Spectral Geometries for Some Diffeologies

Paolo Bertozzini

Department of Mathematics and Statistics - Thammasat University - Bangkok

"Noncommutative Index Theory"
Simons Semester
"Noncommutative Geometry the Next Generation"

Banach Center, IMPAN Warsaw, 28 October 2016.

Abstract 1

Diffeologies, introduced by J.-M.Souriau, are a vast generalization of smooth manifolds that, from the point of view of category theory, is much better behaved.

Not much is known on the possible ways to describe such spaces in the language of spectral geometry and it is the purpose of this quite preliminary talk to explore some of the problems involved in such study.

We will also see how categories of spectral triples might be of help in these matters and more.

Abstract 2

This is a joint work with:

- Roberto Conti (Sapienza Università di Roma),
- Fabian Germ (TU Wien).

Outline

- Manifolds and Diffeologies
- Naive Spectral Triples: Hodge De Rham / Atiyah Singer
- ⋆ Naive Spectral Triples for Diffeologies
- ⋆ Outlook: Categorical Approach

- Manifolds and Diffeologies

Generalizing Manifolds 1

- From the categorical point of view, manifolds are problematic:
  - (co)-limits do not always exist;
  - quotients and other constructs can fail to be manifolds;
  - smooth maps are a non-Cartesian category.
- Several alternatives have been proposed (with different motivations):
  - Differential Spaces [Sikorski, Smith]
  - Frölicher Spaces [Frölicher (see Kriegl-Michor)]
  - Diffeologies [Souriau, (Chen - Differentiable Spaces)]
  - Spectral Triples [Connes]
  - Colombeau Algebras
  - Synthetic Differential Geometry [Kock],
  - Generalized Structured Spaces [Lurie, Schreiber]

Generalizing Manifolds 2

$$
\begin{tikzcd}
\boxed{\text{manifolds}} \arrow[r, hook] & \boxed{\text{manifolds with boundary}} \arrow[d, hook] \\
\boxed{\text{diffeologies}} & \boxed{\text{manifolds with singularities}} \arrow[l, hook']
\end{tikzcd}
$$

Two functors relate these categories of differentiable maps

$$
\begin{tikzcd}
\boxed{\text{Diffeologies}} \arrow[r, bend left] & \boxed{\text{Sikorski Spaces}} \arrow[l, bend left]
\end{tikzcd}
$$

and Frölicher spaces are equivalent to the reflective subcategories.

Generalizing Manifolds 3

<< ... diffeology did not spring up on an empty battlefield. Many solutions have been proposed to these question, from functional analysis to noncommutative geometry, via smooth structures à la Sikorski or à la Frölicher. For what concern us, each of these attempts is unsatisfactory: functional analysis is often an overkilling heavy machinery. ... Noncommutative geometry is unconfortable for the geometer who is not familiar enough with the C-algebra world, where he loses intuituion and sensibility. >>*

[Patrick Iglesias-Zemmour - introduction to the book "Diffeology"]

Diffeologies

A **diffeology** on a set M is a family $\mathcal{D}$ of maps $\phi : U_\phi \to M$, called **plots**, where $U_\phi \subset \mathbb{R}^{n_\phi}$, with $n_\phi \in \mathbb{N}$, such that:

- U_ϕ is open in $\mathbb{R}^{n_\phi}$, for all $\phi \in \mathcal{D}$,
- for all $p \in M$ there exists a $\phi \in \mathcal{D}$ such that $p \in \phi(U_\phi)$,
- $\phi \circ f \in \mathcal{D}$ if $\phi \in \mathcal{D}$ and $f \in C^\infty(V, U_\phi)$, with $V \subset \mathbb{R}^m$ open,
- $\phi \in \mathcal{D}$ when $\phi : \bigcup_{j \in J} U_j \to M$ and $\phi|_{U_j} \in \mathcal{D}$, for all $j \in J$.

A function $f : M \to N$ between two diffeological spaces $(M, \mathcal{D}_M)$, $(N, \mathcal{D}_N)$ is **smooth** if $\phi \in \mathcal{D}_M$ implies $f \circ \phi \in \mathcal{D}_N$.

Smooth maps between diffeologies give a Cartesian closed category.

Manifolds and Diffeologies

- Every manifold M is a diffeology:
  the family of all smooth maps $U_\phi \xrightarrow{\phi} M$, with $U_\phi \subset \mathbb{R}^{n_\phi}$ open, is the **standard diffeology**[1] of the manifold M.
- Smooth maps between manifolds give a full subcategory of the smooth maps of diffeologies.
- Diffeologies are naturally topological spaces: the $\mathcal{D}$-**topology** of a diffeology $\mathcal{D}$ is the strong topology induced on M by $\mathcal{D}$: A is open iff $\phi^{-1}(A)$ open in $\mathbb{R}^{n_\phi}$, for all $\phi \in \mathcal{D}$.
- $f : M \to \mathbb{R}$ is continuous iff $f \circ \phi : U_\phi \to \mathbb{R}$ is continuous.

[1] The (maximal) atlas of a smooth manifold gives a diffeology and the standard diffeology is its saturation.

Further Examples

- Manifolds with borders and manifolds with corners
- Manifolds with singularities (arbitrary unions of submanifolds)
- Quotients of manifolds (irrational tori $\mathbb{R}/(\mathbb{Z}+\theta\mathbb{Z})$)
- Equalizers of smooth maps $f, g : M \to N$ of manifolds
- Smooth maps $C^\infty(M, N)$ between two given manifolds M, N
- Infinite dimensional manifolds
- The set of diffeologies on M is a complete lattice.

  Any family $\mathcal{P}$ of maps $U_\phi \xrightarrow{\phi} M$, with $U_\phi \subset \mathbb{R}^{n_\phi}$ open, determines a diffeology on M: the smallest diffeology containing $\mathcal{P}$.

Categories of Plots 1

Let $\mathcal{S}$ denote the category of smooth maps $U \xrightarrow{f} V$ between open sets $U \subset \mathbb{R}^{n_1}$, $V \subset \mathbb{R}^{n_2}$ and let M be a diffeological space.

