# Sum rules and Killip-Simon problem

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Harmonic analysis, **complex analysis, spectral theory** and all that

Bedlewo, August 2016

# On generalized sum rules for Jacobi matrices

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Operator theory and applications in mathematical physics

Bedlewo, July 2004

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Well known three term recurrence relation

$$x\,P_n(x) = a(n)P_{n-1}(x) + b(n)P_n(x) + a(n+1)P_{n+1}(x), \quad x \cdot L^2_\sigma \to L^2_\sigma.$$

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Jacobi matrix:

$$J_+ = \begin{bmatrix} b(0) & a(1) & & \\ a(1) & b(1) & a(2) & \\ & \ddots & \ddots & \ddots \end{bmatrix}, \quad J_+ : \ell^2_+ \to \ell^2_+.$$

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## Spectral Theorem

We have one-to-one correspondence between $\Sigma$ and $\mathcal{J}$, $\sigma \mapsto J_+$.

## General Problem

Find correspondence between subclasses of $\mathcal{J}$ and $\Sigma$. Which properties of $\sigma$ are responsible for this or that properties of $J_+$ and vice versa?

T. Tao and Ch. Thiele, *Nonlinear Fourier Analysis*.

Using this allusion

# Killip-Simon Theorem [Annals, 2003]: $\ell^2$ perturbation of the matrix with constant coefficients

Let

$$\overset{\circ}{J}_+ = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots \end{bmatrix}, \quad J_+ = \begin{bmatrix} b(0) & a(1) & & \\ a(1) & b(1) & a(2) & \\ & \ddots & \ddots & \ddots \end{bmatrix}.$$

Then

$$\text{(op-c)} \quad \sum (a(n)-1)^2 < \infty, \quad \sum b(n)^2 < \infty.$$

if and only if support of $\sigma$ is $E \cup X$, $E = [-2, 2]$, $X = \{x_k\}$, and

$$\text{(sp-c)} \quad \int_E |\log \sigma'_{a.c.}(x)| \sqrt{4 - x^2} dx < \infty, \quad \sum_X \sqrt{(x_k^2 - 4)^3} < \infty.$$

**Remarks:**

1. **Von Neumann Theorem**. For an arbitrary self-adjoint $A$ there exists $B$ of the Hilbert-Schmidt class, $B \in HS$, such that $\sigma_{a.c.}(A+B) = \emptyset$.

Since $\int_E |\log \sigma'_{a.c.}(x)|\sqrt{4-x^2}dx < \infty$, we get

$$\sigma'_{a.c.} \neq 0 \text{ a.e. on } [-2,2],$$

although $J_+ - \overset{\circ}{J}_+ \in HS$ (an arbitrary Jacobi matrix of the class).

Proved by Deift and Killip in 1999 (Comm. Math. Phys.)

2. Actually Killip and Simon proved "Parseval's identity" for this "non-linear Fourier transform" (**sum rule**).

# Nazarov, Peherstorfer, Volberg, Yuditskii [IMNR, 2005]: sum rules in a very general form

Let for a nonnegative polynomial $A(x)$

$$\Lambda_A(\sigma) = \sum_X F(x_k) + \frac{1}{2\pi}\int_{-2}^{2} \log\left(\frac{\sqrt{4-x^2}}{2\pi\sigma'_{a.c.}(x)}\right) A(x)\sqrt{4-x^2}dx,$$

$$F(x) = \begin{cases} \int_2^x A(x)\sqrt{x^2-4}dx, & x>2 \\ -\int_{-2}^x A(x)\sqrt{x^2-4}dx, & x<-2 \end{cases}$$

**Theorem**. $\Lambda_A(\sigma) < \infty$ if and only if the "naive trace" of

$$\Phi(J_+) - \Phi(\overset{\circ}{J}_+) - \text{diag}\{a \log a(m)\}$$

is finite, where

$$\Phi'(z) = zA(z) - \frac{1}{\pi}\int_{-2}^{2} \frac{A(x)-A(z)}{x-z}\sqrt{4-x^2}dx, \quad a = \frac{1}{\pi}\int_{-2}^{2} A(x)\sqrt{4-x^2}dx$$

**Remarks.**

1. This generalization has sense only if $A(x)$ has zeros on $E = [-2, 2]$.
2. Nobody knows what the "trace condition" means in terms of $\{a(n), b(n)\}$ except for a few special cases.
3. The method is absolutely restricted to the case of perturbations with the "constant background".
4. An extremely interesting probabilistic interpretation via the Large Deviations Principle (LDP).

Gamboa, Nagel, Rouault, JFA, 2016.

