Cuntz-Pimsner Algebras, mapping cones and weighted lens spaces

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- Study principal circle bundles and the associated Gysin sequence.
- Line bundles and the associated bundle construction.
- Recipe to compute the K-theory groups
- Also has applications in mathematical physics (T-duality, Chern-Simons theories).

The classical Gysin sequence in K-theory for circle bundles has the form of a cyclic six term exact sequence:

$$\begin{tikzcd} K^0(X) \arrow[r, "\alpha"] & K^0(X) \arrow[r, "\pi^*"] & K^0(P) \arrow[d, "\delta_{0,1}"] \\ K^1(P) \arrow[u, "\delta_{1,0}"] & K^1(X) \arrow[l, "\pi^*"] & K^1(X) \arrow[l, "\alpha"] \end{tikzcd} , \tag{1}$$

where α is the mutiliplication by the Euler class

$$\chi(L) = 1 - [L] \tag{2}$$

of the line bundle $L \to X$ with associated circle bundle $\pi : P \to X$.

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Definition

A *self Morita equivalence bimodule (SMEB)* over B is a pair (E, ϕ) where E is a full right Hilbert C^*-module over B and

$$\phi : B \to \mathcal{K}(E)$$

is an isomorphism.

Example: $A = C(X)$ and $E = \Gamma(\mathcal{L})$ the module of sections of a Hermitian line bundle $\mathcal{L} \to X$.

Self Morita equivalences over B form a group named the *Picard group* of B, denoted $\mathrm{Pic}(B)$.

Universal C*-algebras constructed out of a C*-correspondence. Defined in terms of creation and annihilation operators on the module

$$\mathcal{F}_E := B \oplus \bigoplus_{n \geq 1} E^{\otimes n}$$

Quotient in a short exact sequence:

$$0 \longrightarrow \mathcal{K}(\mathcal{F}_E) \longrightarrow \mathcal{T}_E \xrightarrow{\pi} \mathcal{O}_E \longrightarrow 0. \tag{3}$$

Generalise: Cuntz and Cuntz-Krieger algebras, Graph algebras, crossed products by $\mathbb{Z}$, crossed products by a partial automorphism.

In the case of a self-Morita equivalence bimodule the Pismer algebras can be represented on the module

$$\mathcal{F}_{E,\mathbb{Z}} := \bigoplus_{n\in\mathbb{Z}} E^{(n)}$$

where $E^{\otimes n} = (E^*)^{\otimes n}$ for $n > 0$, as the smallest C*-algebra generated by creation and annihilation operators thereon.

For a SMEB (more generally for any f.g.p. module): realisation in terms of generators and relations.

Let $\{\eta_i\}_{i=1}^n$ be a finite frame for E, i.e.

$$\xi = \sum_{j=1}^{n} \eta_j \langle \eta_j, \xi \rangle_B, \qquad \forall \xi \in E.$$

Then $\mathcal{O}_E$ is the *universal C^*-algebra* generated by B together with n operators $S_1, \ldots, S_n$, satisfying

$$S_i^* S_j = \langle \eta_i, \eta_j \rangle_B, \quad \sum\nolimits_j S_j S_j^* = 1, \quad \text{and} \quad bS_j = \sum\nolimits_i S_i \langle \eta_i, \phi(b) \eta_j \rangle_B,$$

for $b \in B$, and $j = 1, \ldots, n$.

We have a circle action γ on $\mathcal{O}_E$ called the *gauge action*.
This is defined on generators by

$$\gamma_z S_i = zS_i, \quad \forall i = 1, \ldots, n.$$

We denote by $\mathcal{O}_E^\gamma$ the fixed point for this action.
We have a natural $\mathcal{O}_E^\gamma$-valued conditional expectation.

$$\rho(x) = \int_{S^1} \gamma_z(x) dz. \tag{4}$$

Proposition

E is a self-Morita equivalence bimodule if and only if $\mathcal{O}_E^\gamma \simeq B$.

Let A be a C^*-algebra with an action $\{\sigma_z\}_{z\in S^1}$.