- The **category** $\mathcal{S}/M$ **of plots of** M has objects the plots $U_\phi \xrightarrow{\phi} M$, $\phi \in \mathcal{D}$ and morphism (ψ, f, ϕ) the commutative diagrams $\psi \circ f = \phi$, with $\phi, \psi \in \mathcal{D}$ and $f \in \mathcal{S}$.
- a diffeology M can be seen as a colimit of the functor $(\phi, f, \psi) \mapsto f$ from $\mathcal{S}/M$ to the category of diffeologies.
- the plots of a diffeology M are a sheaf on the site $\mathcal{S}$.
- the category of smooth maps between diffeologies is a pre-topos (not all sheaves on $\mathcal{S}$ are diffeologies).[2]

[2] J.Baez, A.Hoffnung (2008) `arXiv:0807.1704`

Categories of Plots 2

Further properties of the category of plots might become relevant in some classes of examples:[3]

- A diffeology is **weakly filtered** if its category of plots is a directed set.
- A weakly filtered diffeology is **filtered** if every diagram of plots has a co-cone.

[3] J.D.Christensen, E.Wu (2015) `arXiv:1510.09182`

Germs of Spans $\star$

- Manifolds are obtained by “global glueing” a family of smooth spans of charts whose germs constitute a bundle of groupoids.
- Diffeologies are obtained by “global glueing” a family of smooth spans of plots whose germs constitute a category.

Manifolds Inside Diffeologies ⋆

- Given a diffeology $(M, \mathcal{D})$, for all $n \in \mathbb{N}$, by Zorn's lemma we can always consider the family $\{\mathcal{D}^n_\alpha\}_{\alpha\in\Gamma}$ of maximal smooth n-manifold atlases $\mathcal{D}^n_\alpha \subset \widehat{\mathcal{D}}$ inside the saturation[4] of $\mathcal{D}$.
- Each such maximal atlas determines a manifold M_α inside M.
- The maximal manifold M_α might not be closed in the $\mathcal{D}$-topology (dense smooth injection of $\mathbb{R}$ into $\mathbb{T}^2$).
- The same subset $N \subset M$ can be a maximal closed submanifold with different incompatible atlases ($\mathbb{R}$ with diffeology $\mathcal{D} := \{\phi, \psi\}$, where $\phi : t \mapsto t$, $\psi : t \mapsto t^3$).

[4] The **saturation** $\widehat{\mathcal{D}}$ is the diffeology that contains all the differentiable maps $U_\phi \xrightarrow{\phi} M$ defined on open sets $U_\phi \in \mathbb{R}^{n_\phi}$.

• Naive Spectral Triples: Hodge De Rham / Atiyah Singer

Naive Spectral Triples

A naive (unital) **spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ consists of:

- a (unital) C*-algebra $\mathcal{A}$,
- a (unital) $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ as bounded operators on a Hilbert space $\mathcal{H}$,
- a (usually unbounded) operator[5] D on $\mathcal{H}$

such that, for a (unital) dense pre-C*-subalgebra $\mathcal{A}^\infty \subset \mathcal{A}$:

- $\mathrm{Dom}(D)$ is invariant under $\pi(x)$, for all $x \in \mathcal{A}^\infty$,
- $[D, \pi(x)] : \mathrm{Dom}(D) \to \mathcal{H}$ is a bounded operator, if $x \in \mathcal{A}^\infty$.

We consider spectral triples over unital Abelian C*-algebras.

[5] D is usually required to be self-adjoint and have compact resolvent.

Hodge De Rham Spectral Triples 1

Let M be a compact smooth Riemannian finite-dimensional oriented manifold; we construct its Hodge De Rham naive spectral triple $(\mathcal{A}, \mathcal{H}_{HD}, D_{HD})$ as follows:

- $\mathcal{A} := C(M)$ is the Abelian unital C*-algebra of complex-valued continuous functions on M.
- $\mathcal{H}_{HD} := \mathrm{L}^2(\Lambda^{\mathbb{C}}_\bullet(M))$, the Hilbert space of "square integrable" sections of the complexified Grassmann Hermitian bundle $\Lambda^{\mathbb{C}}_\bullet(T(M))$ of M, is obtained by separation/completion of the Hilbert $C(M)$-bimodule $\Omega_\bullet(M) := \Gamma(\Lambda^{\mathbb{C}}_\bullet(T(M)))$ of continuous sections of $\Lambda^{\mathbb{C}}_\bullet(T(M))$ via the measure $\mu : C(M) \to \mathbb{C}$ given by the usual integration of volume forms: $\mu(f) := \int_M \star f$, where $\star$ denotes the Hodge duality operator: $\langle \omega \mid \rho \rangle := \int_M \star \langle \omega \mid \rho \rangle_{C(M)} = \int_M \omega \wedge (\star\rho)$, for $\omega, \rho \in \Omega_\bullet(M)$.

Hodge De Rham Spectral Triples 2

- The representation $\pi : C(M) \to \mathcal{B}(\mathcal{H}_{HD})$ is given by linear continuous extension (under the previous completion) of the usual action by pointwise multiplication of $C(M)$ on $\Omega_\bullet(M)$.
- The Hodge De Rham Dirac operator D_{HD} is obtained, from $\nabla^{\Omega_\bullet(M)} : \Omega^\infty_\bullet(M) \to \Omega^\infty_\bullet(M) \otimes_{C^\infty(M)} \Omega^\infty_1(M)$, the unique extension of the Levi-Civita covariant derivative to the smooth complexified Grassmann algebra of M, and from the Clifford action $c_{\Omega_\bullet(M)} : \mathbb{C}\mathrm{l}(M) \otimes_{C(M)} \Omega_\bullet(M) \to \Omega_\bullet(M)$, where $\mathbb{C}\mathrm{l}(M)$ is the complexified Clifford algebra of M, consisting of continuous sections of the complexified Clifford bundle $\mathbb{C}\mathrm{l}(T^*(M))$ of the Hermitian bundle $T^*(M)$, "by contraction":

$$D_{HD} = (-i) \cdot c_{\Omega_\bullet(M)} \circ \nabla^{\Omega_\bullet(M)}.$$

Atiyah Singer Spectral Triples

When the manifold M is further assumed to be spinorial, with a given complex (Hermitian) spinor bundle $S(M)$, the Atiyah Singer naive spectral triple $(\mathcal{A}, \mathcal{H}_{AS}, D_{AS})$ is given by:

