The Riesz transform, quantitative rectifiabilty, and a two-phase problem for harmonic measure

Xavier Tolsa

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$\mathcal{R}_\mu f(x) = \lim_{\varepsilon \searrow 0} \mathcal{R}_{\mu,\varepsilon} f(x)$, where

$$\mathcal{R}_{\mu,\varepsilon} f(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}} \, f(y) \, d\mu(y).$$

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In $\mathbb{C}$, the Cauchy transform of $f \in L^1_{loc}(\mu)$ is $\mathcal{C}_\mu f(z) = \lim_{\varepsilon \searrow 0} \mathcal{C}_{\mu,\varepsilon} f(z)$, where

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The existence of principal values is not guarantied, in general.

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We also denote

$$\mathcal{R}\mu = \mathcal{R}_\mu 1, \qquad \mathcal{C}\mu = \mathcal{C}_\mu 1.$$

Rectifiability

We say that E is rectifiable if it is $\mathcal{H}^1$-a.e. contained in a countable union of curves of finite length.

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Question: Let $E \subset \mathbb{R}^d$ with $\mathcal{H}^n(E) < \infty$, and $\mu = \mathcal{H}^n_E$.
If $\mathcal{R}_\mu : L^2(\mu) \to L^2(\mu)$ is bounded, is then E n-rectifiable?

The Cauchy transform, curvature, and rectifiability

Theorem (David, Léger, 1998)

Let $E \subset \mathbb{C}$ *with* $\mathcal{H}^1(E) < \infty$, *and* $\mu = \mathcal{H}^1_E$.
If $\mathcal{C}_\mu$ *is bounded in* $L^2(\mu)$, *then* E *is rectifiable.*

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- It involves ideas from quantitative rectifiability which go back to the Jones' traveling salesman theorem.

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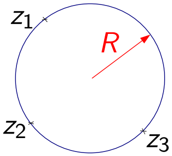
- The proof relies on the relationship between Menger curvature and the Cauchy kernel, found by Melnikov.



The curvature of μ is

$$c^2(\mu) := \iiint \frac{1}{R(x,y,z)^2}\, d\mu(x) d\mu(y) d\mu(z).$$

The David - Léger theorem

Melnikov (1995) found that:

$$\frac{1}{R(z_1,z_2,z_3)^2} = \sum_{s\in S_3} \frac{1}{(z_{s_2}-z_{s_1})\overline{(z_{s_3}-z_{s_1})}}.$$

Melnikov and Verdera (1995): If $\mu(B(z,r)) \leq c\,r$ for all $z \in \mathbb{C}$, $r>0$, then

$$\|\mathcal{C}_\mu 1\|^2_{L^2(\mu)} = \frac{1}{6}c^2(\mu) + O(\mu(\mathbb{C})).$$

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The preceding theorem of David and Léger follows from the next one:

Theorem (David, Léger)

If $\mathcal{H}^1(E) < \infty$ and $c^2(\mathcal{H}^1_{|E}) < \infty$, then E is rectifiable.

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Let $B \subset \mathbb{R}^2$ be a ball such that $C_0^{-1} r(B) \leq \mu(B) \leq C_0 r(B)$.
If

$$c^2(\mu|_B) \leq \varepsilon \, \mu(B) \quad \text{for some } \varepsilon > 0 \text{ small enough,}$$

then there exists a (possibly rotated) Lipschitz graph Γ with slope $\leq 1/10$ such that

$$\mu(\Gamma \cap B) \geq \frac{99}{100} \, \mu(B).$$

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- We are interested in finding a version of this result valid in higher dimensions replacing curvature by "the $L^2(\mu)$ norm of $\mathcal{R}\mu$".

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Theorem (Nazarov, T., Volberg, 2012)

Let $E \subset \mathbb{R}^{n+1}$ with $\mathcal{H}^n(E) < \infty$, and $\mu = \mathcal{H}^n_E$. If $\mathcal{R}_\mu : L^2(\mu) \to L^2(\mu)$ is bounded, then E n-rectifiable.