For each $n \in \mathbb{Z}$, one can define the spectral subspaces

$$A_{(n)} := \left\{\xi \in A \mid \sigma_z(\xi) = z^{-n}\xi \quad \text{for all } z \in S^1\right\}.$$

It is easy to check that $A^*_{(n)} = A_{(-n)}$ and that $A_{(n)}A_{(m)} \subseteq A_{(n+m)}$.

Theorem ([AKL16])

Suppose that the circle action $\{\sigma_z\}$ satisfies

$$A^*_{(1)}A_{(1)} = A_{(0)} = A_{(1)}A^*_{(1)}.$$

Then the Pimsner algebra $\mathcal{O}_{A_{(1)}}$ is isomorphic to A.

Proposition (Gabriel-Grensing)

Let A be a unital, commutative C^*-algebra. Suppose that the first spectral subspace $E = A_{(1)}$ generates A as a C^*-algebra, and that it is finitely generated projective over $B = A_{(0)}$.
Then the following facts hold

1. $B = C(X)$ *for some compact space* X;
2. $E = \Gamma(\mathcal{L})$ *for some line bundle* $\mathcal{L} \to X$;
3. $A = C(P)$, *where* $P \to X$ *is the principal* S^1 *bundle over* X *associated to the line bundle* $\mathcal{L}$.

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[Pim97]: The defining extension is semi-split. Hence it induces six term exact sequences in KK-theory.

These simplify by using:

- The class of the correspondence $E \in KK(B, B)$;
- The class of the Morita equivalence $[\mathcal{F}_E] \in KK(\mathcal{K}_B(\mathcal{F}_E), B)$;
- The class of the KK-equivalence $[\alpha]^{-1} \in KK(\mathcal{T}_E, B)$, which is the inverse to the class of the inclusion $\alpha : B \hookrightarrow \mathcal{T}_E$.

These satisfy:

$$[\mathcal{F}_E] \otimes_B (1 - [E]) = [j] \otimes_{\mathcal{T}_E} [\alpha]^{-1}$$

Let [ext] be the class of the defining extension and
$[\partial] := [\mathrm{ext}] \otimes_{\mathcal{K}(\mathcal{F}_E)} [\mathcal{F}_E] \in KK_1(\mathcal{O}_E, B)$ the class of the product.
For $C = \mathbb{C}$ we get exact sequences in K-theory

$$\begin{tikzcd}
K_0(B) \arrow[r, "1-[E]"] & K_0(B) \arrow[r, "j_*"] & K_0(\mathcal{O}_E) \arrow[d, "{[\partial]}"] \\
K_1(\mathcal{O}_E) \arrow[u, "{[\partial]}"] & K_1(B) \arrow[l, "j_*"] & K_1(B) \arrow[l, "1-[E]"]
\end{tikzcd},$$

and in K-homology

$$\begin{tikzcd}
K^0(B) \arrow[d, "{[\partial]}"] & K^0(B) \arrow[l, "1-[E]"'] & K^0(\mathcal{O}_E) \arrow[l, "j^*"'] \\
K^1(\mathcal{O}_E) \arrow[r, "j^*"] & K^1(B) \arrow[r, "1-[E]"] & K^1(B) \arrow[u, "{[\partial]}"']
\end{tikzcd}. \tag{5}$$

In the case of a self-Morita equivalence bimodule, the conditional expectation ρ defines a B-valued inner product on $\mathcal{O}_E$.
We denote the completion with Ξ_B.

Then the generator of the circle action, i.e the the number operator, defines an unbounded self-adjoint regular operator D on Ξ_B.
Well defined unbounded Kasparov module $(\mathcal{O}_E, \Xi_B, D)$
The connecting homomorphism is realised as a Kasparov product with the class $[(\mathcal{O}_E, \Xi_B, D)] \in KK^1(\mathcal{O}_E, B)$.