- $\mathcal{A} := C(M)$, the same previous unital C*-algebra,
- $\mathcal{H}_{AS} := \mathrm{L}^2(S(M))$ is the Hilbert space of "square integrable" sections of the Hermitian spinor bundle $S(M)$, obtained by separation/completion of the $C(M)$ Hilbert C*-bimodule $\Gamma(S(M))$, under the same measure $\mu : C(M) \to \mathbb{C}$ induced by integration of volume forms $\mu(f) := \int_M \star f$.
- $D_{AS} := (-i) \cdot c_{\Gamma(S(M))} \circ \nabla^{\Gamma(S(M))}$ is the Atiyah Singer Dirac operator, where $\nabla^{\Gamma(S(M))}$ is the unique Hermitian extension of the Levi-Civita connection to the Hermitian $\mathbb{C}\mathrm{l}^\infty(M)$-module $\Gamma^\infty(S(M))$ of smooth sections of the spinor bundle and $c_{\Gamma(S(M))}$ is the Clifford action.

Reconstruction Theorems 1

Under a family of further technical assumptions “geometries” can be recovered from “spectral data”:

Theorem (Connes reconstruction theorem)

Connes A (2013) **On the Spectral Characterization of Manifolds**
J Noncommut Geom 7(1):1-82 `arXiv:0810.2088`

A commutative spectral triple (that is irreducible, real, graded, strongly regular, m-dimensional, finite, absolutely continuous, orientable with totally antisymmetric Hochschild cycle in the last m entries, and satisfying Poincaré duality) is naturally isomorphic to the canonical Atiyah-Singer spectral triple of a spinorial Riemannian manifold with a given Hermitian spinor bundle equipped with charge conjugation.

Reconstruction Theorems 2

The reconstruction theorem has been also extended to cover:

- Riemannian spectral triples
  Lord S, Rennie A, Varilly J (2012) **Riemannian Manifolds in Noncommutative Geometry** J Geom Phys 62(7):1611-1638 `arXiv:1109.2196`
- almost commutative spectral triples
  Ćaćić B (2012) **A Reconstruction Theorem for Almost-Commutative Spectral Triples** Lett Math Phys 100(2):181-202 `arXiv:1101.5908`

Naive Spectral Modules 1 ⋆

A naive (unital) **spectral module triple** $(\mathcal{A}, \mathcal{M}, D_{\mathcal{M}})$ consists of:

- a (unital) C*-algebra $\mathcal{A}$,
- a (unital) left Hilbert C*-module $\mathcal{M}$ over $\mathcal{A}$,
- an (usually unbounded) regular[6] operator $D_{\mathcal{M}}$ on $\mathcal{M}$

such that, for a (unital) dense pre-C*-subalgebra $\mathcal{A}^\infty \subset \mathcal{A}$:

- $\mathrm{Dom}(D_{\mathcal{M}})$ is invariant under $\pi(x)$, for all $x \in \mathcal{A}^\infty$,
- $[D_{\mathcal{M}}, \pi(x)] : \mathrm{Dom}(D_{\mathcal{M}}) \to \mathcal{M}$ is adjointable, if $x \in \mathcal{A}^\infty$.

When a state $\omega : \mathcal{A} \to \mathbb{C}$ is given on the C*-algebra $\mathcal{A}$ and $D_{\mathcal{M}}$ leaves invariant the ω-null-space in $\mathcal{M}$, by separation-completion of the quotient, we obtain a naive spectral triple.

[6] This means that $\mathrm{Dom}\, D_{\mathcal{M}}$, $\mathrm{Dom}\, D^*_{\mathcal{M}}$ and range of $\mathrm{Id}_{\mathcal{M}} + D^*_{\mathcal{M}} D_{\mathcal{M}}$ are dense.

Naive Spectral Modules 1 $\star$

Proposition

Every Hilbert bundle (H, π_H, M) of Hermitian Clifford modules for the complexified Clifford bundle $(\mathbb{C}\mathrm{l}(T(M)), \pi_{\mathbb{C}\mathrm{l}(T(M))}, M)$ over a (compact) Riemannian manifold M, equipped with a metric Koszul connection $\nabla^H : \Gamma^\infty(H) \to \Gamma^\infty(H) \otimes_{C^\infty(M)} \Omega_1(M)$, induces a naive spectral module triple.

Hodge De Rham and Atiyah Singer spectral triples are obtained by separation-completion of their respective spectral module triples via the unique state provided by integration $\omega(f) := \frac{1}{\mathrm{vol}(M)} \int_M \star f$, that exists when we assume M compact Riemannian orientable.

• Spectral Triples and Diffeologies

Hodge De Rham Spectral Triples: Algebra

Supose that $(M, \mathcal{D})$ is a diffeological space that is *compact Hausdorff* in the induced $\mathcal{D}$-topology.

- Consider the algebra $C(M, \mathbb{R})$ of continuous real-valued functions on M.
- Its complexification $C(M) := C(M, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ is the unital Abelian C*-algebra of complex-valued continuous maps on M.

$$\boxed{\mathcal{A} := C(M)}$$

- Smooth complex-valued functions $C^\infty(M) = C^\infty(M, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ give a dense pre-C*-algebra in $C(M)$.
- By Gel'fand-Naĭmark duality, M is homeomorphic to $\mathrm{Sp}(C(M))$, the spectrum of $C(M)$.