Distribution on the set of random matrices

$$\frac{1}{Z_\Phi^n} e^{-n\beta' \sum_{k=1}^n \Phi(\lambda_k)} \prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^\beta d\lambda_1 \cdots d\lambda_n \quad (*)$$

can be reduced to the distribution on (finite dimensional) Jacobi matrices $J^{(n)}$ (proportional to):

$$e^{-n\beta'\left[\mathrm{tr}\,\Phi(J^{(n)}) - 2\sum_{k=1}^{n-1}(1-\frac{k}{n}-\frac{1}{n\beta})\log a(k)\right]}$$

or in terms of measures $\sigma(\lambda_k) = \sigma_k$: (*) plus

$$\frac{1}{Y_{\beta'}^n}(\sigma_1 \cdots \sigma_n)^{\beta'-1}, \quad \sum \sigma_k = 1.$$

Thus, for a fixed $n$ we have the **probability** written in the **two coordinate systems** (spectral and coefficients).

**Definition.** We say that a sequence $(\mathcal{P}_n)_n$ of probability measures on $(U, \mathcal{F})$ satisfies (LDP) with speed $\tau_n$ and rate function $\mathcal{I} : U \to [0, \infty]$ if:

(i) $\mathcal{I}$ is lower semicontinuous.

(ii) For all closed set $F \subset U$:

$$\limsup_{n\to\infty} \frac{1}{\tau_n} \log \mathcal{P}_n(F) \leq - \inf_{x\in F} \mathcal{I}(x)$$

(iii) For all open sets $O \subset U$:

$$\liminf_{n\to\infty} \frac{1}{\tau_n} \log \mathcal{P}_n(O) \geq - \inf_{x\in O} \mathcal{I}(x)$$

## GNR Theorem

The sum rule identity reflects the fact that (LDP) holds (with speed $\beta' n^2$) and we can write the *rate function* $\mathcal{I}$ either in the spectral $(\sigma)$ or the coefficient sequences $(J_+)$ "coordinates".

# Damanik-Killip-Simon Theorem [Annals, 2010]: periodic background

In the periodic case, $a(k+N) = a(k), b(k+N) = b(k)$, it is natural and very convenient consider two sided Jacobi matrices

$$\overset{\circ}{J} = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & a(0) & b(0) & a(1) & & \\ & & a(1) & b(1) & a(2) & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}, \quad \overset{\circ}{J} : \ell^2 \to \ell^2.$$

**Isospectral set**:

$$J(E) = \{\overset{\circ}{J} : \sigma(\overset{\circ}{J}) = E\}.$$

**Two facts**:

1. $E$ is the spectral set of a periodic Jacobi matrix if and only if

$$\exists T_N : E = T_N^{(-1)}([-2,2]) \text{ (in } \mathbb{C}).$$

2. For the given $E$ there exist two special functions $\mathcal{A}(\alpha)$ and $\mathcal{B}(\alpha)$ on

$\mathbb{T}^g \simeq \mathbb{R}^g/\mathbb{Z}^g$ and $\mu \in \mathbb{R}^g/\mathbb{Z}^g$, $N\mu = 0_{\mathbb{R}^g/\mathbb{Z}^g}$, s.t.

$$J(E) = \{J(\alpha) : \alpha \in \mathbb{R}^g/\mathbb{Z}^g\}$$

where the coefficients of $J(\alpha)$ are of the form

$$a_\alpha(n) = \mathcal{A}(\alpha - n\mu), \ b_\alpha(n) = \mathcal{B}(\alpha - n\mu)$$

So, usually we say $J(E)$ is an isospectral torus.

Non-degenerated case $g = N$, also

$$E = T_g^{(-1)}([-2,2]) = [\mathbf{b}_0, \mathbf{a}_0] \setminus \cup_{j=1}^{g} (\mathbf{a}_j, \mathbf{b}_j).$$

**DKS spectral condition**:

$\sigma$ is supported on $E \cup X$, $X = \{x_k\}$, and

$$\text{(p-sp-c)} \quad \int_E |\log \sigma'_{a.c.}(x)| \mathrm{dist}(x, \mathbb{R}\setminus E)^{1/2} dx < \infty, \quad \sum_X \mathrm{dist}(x_k, E)^{3/2} < \infty.$$

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Compere to

$$\text{(sp-c)} \quad \int_E |\log \sigma'_{a.c.}(x)| \sqrt{4 - x^2} dx < \infty, \quad \sum_X (x_k^2 - 4)^{3/2} < \infty,$$

for $E = [-2, 2]$.