- In the case of SMEBs, the Pimsner algebra can be thought of as a noncommutative associated circle bundle construction.
- The corresponding six-term exact can be interpreted as a *Gysin sequence* in K-theory and K-homology for the 'line bundle' E over the 'noncommutative base space' B.
- Multiplication by the Euler class is replaced with the Kasparov product with $1 - [E]$.

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Aim: ccompare the Gysin exact sequences with the exact sequences associated to the mapping cone of the inclusion $B \to \mathcal{O}_E$.

$$0 \longrightarrow S\mathcal{O}_E \xrightarrow{j_*} M(B, O_E) \xrightarrow{\mathrm{ev}} B \longrightarrow 0, \tag{6}$$

where $\mathrm{ev}(f) = f(0)$ and $j(g \otimes b)(t) = g(t)b$.

$$\begin{tikzcd} K_0(B) \arrow[r, "\partial'"] & K_0(S\mathcal{O}_E) \arrow[r, "j_*"] & K_1(M) \arrow[d, "\mathrm{ev}_*"] \\ K_0(M) \arrow[u, "\mathrm{ev}_*"] & K_1(S\mathcal{O}_E) \arrow[l, "j_*"] & K_1(B) \arrow[l, "\partial'"] \end{tikzcd} \tag{7}$$

We use the identification $\mathrm{Bott} : K_j(\mathcal{O}_E) \to K_{j+1}(S\mathcal{O}_E)$ to define a map $j_*^B : K_i(\mathcal{O}_E) \to K_{i+1}(M)$ given by $j_* \circ \mathrm{Bott}$.

We now compare the six term exact sequences induced by the mapping cone of the inclusion with the Gysin six term exact sequences.

$$\begin{tikzcd}
\cdots \arrow[r, "\iota_*"] & K_0(\mathcal{O}_E) \arrow[r, "j_*^B"] \arrow[d, "="] & K_1(M) \arrow[r, "\mathrm{ev}_*"] \arrow[d, "?"] & K_1(B) \arrow[r, "\iota_*"] \arrow[d, "="] & K_1(\mathcal{O}_E) \arrow[r, "j_*^B"] \arrow[d, "="] & \cdots \\
\cdots \arrow[r, "\iota_*"] & K_0(\mathcal{O}_E) \arrow[r, "\partial"] & K_1(B) \arrow[r, "1-[E]"] & K_1(B) \arrow[r, "\iota_*"] & K_1(\mathcal{O}_E) \arrow[r, "\partial"] & \cdots
\end{tikzcd} \tag{8}$$

Meyer & Nest ([MR06]): the KK category is a *triangulated category*, whose exact triangles are mapping cone triangles with isomorphisms given by KK-equivalence (cf. [MR06]). Indeed, for every semisplit extension with quotient map π, one has an isomorphism of triangles where all vertical arrows are KK-equivalences.

The KK-equivalence between B and $\mathcal{T}_E$ and the natural Morita equivalence between B and $\mathcal{K}(\mathcal{F}_E)$, together with the axioms of a triangulated category which imply that the mapping cone of $B \to \mathcal{O}_E$ is unique up to a (non-canonical) isomorphism in KK.

The operator D in the extension class has discrete spectrum and commutes with the left action of B, hence we have $\iota^*_{B,O_E}[(\mathcal{O}_E, \Xi_B, D)] = 0$.
There is a class $[\widehat{D}] \in KK(M(B, O_E), B)$ such that $j^{B*}[\widehat{D}] = [(\mathcal{O}_E, \Xi_B, D)]$.
An explicit unbounded representative for the class $[\widehat{D}]$, provided by the main result of [CPR10]. One obtains commutativity of

$$\begin{tikzcd}
\cdots \arrow[r, "\iota_*"] & K_0(\mathcal{O}_E) \arrow[r, "j^B_*"] \arrow[d, "="] & K_1(M) \arrow[r, "\mathrm{ev}_*"] \arrow[d, "\widehat{D}"] & \cdots \\
\cdots \arrow[r, "\iota_*"] & K_0(\mathcal{O}_E) \arrow[r, "\partial"] & K_1(B) \arrow[r, "1-[E]"] & \cdots
\end{tikzcd} \tag{9}$$

$$
\begin{tikzcd}
\cdots \arrow[r, "j_*^B"] & K_i(M) \arrow[r, "\mathrm{ev}_*"] \arrow[d, "\widehat{D}"] & K_i(B) \arrow[r, "\iota_*"] \arrow[d, "="] & \cdots \\
\cdots \arrow[r, "\partial"] & K_i(B) \arrow[r, "1-[E]"] & K_i(B) \arrow[r, "\iota_*"] & \cdots
\end{tikzcd}
\tag{10}
$$