Algebra Problems $\star$

- Assuming T_2 (local) compactness for diffeologies is not as innocuous as in the manifold case: T_2 can fail for quotients! $\{0\} \cup \left(\bigcup_{n=1}^{\infty} \frac{1}{n}(\mathbb{T})\right) \subset \mathbb{C}$ with the diffeology generated by $\{\phi_n(t) = \frac{1}{n}(\exp(it)) \mid n \in \mathbb{N}_0\}$ is not compact.
- Since the category of C*-algebras is complete and co-complete, one can work with the C*-limits of the categories of restriction morphisms between the C*-algebras $C(M_\alpha)$ for every closed maximal manifold $M_\alpha \subset M$.
- Problems in "reconstruction attempts": the Gel'fand spectrum of the limit is necessarily locally compact Hausdorff and hence not always homeomorphic to M in its $\mathcal{D}$-topology.
- Solution: either use spectral geometries over "topological $*$-algebras" or work with the bare "category of spectral geometries" over the maximal closed manifolds $M_\alpha \subset M$.

Linear Bundles Functorialities: Grassmann

Let $\mathscr{L}_X^{\mathbb{K}}$ be a category with objects "linear bundles" (E, π_E, X) on X (i.e. surjective maps $\pi_E : E \to X$ whose fibers $E_p := \pi^{-1}(p)$, $p \in X$, are $\mathbb{K}$-linear spaces) and with morphisms fiber preserving fiberwise $\mathbb{K}$-linear maps.

The covariant Grassmann functor $\mathscr{L}_X^{\mathbb{K}} \xrightarrow{\Lambda} \mathscr{L}_X^{\mathbb{K}}$:

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] & \mapsto & \Lambda(E) \arrow[rr, "\Lambda(\phi)"] \arrow[dr, "\pi_{\Lambda(E)}"'] & & \Lambda(F) \arrow[dl, "\pi_{\Lambda(F)}"] \\ & X & & & & X & \end{tikzcd}$$

associates to every bundle (E, π_E, X) with p-fibers E_p its "Grassmann bundle" $(\Lambda(E), \pi_{\Lambda(E)}, X)$ with p-fibers $\Lambda(E)_p := \Lambda(E_p)$ the exterior algebra of E_p; and to every linear $\phi_p : E_p \to F_p$ its Grassmann homomorphism $\Lambda(\phi_p)$.

Linear Bundles Functorialities: Clifford

Let $\mathscr{H}_X^{\mathbb{R}}$ be the subcategory of $\mathscr{L}_X^{\mathbb{R}}$ with objects "Hermitian linear bundles" (i.e. linear bundles with fibers equipped with a $\mathbb{R}$-valued inner product) and with morphisms those morphisms of $\mathscr{L}_X^{\mathbb{R}}$ that are fiberwise inner product preserving.

The covariant Clifford functor $\mathscr{H}_X^{\mathbb{R}} \xrightarrow{\mathrm{Cl}} \mathscr{L}_X^{\mathbb{R}}$:

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & X & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathrm{Cl}(E) \arrow[rr, "\mathrm{Cl}(\phi)"] \arrow[dr, "\pi_{\mathrm{Cl}(E)}"'] & & \mathrm{Cl}(F) \arrow[dl, "\pi_{\mathrm{Cl}(F)}"] \\ & X & \end{tikzcd}$$

associates to every Hermitian $\mathbb{R}$-bundle (E, π_E, X) with p-fibers E_p its "Clifford bundle" $(\mathrm{Cl}(E), \pi_{\mathrm{Cl}(E)}, X)$ with p-fibers $\mathrm{Cl}(E)_p := \mathrm{Cl}(E_p)$ the Clifford algebra of E_p; and to every isometry $\phi_p : E_p \to F_p$ its Bogolyubov homomorphisms $\mathrm{Cl}(\phi_p)$.

Linear Bundles Functorialities: Clifford Action

For Hermitian $\mathbb{R}$-bundles, we have a "fibered version" of the usual left Clifford action: $c_E : \mathrm{Cl}(E) \to \mathrm{End}_X(\Lambda(E))$, defined by universal factorization property on elements $v \in E_x$, with $x \in X$ as $c_E(v) := \epsilon_E(v) + \iota_E(v)$ where the "creation and annihilation operators" are defined, for $w_1, \dots, w_k \in E_x$ by:

$$\epsilon_E(v)(w_1 \wedge \cdots \wedge w_k) := (v \wedge w_1 \wedge \cdots \wedge w_k),$$

$$\iota_E(v)(w_1 \wedge \cdots \wedge w_k) := \sum_{j=1}^{k} \langle v \mid w_j \rangle_{E_x} (-1)^{j-1} (w_1 \wedge \cdots \wedge \check{w}_j \wedge \cdots \wedge w_k).$$

Our Clifford algebras satisfy: $[c_E(v), c_E(w)]_+ = 2\langle v \mid w \rangle_E \cdot I_{\Lambda(E)}$.

$\Lambda(E)$ is an Hermitian linear $\mathbb{R}$-bundle with inner product defined fiberwise via universal factorization property by:

$$\langle v_1 \wedge \cdots \wedge v_k \mid w_1 \wedge \cdots \wedge w_j \rangle_{\Lambda(E)} := \delta_{kj} \cdot \det[\langle v_\alpha \mid w_\beta \rangle_E].$$

Linear Bundles Functorialities: Complexification

We have a fibered version of the usual complexification functor $\mathscr{L}_X^{\mathbb{R}} \to \mathscr{L}_X^{\mathbb{C}}$, $E \mapsto E^{\mathbb{C}}$ fiberwise defined as $E_x^{\mathbb{C}} \simeq E_x \otimes_{\mathbb{R}} \mathbb{C}$, that also restricts to a functor $\mathscr{H}_X^{\mathbb{R}} \to \mathscr{H}_X^{\mathbb{C}}$.

Hence for Hermitian $\mathbb{R}$-bundles $(E, \langle \cdot \mid \cdot \rangle_E)$, we obtain:

- a complexified Hermitian Grassmann bundle $\Lambda^{\mathbb{C}}(E)$,
- a complexified Clifford algebra bundle $\mathbb{C}\mathrm{l}(E)$,
- a complexified Clifford action $c_E^{\mathbb{C}} : \mathbb{C}\mathrm{l}(E) \to \mathrm{End}_X(\Lambda^{\mathbb{C}}(E))$, such that $c_E^{\mathbb{C}}(v)$ is Hermitian, for all $v \in E$.