**In terms of coefficients**:

Let

$$\mathrm{dist}^2(J_+^{(1)}, J_+^{(2)}) := \sum_{n\in\mathbb{Z}_+} \{(a_n^{(1)} - a_n^{(2)})^2 + (b_n^{(1)} - b_n^{(2)})^2\}2^{-n}$$

$$\mathrm{dist}(J_+, J(E)) := \inf_{\overset{\circ}{J}\in J(E)} \mathrm{dist}(J_+, \overset{\circ}{J}_+)$$

## Theorem

*The spectral condition (p-sp-c) is equivalent to*

$$\text{(p-op-c)} \quad \sum_{m\in\mathbb{Z}^+} \mathrm{dist}^2((S_+^m)^* J_+ S_+^m, J(E)) < \infty.$$

# Main result

Let $E = [\mathbf{b}_0, \mathbf{a}_0] \setminus \cup_{j=1}^{g} (\mathbf{a}_j, \mathbf{b}_j)$ be an arbitrary system of intervals.

**Finite gap Jacobi matrices**:
[Akhiezer, Novikov, Dubrovin, Its, Matveev, Krichever, Aptekarev,......]

$$J(E) = \{\overset{\circ}{J} \text{ is almost periodic} : \sigma(\overset{\circ}{J}) = \sigma_{a.c.}(\overset{\circ}{J}) = E\}$$

Still we have an isospectral torus: for the given $E$ there exist two special functions $\mathcal{A}(\alpha)$ and $\mathcal{B}(\alpha)$ on $\mathbb{R}^g/\mathbb{Z}^g$ and $\mu \in \mathbb{R}^g/\mathbb{Z}^g$, s.t.

$$J(E) = \{J(\alpha) : \alpha \in \mathbb{R}^g/\mathbb{Z}^g\}$$

where the coefficients of $J(\alpha)$ are of the form

$$\overset{\circ}{a}(n) = \mathcal{A}(\alpha - n\mu), \ \overset{\circ}{b}(n) = \mathcal{B}(\alpha - n\mu)$$

**Definition** (copy and paste (p-sp-c)). $J_+ \in \mathrm{KS}(E)$ if $\sigma(J_+) = E \cup X$ and

$$\text{(f.g.-sp-c)} \quad \int_E |\log \sigma'(x)| \operatorname{dist}^{1/2}(x, \mathbb{R} \setminus E) dx + \sum_{x_k \in X} \operatorname{dist}^{3/2}(x_k, E) < \infty.$$

### Theorem

*$J_+$ belongs to $KS(E)$ if and only if there exist $\epsilon_\alpha(n) \in \ell^2_+(\mathbb{R}^g)$ and $\epsilon_a(n) \in \ell^2_+$, $\epsilon_b(n) \in \ell^2_+$ such that*

$$\begin{aligned} a(n) &= \mathcal{A}(\alpha(n) - n\mu) + \epsilon_a(n), \quad \alpha(n) = \sum_{k=0}^{n} \epsilon_\alpha(k) \\ b(n) &= \mathcal{B}(\alpha(n) - n\mu) + \epsilon_b(n), \end{aligned}$$

# Two words about the method

1. DKS was based on the *Magic Formula*: $E = T_g^{-1}([-2,2])$

$$\overset{\circ}{J} \in J(E) \Leftrightarrow T_g(\overset{\circ}{J}) = S^{g+1} + S^{-(g+1)}$$

and reduction to the block-matrix Jacobi matrix with constant coefficients

$$T_g(J_+) - (S_+^{g+1} + (S_+^*)^{g+1}) \in HS.$$

2. For an arbitrary $E$ there exists a unique

$$\Delta(x) = \lambda_0 x + \mathbf{c}_0 + \sum_{j=1}^{g} \frac{\lambda_j}{\mathbf{c}_j - x}, \ \lambda_j > 0,$$

such that

$$E = \Delta^{-1}([-2,2])$$

3. Orthogonalization of the family w.r.t. $\sigma$

$$1, \frac{1}{\mathbf{c}_g - x}, \dots \frac{1}{\mathbf{c}_1 - x}, x, \frac{1}{(\mathbf{c}_g - x)^2}, \dots, \frac{1}{(\mathbf{c}_1 - x)^2}, \dots$$

generates **GMP** matrices $A_+$.

4. Magic formula for them

$$\overset{\circ}{A} \in A(E) \Leftrightarrow \Delta(\overset{\circ}{A}) = S^{g+1} + S^{-(g+1)}$$

5. Back to Jacobi via *Jacobi flow on GMP matrices* (a new integrable system)

$$\begin{array}{ccc} & d\sigma & \\ \swarrow & & \searrow \\ A_+ & \xrightarrow{\mathcal{F}} & J_+ \\ \mathcal{J}\downarrow & & \mathcal{S}\downarrow \\ A_+ & \xrightarrow{\mathcal{F}} & J_+ \end{array}$$