- We use the characterisation of the K-theory group $K_0(M)$ due to Putnam.
- For any $v \in K_*(M)$, we need to evaluate the product $[v] \otimes_{\mathcal{O}_E} [\widehat{D}] \otimes_B ([\mathrm{Id}_{KK(B,B)}] - [E])$. Our strategy is to use [CPR10], to find that the latter product is given by an index.
- This works for $i = 0$. For $i = 1$ we have to adapt the argument to suspended algebras.

Theorem (A.-Rennie 16)

Let $(\mathcal{O}_E, \Xi_B, D)$ be the unbounded representative of the defining extension and $(M(B, \mathcal{O}_E), \widehat{\Xi}_B, \widehat{D})$ the lift to the mapping cone.
Then

$$\cdot \otimes_{M(B,\mathcal{O}_E)} [(M(B, \mathcal{O}_E), \widehat{\Xi}_B, \widehat{D})] : \; K_*(M(B, \mathcal{O}_E)) \to K_*(B)$$

is an isomorphism that makes diagrams in K-theory commute.
If furthermore the algebra B belongs to the Bootstrap class, the Kasparov product with the class $[(M(B, \mathcal{O}_E), \widehat{\Xi}_B, \widehat{D})] \in KK(M(B, \mathcal{O}_E), \mathcal{O}_E)$ is a KK-equivalence.

- The result is valid for more general bimodules: bi-Hilbertian bimodules of finite Jones-Watatani index, satisfying some additional assumption.
- Relies on results by Goffeng, Mesland, Rennie ([GMR15]) on unbounded representatives for the extension class.
- In order to deal with suspensions we generalised their construction to *nonunital* C^*-algebras.

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The C^*-algebra of the odd-dimensional quantum sphere $C(S_q^{2n+1})$ is the universal C^*-algebra generated by $n+1$ elements $\{z_i\}_{i=0,\ldots,n}$ and relations:

$$z_i z_j = q^{-1} z_j z_i \qquad 0 \le i < j \le n\,,$$

$$z_i^* z_j = q z_j z_i^* \qquad i \neq j\,,$$

$$[z_n^*, z_n] = 0, \qquad [z_i^*, z_i] = (1-q^2) \sum_{j=i+1}^{n} z_j z_j^* \qquad i = 0, \ldots, n-1\,,$$

and a sphere relation:

$$z_0 z_0^* + z_1 z_1^* + \ldots + z_n z_n^* = 1\,.$$

This C^*-algebra can be realised as a graph C^*-algebra.

Let $\mathrm{m} = (m_0, \ldots, m_n)$ any weight vector.
Weighted circle action on $C(S_q^{2n+1})$, whose fixed point algebra is the algebra of functions on the weighted projective spaces $C(\mathbb{WP}^n(\mathrm{m}))$:

$$\sigma_\xi^{\mathrm{m}}(z_i) = \xi^{m_i} z_1 \qquad \xi \in \mathbb{T}^1. \tag{11}$$

Brzeziński-Szymański (BS16): let m be a weight vector such that there exists $0 \leq j \leq n-1$ with m_j coprime with m_n. Then there exists an exact sequence of C*-algebras

$$0 \longrightarrow \mathcal{K}^{\oplus m_n} \longrightarrow C(\mathbb{WP}^n(\mathrm{m})) \longrightarrow C(\mathbb{WP}_q^{n-1}(\mathrm{m}_n)) \longrightarrow 0, \tag{12}$$

where m_n denotes the weight vector $(m_0, \ldots, m_{n-1})$.