Linear Bundles Functorialities: Duals

There is a contravariant endofunctor $\mathscr{L}_X^{\mathbb{K}} \xrightarrow{\widehat{\ }} \mathscr{L}_X^{\mathbb{K}}$ that to every bundle with p-fibers E_p associates a bundle with p-fibers the "dual spaces" $\hat{E}_p := \mathcal{L}(E_p; \mathbb{K})$ of E_p; and that to every morphism of bundles given on p-fibers by $\phi_p : E_p \to F_p$ associates the bundle of "transposed maps" $\hat{\phi}_p : \hat{F}_p \to \hat{E}_p$ defined as $\hat{\phi}_p(\omega) := \omega \circ \phi_p$.

- We have a natural isomorphism $\widehat{\Lambda(E)} \simeq \Lambda(\widehat{E})$;
- a natural embedding $E \xrightarrow{\theta} \hat{\widehat{E}}$, given as $(\theta_p(e))(\omega) := \omega(e)$, for $e \in E_p$, $\omega \in \widehat{E}_p$;
- a Riesz natural transformation $E \xrightarrow{\gamma} \widehat{E}$ for Hermitian bundles: $(\gamma_p(e))(x) := \langle e \mid x \rangle_p$, for $e, x \in E_p$; this is a natural isomorphism when the fibers are Hilbert spaces.

Section Functor

- There is a covariant functor Γ that to $\mathbb{K}$-linear bundles (E, π, X) associates their modules of sections $\Gamma(E) := \{\sigma : X \to E \mid \pi \circ \sigma = \mathrm{Id}_X\}$ over the algebra $\mathbb{K}^X$.
- There is a (complexified) Grassmann endofunctor of $\mathbb{K}^X$-modules $\mathcal{M} \mapsto \Lambda^{\mathbb{C}}(\mathcal{M})$.
- There is a (complexified) Clifford endofunctor of (pseudo)-Hermitian $\mathbb{K}^X$-modules $(\mathcal{M}, g) \mapsto \mathbb{C}\mathrm{l}(\mathcal{M})$ and a Clifford action $c_{\mathcal{M}} : \mathbb{C}\mathrm{l}(\mathcal{M}) \otimes_{\mathbb{K}^X} \Lambda^{\mathbb{C}}(\mathcal{M}) \to \Lambda^{\mathbb{C}}(\mathcal{M})$.

We have natural isomorphisms: $\Gamma(\Lambda^{\mathbb{C}}(E)) \simeq \Lambda^{\mathbb{C}}(\Gamma(E))$, $\Gamma(\mathbb{C}\mathrm{l}(E)) \simeq \mathbb{C}\mathrm{l}(\Gamma(E))$ intertwining the Clifford actions c_E and $c_{\Gamma(E)}$.

Modules of Sections - Takahashi Equivalence

There is an equivalence between the monoidal category of $\mathscr{H}_X$ Hilbert bundles over a compact space X and the monoidal category $\mathscr{M}_{C(X)}$ of Hilbert C*-modules over the C*-algebra $C(X)$:

- to every Hilbert bundle (E, π, X) we associate its $C(X)$ Hilbert C*-module $\Gamma(E)$ of continuous sections (with poitwise $C(X)$-valued inner product and pointwise action of $C(X)$),
- to a bundle morphism $\phi : E \to F$, we associate the $C(X)$-linear map $\Gamma_\phi : \Gamma(E) \to \Gamma(F)$, $\Gamma_\phi(\sigma) := \phi \circ \sigma$,
- fiberwise tensor products of Hilbert bundles $E \otimes_X F$ transform into Rieffel tensor products of Hilbert $C(X)$-modules $\Gamma(E) \otimes_{C(X)} \Gamma(F)$.

This a "Hilbert bundle version" of the usual Serre-Swan equivalence beween vector bundles and finite projective modules.

Immediate Consequences of Bundle Functorialities 1

From the previous linear bundles functorialities:

- as long as a "tangent linear bundle" $T(M)$ is available for a diffeology M ... we can construct the $C(M)$ module $\Omega_\bullet(M)$,
- as long as $T(M)$ is (pseudo) Hermitian, we can construct a complex Clifford action $c_{\Omega_\bullet(M)}$ of $\mathbb{C}\mathrm{l}(M)$ on $\Omega_\bullet(M)$.

Care should be taken to assure:

- that the fibers $T_p(M)$ are Hilbertian ... if we need to preserve Riesz isomorphism, and to remain within Takahashi duality,
- some "uniformity" on the topology of $T(M)$ to allow the existence of completions of $\Gamma(T(M))$,
- some "saturation" condition for sections of $T(M)$ in view of future "reconstruction theorems" from modules of sections.

Immediate Consequences of Bundle Functorialities 2

An alternative construction can be performed using directly modules of forms:[7]

- as long as a $C(M)$-module $\Omega_1^{\mathbb{R}}(M)$ of 1-forms is available, we can still construct its exterior $C(M)$-module $\Lambda(\Omega_1^{\mathbb{R}}(M)) \simeq \Omega_\bullet^{\mathbb{R}}(M)$ of forms and its complexification $\Omega_\bullet(M)$;
- as long as $\Omega_1^{\mathbb{R}}(M)$ is (pseudo)-Hilbertian, we can still construct its Clifford $C(M)$-module $\mathrm{Cl}(\Omega_1^{\mathbb{R}}(M))$ and its complexified $\mathbb{C}\mathrm{l}(\Omega_1^{\mathbb{R}}(M)) \simeq \Gamma(\mathbb{C}\mathrm{l}(T^*(M))) =: \mathbb{C}\mathrm{l}(M)$ and we still have a Clifford action $c_{\Omega_\bullet(M)} : \mathbb{C}\mathrm{l}(M) \otimes_{C(X)} \Omega_\bullet(M) \to \Omega_\bullet(M)$.