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- In the case that μ is AD-regular, i.e.,

$$\mu(B(x,r)) \approx r^n \quad \text{for all } x \in \operatorname{supp}\mu,\ 0 < r \leq \operatorname{diam}(E),$$

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  it follows that E is uniformly n-rectifiable.
- A previous case solved by Hofmann, Martell, Mayboroda: For $\mu = \mathcal{H}^n|_{\partial\Omega}$, where Ω is a uniform domain, using harmonic measure.

Uniform rectifiability

Let $E \subset \mathbb{R}^d$.
E is uniformly n-rectifiable if it is AD-regular and there are $M, \theta > 0$ such that for all $x \in E$, $0 < r \leq \text{diam}(E)$, there exists a Lipschitz map

$$g : \mathbb{R}^n \supset B_n(0,r) \to \mathbb{R}^d, \qquad \|\nabla g\|_\infty \leq M,$$

such that

$$\mathcal{H}^n\big(E \cap B(x,r) \cap g(B_n(0,r))\big) \geq \theta\, r^n.$$

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Uniform n-rectifiability is a quantitative version of n-rectifiability introduced by David and Semmes.

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Recall:

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Let μ be a Radon measure on $\mathbb{R}^{n+1}$ and $B \subset \mathbb{R}^{n+1}$ a ball satisfying:

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(b) *$\mu(B(x,r)) \leq C_0\, r^n$ for all x, r.*

(c) *There is some n-plane L passing through the center of B such that for some $0 < \delta \ll 1$, it holds $\beta_{\mu,1}^L(B) \leq \delta$.*

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(e) *For some $0 < \varepsilon \ll 1$, $\int_B |\mathcal{R}\mu(x) - m_{\mu,B}(\mathcal{R}\mu)|^2\, d\mu(x) \leq \varepsilon\, \mu(B)$.*

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Then there exists some $\tau > 0$ such that if δ, ε are small enough, then there is a uniformly n-rectifiable set $\Gamma \subset \mathbb{R}^{n+1}$ such that

$$\mu(B \cap \Gamma) \geq \tau\, \mu(B).$$

Remarks

- We have denoted

$$\beta_{\mu,1}^{L}(B) = \frac{1}{r(B)^n} \int_B \frac{\mathrm{dist}(x, L)}{r(B)} \, d\mu(x).$$

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- It is not natural to assume

$$\int |\mathcal{R}\mu(x)|^2\, d\mu(x) \leq \varepsilon\, \|\mu\|$$

instead of (e).

Key steps of the proof

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3. For the step 1.:
   - We approximate μ by a periodic measure ν.
   - We apply a variational argument (inspired by Eiderman, Nazarov, Volberg), by taking a minimizer of:

$$F(b) = c\,\|b\|_\infty\,\nu(B) + \int_B |\mathcal{R}_\nu b|^2\, b\, d\nu,$$

   with $b \in L^\infty(\nu)$, periodic, such that $\nu(B) = \int_B b\, d\nu$.
   Applying a maximum principle we get a contradiction if $\nu(LD)$ is big.

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For $p \in \Omega$, ω^p is the harmonic measure in Ω with pole in p.

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Which is the connection with rectifiability?

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Which is the connection with rectifiability?

- In the plane if Ω is simply connected and $\mathcal{H}^1(\partial\Omega) < \infty$, then $\mathcal{H}^n \approx \omega^p$. (F.& M. Riesz)
- Many classical results in $\mathbb{C}$ using complex analysis.
- In higher dimension, real analysis techniques. Connection with uniform rectifiability studied recently by Hofmann, Martell, Uriarte-Tuero, Mayboroda, Badger, Bortz, Akman, etc.