Let E denote the Hilbert C*-module given by the first spectral subspace for the weighted circle action on $C(S_q^{2n+1})$.
The Pimsner algebra over $B = C(\mathbb{WP}^n)$ for the module $E^{\otimes d}$ is the C*-algebra of the quantum lens space, i.e.

$$\mathcal{O}_{E^{\otimes d}} \simeq C(L_q^{2n+1}(d \cdot N; \mathrm{m}))$$

Free action (principal circle bundle) for $N_{\mathrm{m}} = \prod_{i=0}^{n} m_i$.

For any separable C^*-algebra C, Pimsner exact sequences:

$$\begin{tikzcd}
KK_0(C, C(\mathbb{WP}^n_q(\mathrm{m}))) \arrow[r, "(1-[E^{\otimes d}])\otimes"] & KK_0(C, C(\mathbb{WP}^n_q(\mathrm{m}))) \arrow[r, "i_*"] & KK_0(C, C(L_q(d))) \arrow[d, "\partial"] \\
KK_1(C, C(L_q(d))) \arrow[u, "\partial"] & KK_1(C, C(\mathbb{WP}^n_q(\mathrm{m}))) \arrow[l, "i_*"] & KK_1(C, C(\mathbb{WP}^n_q(\mathrm{m}))) \arrow[l, "(1-[E^{\otimes d}])\otimes"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
KK_0(C(\mathbb{WP}^n_q(\mathrm{m})), C) \arrow[d, "\partial"] & KK_0(C(\mathbb{WP}^n_q(\mathrm{m})), C) \arrow[l, "\otimes(1-[E^{\otimes d}])"'] & KK_0(C(L_q(d)), C) \arrow[l, "i^*"'] \\
KK_1(C(L_q(d)), C) \arrow[r, "i^*"] & KK_1(C(\mathbb{WP}^n_q(\mathrm{m})), C) \arrow[r, "\otimes(1-[E^{\otimes d}])"] & KK_1(C(\mathbb{WP}^n_q(\mathrm{m})), C) \arrow[u, "\partial"]
\end{tikzcd}$$

Proposition ([BS16, Corollary 3.2])

Let m *be a weight vector with the property that for each* $j \geq 1$ *there exists* $i < j$ *such that* $\gcd(m_i, m_j) = 1$*. Then the K-theory groups of the quantum weighted projective spaces are given by*

$$K_0(C(\mathbb{WP}^n_q(\mathrm{m})) = \mathbb{Z}^{1+\sum_{i=1}^n m_i}, \quad K_1(C(\mathbb{WP}^n_q(\mathrm{m})) = 0.$$

Proposition

Let m *be a weight vector satisfying the assumptions of Proposition 5.1,* $M := m_1 + \cdots + m_n$*. Then the* C^**-algebra* $C(\mathbb{WP}^n(\mathrm{m}))$ *is KK-equivalent to* $\mathbb{C}^{1+M}$*.*

Let $[I] \in KK(\mathbb{C}^{M+1}, C(\mathbb{WP}^n_q(\mathrm{m})))$ and $[\Pi] \in KK(C(\mathbb{WP}^n_q(\mathrm{m})), \mathbb{C}^M)$ implement the KK-equivalence between $\mathbb{C}^{M+1}$ and $C(\mathbb{WP}^n_q(\mathrm{m}))$, i.e.

$$[I] \otimes_{C(\mathbb{WP}^n_q(\mathrm{m}))} [\Pi] = 1_{KK(\mathbb{C}^{M+1}, \mathbb{C}^{M+1})}, \quad [\Pi] \otimes_{\mathbb{C}^{M+1}} [I] = 1_{KK(C(\mathbb{WP}^n_q(\mathrm{m})), C(\mathbb{WP}^n_q(\mathrm{m})))}. \tag{13}$$

Simplify the exact sequences (24) and (24):

$$KK_i(C, \mathbb{C}^M) \simeq \bigoplus_{k=0}^{M+1} K^i(C) \quad \text{and} \quad KK_i(\mathbb{C}^M, C) \simeq \bigoplus_{k=0}^{M+1} K_i(C), \quad i = 0, 1.$$