The link between the two constructions is provided by Riesz morphisms $\gamma : T(M) \to T^*(M)$ and $\Gamma_\gamma : \Gamma(T(M)) \to \Omega_1^{\mathbb{R}}(M)$.

[7] Iglesias-Zemmour P **Diffeology** American Mathematical Society (2013).

The Hector Tangent Space / Tangent Functor

- Given a point $p \in M$ in a diffeological space $(M, \mathcal{D})$, consider the category $\mathcal{D}_p$ of smooth maps $f : U_\phi \to V_\psi$ between domains of plots ϕ, ψ covering p, such that $\phi = \psi \circ f$.
- The covariant tangent functor $\mathcal{T}$ produces a category $\mathcal{T}(\mathcal{D}_p)$ of $\mathbb{R}$-linear maps $T_u(U_\phi) \xrightarrow{Df_u} T_{f(u)}(V_\psi)$, for all $f \in \mathcal{D}_p$ and for all $u \in \mathrm{Dom}(f)$ such that $f(u) = p$
- Since the category of $\mathbb{R}$-linear maps admits arbitrary (co)-limits, we define the **Hector tangent space** in $p \in M$ as
$$T_p(M) := \varinjlim \mathcal{T}(\mathcal{D}_p),$$
the colimit of the diagram of linear maps in $\mathcal{T}(\mathcal{D}_p)$.
- We have a covariant **Hector tangent bundle functor** $\mathcal{T} : \mathscr{D} \to \mathscr{L}^{\mathbb{R}}$ from diffeologies to $\mathbb{R}$-linear bundles.

Problems 1

To discuss smooth fields we need to define a diffeology on $T(M)$.

- The natural Hector diffeology on $T(M)$ does not make it a diffeological linear bundle!
  The problem has been corrected by Christensen and Wu.[8]
- The fibers $T_p(M)$ are diffeological vector spaces, but in general $T_p(M)$ can be infinite dimensional.
- If $T_p(M)$ is finite dimensional for all $p \in M$, the dimensions of the fibers are generally different and the bundle is not usually locally trivial. We are out of the category of vector bundles.

[8] Christensen JD, Wu E (2014) `arXiv:1411.5425`

Problems 2

- Even if $T_p(M)$ is finite dimensional (and hence with its $\mathcal{D}$-topology is isomorphic as TVS to a certain $\mathbb{R}^n$), in principle the smooth dual, might have smaller dimension and hence we might lose finite dimensional "smooth" reflexivity and (in the Hermitian case) "smooth" Riesz isomorphisms.
- If a finite dimensional $T_p(M)$ is equipped with an inner product that is smooth, than its diffeology is necessarily the usual one. Hence it might be impossible to define smooth inner products on $T(M)$ (unless their diffeology is standard).

  This problem disappears at least for *filtered diffeologies*, since in this case Christensen and Wu proved that $T_p(M)$ has the usual diffeology.[9]

[9] Christensen JD, Wu E (2015) `arXiv:1510.09182`

Topological / Diffeological Vector Spaces

A **topological vector space** is a vector space $(V, +, \cdot)$ over $\mathbb{K} := \mathbb{R} / \mathbb{C}$ such that:

- V is equipped with a topology $\mathcal{T}_V$ making
- the maps $+ : V \times V \to V$ and $\cdot : \mathbb{K} \times V \to V$ continuous.

A **diffeological vector space** is a vector space $(V, +, \cdot)$ over $\mathbb{K} := \mathbb{R} / \mathbb{C}$ such that:

- V is equipped with a diffeology $\mathcal{D}_V$ making
- the maps $+ : V \times V \to V$ and $\cdot : \mathbb{K} \times V \to V$ smooth.

A diffeological vector space is a TVS for the $\mathcal{D}_V$-topology.

The notions of **diffeological algebra** and **diffeological module** are defined similarly via smoothness of the product / action.

Topological Linear Bundles ⋆

A **topological bundle** is bundle (E, π, X) with π continuous.

A **linear topological bundle** is a linear bundle that is topological such that $+ : E \times_X E \to E$, $\quad \cdot : \mathbb{K} \times E \to E$ and $p \mapsto 0_{E_p}$ are continuous maps and this "bundle uniformity condition" holds:

there is a base of neighborhoods of the zero section $\{W_\theta\}_{\theta \in \Theta}$ such that, for every point $e_o \in E$, a base of neighborhoods of e_o in the topology of E is given by the family of "tubular sets" of the form:

$$W_\theta \cup \pi^{-1}(O) + \sigma := \{e' \in E \mid e' - \sigma(\pi(e')) \in W_\theta, \ \pi(e') \in O\}$$

with $\sigma \in \Gamma^0(E)$ any continuous section of E with $\sigma(\pi(e')) = e'$, W_θ any of the given base neighborhoods of the zero section of E, and $O \subset X$ an open set in X such that $\pi(e_o) \in O$.

Diffeological Bundles $\star$

A **diffeological bundle** is a bundle (E, π, X) where π is a smooth map between diffeologies $(E, \mathcal{D}_E)$ and $(X, \mathcal{D}_X)$.[10]

A **linear diffeological bundle**[11] is a linear bundle, that is a diffeological bundle and furthermore such that:
$+ : E \times_X E \to E, \quad \cdot : \mathbb{K} \times E \to E, \quad p \mapsto 0_{E_p}$ are smooth,
and the following "smooth bundle uniformity" condition holds:

(E, π, X) is a linear topological bundle admitting $\{W_\theta \cup \pi^{-1}(O) + \sigma \mid \sigma \in \Gamma^\infty(E),\ \sigma(\pi(e_o)) = e_o\}$ as "smooth" subfamily of tubular neighborhood of $e_0 \in E$.

[10] P.Iglesias Zemmour and D.Christensen-E.Wu further require that pull-backs $\phi^\bullet(E)$ over plots $\phi \in \mathcal{D}_X$ are locally trivial.
In this case the Hector tangent bundle is not always a diffeological bundle!

[11] With spectral theory of diffeological modules in view, this notion is more restrictive compared to Pervova's definition of *diffeological pseudo-bundle*.

