) with an application of our upper GCD sum bound for $\sigma = 1/2$ to the pair correlation statistics of sequences on the unit interval. Their main technical lemma is the following variance estimate:

Lemma (Aistleitner–Larcher–Lewko (2016))

$$\int_0^1 \left(R_2([-s,s],\alpha,N) - \frac{2(N-1)s}{N}\right)^2 d\alpha \ll E(A_N) N^{-3} \exp\left(\kappa\sqrt{\frac{\log N \log_3 N}{\log_2 N}}\right),$$

where $N \cdot R_2([-s,s],\alpha,N)$ *counts the metric pair correlations of* $(\{\alpha a_n\})$ *of size* $\leq s/N$ *and* $E(A_N)$ *is the additive energy of* $a_1,\ldots,a_N$ *for the sequence* $A := (a_n)$.

Application of nearly optimal GCD sets to $\zeta(s)$

We are interested in the Riemann zeta function which for $\sigma = \operatorname{Re} s > 1$ is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s};$$

it has a meromorphic extension to the whole complex plane. In particular, we have the classical approximation ($s = \sigma + it$)

$$\zeta(s) = \sum_{n \le T} n^{-s} - \frac{N^{1-s}}{1-s} + O(T^{-\sigma}),\ |t| \le T, 0 < \sigma < 1.$$

We have seen that the size of $\Gamma_\sigma(N)$ and $\Lambda_\sigma(N)$ suggests that there could be a link to extreme values of $t \to |\zeta(\sigma + it)|$ for $1/2 < \sigma \le 1$ and possibly also for $\sigma = 1/2$. Aistleitner (2015) pursued this idea and was able to recapture Montgomery's lower estimates (1977) for extreme values in the range $1/2 < \sigma < 1$, by using a method of Hilberdink (2009).

How big can $|\zeta(1/2+it)|$ be?

Theorem (Bondarenko-S (arXiv 2015))

$$\max_{\sqrt{T}\le t\le T}\left|\zeta\left(\frac{1}{2}+it\right)\right| \ge \exp\left((1/\sqrt{2}+o(1))\sqrt{\frac{\log T\log_3 T}{\log_2 T}}\right),\ T\to\infty.$$

Here the novelty is the triple log:

- Montgomery (assuming RH) and Balasubramanian and Ramachandra (uncondionally) proved in 1977 that there exist arbitrarily large t such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \gg \exp\left(c\sqrt{\frac{\log t}{\log\log t}}\right).$$

- In 2008, Soundararajan improved c to $1+o(1)$ and found that such a large value could be found on $[T,2T]$ for every T large enough.

Upper bound for $|\zeta(1/2+it)|$

The best upper bound is

$$|\zeta(1/2+it)| \ll |t|^{13/84+\varepsilon},$$

obtained by Bourgain (2016). On the Riemann hypothesis, it is known that

$$|\zeta(1/2+it)| \ll \exp\Big(c\frac{\log|t|}{\log\log|t|}\Big).$$

Farmer–Gonek–Hughes (2007) have conjectured, by use of random matrix theory, that the right bound is

$$\exp\big((1/\sqrt{2}+o(1))\sqrt{\log|t|\log\log|t|}\big).$$

Proof: Soundararajan's resonance method (2008)

Basic idea: Let S be a nonnegative function and consider

$$M_1(T) := \int_{T^{1/2} \le |t| \le T} S(t)dt,$$
$$M_2(T) := \int_{T^{1/2} \le |t| \le T} \zeta(1/2+it)S(t)dt.$$

Then obviously

$$\max_{T^{1/2} \le t \le T} \left|\zeta(1/2+it)\right| \ge \frac{|M_2(T)|}{M_1(T)};$$

to proceed, we need an upper bound for $M_1(T)$ and a lower bound for $M_2(T)$. In principle, we may catch the maximum by choosing the "right" S, but this is not easy: We do not know where the maximum is!

Resonating Dirichlet polynomial

Following Soundararajan, we choose

$$S(t) = |R(t)|^2 \Phi\Big(\frac{\log T}{T} t\Big),$$

where

$$R(t) = \sum_{m \in \mathscr{M}'} r(m) m^{-it}$$

is a Dirichlet polynomial, with $\mathscr{M}'$ a “good” set of integers and positive coefficients $r(m)$, that “resonates” well with $\zeta(1/2 + it)$, and $\Phi(t)$ is a positive, smooth, and well localized function.

Question

How to choose R and Φ*?*

Choice of R and Φ

Soundararajan chose R to be of length $\leq T$ (i.e. no integer from $\mathscr{M}'$ larger than T) and Φ a bump function localized to $[T, 2T]$. He then solved an optimization problem to find the best coefficients $r(m)$, and thus his result is the best you can obtain with a resonator of length $\leq T$.

We made two observations:

- Since we integrate $\zeta(1/2 + it)$ against $|R(t)|^2$, the frequencies that pick out large contributions are of the form $\log(m/n)$ with m, n in $\mathscr{M}'$. Thus what really matters are the ratios m/n less than T rather than m and n themselves.
- The analysis becomes harder with a longer R, but a remedy is: Choose $\Phi \geq 0$ with $\widehat{\Phi} \geq 0$, e.g. $\Phi(t) := e^{-t^2/2}$. Then after integration the main terms in $M_2(T)$ are positive (coming from the approximative formula).

$M_2(T)$ after integration

A major difficulty is to estimate $M_2(T)$ from below. Integrating, using that Φ is a Gaussian, we get that

$$M_2(T) = \frac{\sqrt{2\pi}T}{\log T} \sum_{m,n\in\mathscr{M}} \sum_{k\leq T} \frac{r(m)r(n)}{\sqrt{k}} \Phi\Big(\frac{T}{\log T} \log \frac{km}{n}\Big) + \text{neglible terms.}$$

Roughly, what we do next[1] is to retain only those k that are divisors of some n. Reason: the main contribution comes from terms for which $km/n \approx 1$.

[1] Not completely true, we retain k in a larger set $\mathscr{M}$; this has to do with possible clusters of the set $\mathscr{M}$, which comes from the GCD problem and is defined by a multiplicative recipe. See the next slide.

Indication of choice of $\mathscr{M}'$ and $r(m)$

$\mathscr{M}'$ and $r(m)$ come from our work on GCD sums:

- $\mathscr{M}$ is the set of square-free numbers n that have at most $\frac{a \log N}{k^2 \log_3 N}$ prime divisors p satisfying

  $$e^k \log N \log_2 N < p \le e^{k+1} \log N \log_2 N, \ k = 1, \ldots, [(\log_2 N)^\beta],$$

  where $1 < a < 1/\beta$ and $N \approx \sqrt{T}$. It is clear that $\mathscr{M}$ is divisor closed, and it is close enough to having the desired completeness property. ($\mathscr{M}'$ is a modification of $\mathscr{M}$ which we need to make because of possible "clusters" in $\mathscr{M}$.)
- $r(m)$ is (essentially) a special multiplicative function restricted to $\mathscr{M}'$.

Two obvious questions

- How close does the resonator take us to the maximum of $|\zeta(1/2+it)|$?
- Is there a better choice of resonator, allowing for improvements?