Replace $[E]$ by the class

$$[I] \otimes_{C(\mathbb{WP}^n_q(\mathrm{m}))} [E] \otimes_{C(\mathbb{WP}^n_q(\mathrm{m}))} [\Pi] \in KK(\mathbb{C}^{M+1}, \mathbb{C}^{M+1}). \tag{14}$$

The six term exact sequence in (24) becomes

$$\begin{tikzcd}
\oplus_{i=0}^{M+1} K^0(C) \arrow[r, "1-\mathrm{A}^d"] & \oplus_{i=0}^{M+1} K^0(C) \arrow[r] & KK_0\big(C, C(L_q(d))\big) \arrow[d] \\
KK_1(C, C(L_q(d))) \arrow[u] & \oplus_{i=0}^{M+1} K^1(C) \arrow[l] & \oplus_{i=0}^{M+1} K^1(C) \arrow[l, "1-\mathrm{A}^d"]
\end{tikzcd} ,$$

while, denoting with A^t the transpose of A, the six term exact sequence in (24) becomes

$$\begin{tikzcd}
\oplus_{i=0}^{M+1} K_0(C) \arrow[d] & \oplus_{r=0}^{M+1} K_0(C) \arrow[l, "1-(\mathrm{A}^t)^d"'] & KK_0\big(C(L_q(d)), C\big) \arrow[l] \\
KK_1\big(C(L_q(d)), C\big) \arrow[r] & \oplus_{i=0}^{M+1} K_1(C) \arrow[r, "1-(\mathrm{A}^t)^d"] & \oplus_{i=0}^{M+1} K_1(C) \arrow[u]
\end{tikzcd} .$$

Where $\mathrm{A} \in Mat_{M+1}(\mathbb{Z})$ is the map implementing the tensor product with $[E]$.

For $C = \mathbb{C}$, using the fact that $K_1(\mathbb{WP}^n_q(\mathrm{m})) = K^1(\mathbb{WP}^n_q(\mathrm{m}) = 0$, we obtain exact sequences

$$0 \longrightarrow K_1\big(C(L_q(d)\big) \longrightarrow \mathbb{Z}^{M+1} \xrightarrow{1-\mathrm{A}^d} \mathbb{Z}^{M+1} \longrightarrow K_0\big(C(L_q(d)\big) \longrightarrow 0 \ ,$$

and

$$0 \longleftarrow K^1\big(C(L_q(d))\big) \longleftarrow \mathbb{Z}^{M+1} \xleftarrow[1-(\mathrm{A}^t)^d]{} \mathbb{Z}^{M+1} \longleftarrow K^0\big(C(L_q(d))\big) \longleftarrow 0 \ .$$

Computation of the K-theory and K-homology groups of the quantum lens spaces.

Theorem

Let m *be a weight vector satisfying the assumptions of Proposition 5.1. Then for any* $d \in \mathbb{N}$ *we have that*

$$K_0\big(C(L_q(d))\big) \simeq \mathrm{Coker}(1 - \mathrm{A}^d), \qquad K_1\big(C(L_q(d))\big) \simeq \mathrm{Ker}(1 - \mathrm{A}^d)$$

and

$$K^0\big(C(L_q(dlk; k, l))\big) \simeq \mathrm{Ker}(1-(\mathrm{A}^t)^d), \qquad K^1\big(C(L_q(d))\big) \simeq \mathrm{Coker}(1-(\mathrm{A}^t)^d)\,.$$

It remains an open problem to describe the precise relationship of our matrix A with the matrix used in [BS16] to compute the K-theory of quantum lens spaces.

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- Pimsner algebras for SMEBs are the analogue of associated circle bundles.
- We made the relationship between Pimsner's exact sequences and mapping cone exact sequences explicit.
- We showed how Pimsner's exact sequences allow us to compute the K-theory and K-homology of quantum lens spaces using a different Cuntz-Pimsner model.

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