Riemannian Diffeologies / Diffeological Connections ⋆

A **Riemannian diffeological space** is a diffeological space $(M, \mathcal{D})$ whose Hector-Christensen-Wu tangent diffeological bundle $T(M)$ is a Hilbert bundle with smooth metric

$$\Gamma(T(M)) \otimes_{C(M)} \Gamma(T(M)) \xrightarrow{g_M} C(M).$$

A **diffeological Koszul connection** on a diffeological bundle (E, π, M) is a smooth $C^\infty(M)$-linear map of diffeological modules: ${}_{C^\infty(M)}\Gamma^\infty(T(M)) \xrightarrow{\nabla^E} {}_{C^\infty(M)}\mathrm{End}_{\mathbb{R}}(\Gamma^\infty(E))$, $\xi \mapsto \nabla^E_\xi$ such that:[12]

$$\nabla^E_\xi(\sigma_1 + \sigma_2) = \nabla^E_\xi(\sigma_1) + \nabla^E_\xi(\sigma_2),$$
$$\nabla^E_\xi(f \cdot \sigma) = Df(\xi) \cdot \sigma + f \cdot \nabla^E(\sigma), \quad \nabla^E_{f \cdot \xi}(\sigma) = f \cdot \nabla^E_\xi(\sigma).$$

[12] In the Hilbertian bundle case, by Riesz duality, this is equivalent to: $\nabla^E : \Gamma^\infty(E) \to \Omega_1(M) \otimes_{C^\infty(M)} \Gamma^\infty(E)$ with the usual "derivation properties".

Problems ⋆

- The condition on the existence of a Riemannian metric on a diffeological linear bundle can be very restrictive (if the fiber are finite dimensional it implies the standard smooth structure on the fibers).
  As said, at least for the Cartesian sub-category of *filtered diffeologies* the condition is not restrictive.
- Similarly, since diffeological linear spaces are not always locally trivial and finite, the existence of a Koszul connection is not guaranteed, by the usual results.
  Note that the Cuntz-Quillen theorem on the existence of *universal* connections on modules if and only if the bimodules are finite projective, does not exclude the existence of connections that are not via the universal differential calculus of the algebra $C(M)$.

Pervova's Multilinear Diffeological Pseudo-bundles

In the past few months a remarkable serie of preprints by E.Pervova[13] has specifically addressed the study of:

- diffeological multilinear algebra,
- diffeological (pseudo)-bundles,
- pseudo Riemannian metrics on diffeological pseudo-bundles,
- diffeological Clifford algebras and Clifford modules.

The theory is generally developed for "diffeological pseudo-bundles" (defined as: vector space objects in the category of "parametrized" diffeologies).

[13] Pervova E arXiv:1504.08186, arXiv:1505.06894v3, arXiv:1507.03787, arXiv:1509.03023, arXiv:1601.00170.

Bundle Functorialities for Diffeological Tangent Bundles

When the Hector-Christensen-Wu tangent bundle $T(M)$ is a *Hilbert bundle*, we can define its complexified Grassmann Hilbert bundle $\Lambda^{\mathbb{C}}_{\bullet}(T(M))$, the complexified Clifford bundle $\mathbb{C}\mathrm{l}(T^*(M))$ and a Clifford action $c_{\Lambda^{\mathbb{C}}_{\bullet}T(M)}$ of $\mathbb{C}\mathrm{l}(T^*(M))$ on $\Lambda^{\mathbb{C}}_{\bullet}(T(M))$.

Passing to the diffeological modules of sections we get:

- the **Grassmann algebra of** M: the $C(M)$ bimodule $\Omega_{\bullet}(M) := \Gamma(\Lambda^{\mathbb{C}}_{\bullet}(T(M)))$ with pointwise multiplication,
- the **Clifford algebra of** M: the $C(M)$ bimodule $\mathbb{C}\mathrm{l}(M) := \Gamma(\mathbb{C}\mathrm{l}(T^*(M)))$ of continuous Clifford forms,
- a **Riesz isomorphism** $\Gamma_{\gamma_{T(M)}} : \Gamma(T(M)) \to \Gamma(T^*(M))$.
- a **Clifford action** $c_{\Omega_{\bullet}(M)} := \mathbb{C}\mathrm{l}(M) \times \Omega_{\bullet}(M) \to \Omega_{\bullet}(M)$.

Naive Hodge - De Rham Module Triples ⋆

To a *compact* diffeological space whose Hector-Christensen-Wu tangent bundle is equipped with a *Riemannian metric* and a *connection*, we associate a naive module triple $(\mathcal{A}, \mathcal{M}, D_{\mathcal{M}})$ where:

- $\mathcal{A} := C(M)$ is the C*-algebra of continuous $\mathbb{C}$-valued maps,
- $\mathcal{M} := \Gamma(\Omega_\bullet(M))$ is the C* Hilbert $\mathcal{A}$-module of continous sections of the complex Grassmann Hilbert bundle of M,
- $D_{\mathcal{M}} := (-i) \cdot c_{\Omega_\bullet(M)} \circ \nabla^{\Omega^\infty_\bullet(M)}$.

Problem: we are not aware of a general way to define integration of forms on a diffeological space and hence a "volume state" on $C(M)$. Plots have different dimensions and they are not always locally diffeomorphically related by change of coordinates.

Hence it is not usually possible to get just a naive spectral triple.

Integration on Diffeologies ⋆⋆

- A De Rham cohomology for diffeologies has been early developed by Iglesias-Zemmour.
  Forms are locally integrated via pull-backs on the plots.
- There are some very specific diffeologies, a subclass of Christensen-Wu filtered diffeologies, (that are still more general than ordinary compact orientable manifolds), whose Hector tangent bundle is actually an orientable vector bundle,[14] where integration can be performed.
- As a consequence, there *might be* a family of compact *Hausdorff* oriented Riemannian diffeological spaces that can be immediately associated to commutative spectral triples.

[14] The irrational torus has a Hector tangent bundle of rank 1 that is a trivial vector bundle ... unfortunately its algebra is degenerate!