Relationship between the Riesz transform and harmonic measure

Let $\mathcal{E}(x)$ the fundamental solution of the Laplacian in $\mathbb{R}^{n+1}$:

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$$G(x,p) = \mathcal{E}(x-p) - \int \mathcal{E}(x-y) \, d\omega^p(y).$$

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Therefore, for $x \in \Omega$:

$$c \, \nabla_x G(x,p) = K(x-p) - \int K(x-y) \, d\omega^p(y).$$

That is, $$\mathcal{R}\omega^p(x) = K(x-p) - c \, \nabla_x G(x,p).$$

Harmonic measure and rectifiability

By using the Nazarov-T.-Volberg theorem on Riesz transforms and rectifiability, we get a converse to the the F.&M. Riesz theorem:

Theorem
Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain. Suppose that there exists $E \subset \partial\Omega$ with $0 < \mathcal{H}^n(E) < \infty$. If $\omega|_E \approx \mathcal{H}^n|_E$, then E is n-rectifiable.

- Proof by
  [Azzam, Mourgoglou, T.]
  + [Hofmann, Martell, Mayboroda, T., Volberg].
- It solves a question posed by Bishop in the 1990's.

Application to a two-phase problem

We say that $\Omega \subset \mathbb{R}^{n+1}$ satisfies the CDC if

$$\mathcal{H}^s_\infty(\Omega^c \cap B(x,r)) \approx r^s \quad \text{for all } x \in \partial\Omega \text{ and } 0 < r \leq r_0, \tag{1}$$

for some fixed $s \in (n-1, n+1)$, $r_0 > 0$.

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Theorem (Azzam, Mourgoglou, T.)

Let $\Omega_1 \subset \mathbb{R}^{n+1}$ and $\Omega_2 = ext(\Omega_1)$ be disjoint connected domains satisfying (1), with $\partial\Omega_1 = \partial\Omega_2$, with harmonic measures ω^1, ω^2. Let $E \subseteq \partial\Omega_1 \cap \partial\Omega_2$ be such that $\omega^1 \ll \omega^2 \ll \omega^1$ on E.

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$$\mathcal{H}^s_\infty(\Omega^c \cap B(x,r)) \approx r^s \quad \text{for all } x \in \partial\Omega \text{ and } 0 < r \leq r_0, \tag{1}$$

for some fixed $s \in (n-1, n+1)$, $r_0 > 0$.

Theorem (Azzam, Mourgoglou, T.)

Let $\Omega_1 \subset \mathbb{R}^{n+1}$ and $\Omega_2 = ext(\Omega_1)$ be disjoint connected domains satisfying (1), with $\partial\Omega_1 = \partial\Omega_2$, with harmonic measures ω^1, ω^2.
Let $E \subseteq \partial\Omega_1 \cap \partial\Omega_2$ be such that $\omega^1 \ll \omega^2 \ll \omega^1$ on E.
Then E contains an n-rectifiable subset F with $\omega^1(E \setminus F) = \omega^2(E \setminus F) = 0$ on which $\omega^1 \ll \omega^2 \ll \mathcal{H}^n \ll \omega^1$.

Remarks

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- Up to now, the theorem was only known when Ω_1, Ω_2 are planar domains by Bishop. Previous case of Jordan domains by Bishop, Carleson, Garnett, Jones.
- A partial result in higher dimensions by Kenig, Preiss and Toro: harmonic measure is concentrated in a subset of dimension n.

Main steps of the proof

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$$\frac{\omega^1(B(x,r))}{r^n} \lesssim 1 \quad \text{for all } 0 < r \leq r_0(x).$$
- To show that $\omega^1 \approx \mathcal{H}^n$ on E we apply the Girela-Sarrión - T theorem with $\mu = \omega^1|_G$, for suitable subsets $G \subset E$, and we deduce that $\omega^1 \approx \mathcal{H}^n$ on E.

Thank you.

Happy birthday Sasha!