- Outlook - Categorical Approach

General Plan

- For every oriented manifold "inside" a diffeology, we can construct a naive spectral triple:
  - we can study categories of (non-unital) spectral triples defined on domains of local plots of the diffeology,
  - or we can study the category of spectral triples defined on global compact oriented submanifolds of the diffeology.
- When an appropriate notion of "limit" exists, we can "assemble" the data into a unique naive spectral triple.

Obstacles

In order to pursue these lines, we need to:

- carefully study how metrics and connections behave via pull-backs on the category of plots (starting with manifolds),
- define a notion of **orientation** for diffeological spaces that induces "compatible" orientations on every plot,
- describe **morphisms** of spectral triples between diffeological submanifolds of different dimension that are smoothly related.

Riemannian Pseudo-metrics

- Imposing a metric on $T(M)$, by pull-back, imposes only **pseudo-metrics** on each $T(U_\phi)$ for every plot ϕ.
- By commutativity of all the diagrams $D\phi = D\psi \circ Df$ for every smooth commutative diagram $\phi = \psi \circ f$ involving plots ϕ, ψ, we see that $(\ker D\phi) = (Df)^{-1}(\ker D\psi)$; $(\ker Df) \subset (\ker D\phi)$ and the Df are a category of partial isometries.[15]
- It is hence possible (as does Pervova) to generalize imposing on $T(M)$ only a smooth pseudo-metric.
  The co-limit of isometries of pseudo-metric spaces exists.
- $T^*(M)$ and hence $\Lambda^{\mathbb{C}}_\bullet(M)$ are still *pseudo* Hermitian and so $\Omega_\bullet(M)$ is still a *pseudo* Hilbert C*-module over $C(M)$.
- $\Omega_\bullet(M)$ is still a Clifford module over the algebra $\mathbb{C}\mathrm{l}(M)$, although now γ_M is only a homomorphism.

[15] In general composition of partial isometries is not a partial isometry!

Problems ⋆

- E.Pervova[16] gave examples of diffeological bundles that cannot admit even a smooth pseudo-metric (contrary to smooth vector bundles that always admit smooth metrics).
- Since spectral geometry, as A.Connes elaborated it via spectral triples, is based on metric information encoded into the Dirac operator, it seems that some diffeologies, in the commutative geometry context, might elude a description via "usual" spectral triples.[17]

[16] Pervova E (2016) Diffeological Vector Pseudo-bundles *Topology and Its Applications* 202:269-300.

[17] Unless the diffeology of $T(M)$ is modified in such a way that the pseudo metric become smooth.

Connections on Diffeologies

- If we assume that $T(M)$ is equipped with a connection ∇, every plot ϕ determines a pull-back connection matrix:

$$\begin{bmatrix} (\nabla_\xi Z^{\|})^{\|} & (\nabla_\xi Z^{\perp})^{\|} \\ (\nabla_\xi Z^{\|})^{\perp} & (\nabla_\xi Z^{\perp})^{\perp} \end{bmatrix} \qquad Z \in \Gamma^\infty(T(M)), \ \xi \in T(M)|_{\phi(U_\phi)},$$

  where the upper-left entry is an induced connection on $T(U_\phi)$ and the off-diagonal "extrinsinc Gauss" terms disappear for totally geodesic plots.
- For a differentiable map f of plots $\phi = \psi \circ f$, since in the metric case, f are isometries on any complement of the kernel of the plot, such matrix decompositions are "stable" under f. Hence it is possible to have co-limits as connections on $T(M)$.

Orientations ⋆

- In some cases it seems possible to define an "orientation" for a diffeology by continouosly (smoothly) selecting an orientation of $T_p(M)$, for all $p \in M$.
- An alternative is to impose a (partial) orientation on each domain plot. These data will uniquely determine pairs of compatible "complement orientations" for every morphism (smooth span) of plots with different dimension.
- For pairs of plots with same dimension whose span is equidimensional, we will have obstuctions to globalization of orientation (as in the case of manifolds).
- Each orientable closed embedded submanifold will get a unique orientation and morphisms of plots will provide unique pairs of "complement orientations".

Totally Geodesic Morphisms of Naive Triples

Given a pair of spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, with $j = 1, 2$, a **totally geodesic morphism** is a pair (ϕ, Φ), where

- $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a (unital) $*$-homomorphism of C*-algebras,
- $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear map that "intertwines" the representations $\pi_1, \pi_2 \circ \phi$ and the Dirac operators D_1, D_2 i.e.:

$$\pi_2(\phi(x)) \circ \Phi = \Phi \circ \pi_1(x), \quad \forall x \in \mathcal{A}_1,$$
$$D_2 \circ \Phi(\xi) = \Phi \circ D_1(\xi), \quad \forall \xi \in \mathrm{Dom}\, D_1.$$

Riemannian Morphisms of Naive Spectral Triples

Given two spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j)$, with $j = 1, 2$, a **Riemannian morphism** is a pair (ϕ, Φ), where

- $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a (unital) $*$-homomorphism of C*-algebras,
- $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear map that "intertwines" the representations $\pi_1, \pi_2 \circ \phi$ and the "Dirac commutators":

$$\pi_2(\phi(x)) \circ \Phi = \Phi \circ \pi_1(x), \quad \forall x \in \mathcal{A}_1,$$
$$[D_2, \pi_2(\phi(x))] \circ \Phi = \Phi \circ [D_1, \pi_1(x)], \quad \forall x \in \mathcal{A}_1.$$

The adjoint action of Φ on the algebra $\Omega_D(\mathcal{A})$ generated by $\pi(\mathcal{A})$ and the commutators $[D, \pi(x)]$, $x \in \mathcal{A}$ (the "non-commutative Clifford algebra") is a $*$-homomorphism extending ϕ.

Morita-Connes Transport of Naive Spectral Triples

A **Morita-Connes transport bimodule** between the spectral triples $(\mathcal{A}_j, H_j, D_j)$, $j = 1, 2$ is given by

- a left-$\mathcal{A}_2$ right-$\mathcal{A}_1$ bimodule X that is Hilbert C* over $\mathcal{A}_1$,
- a Hermitian connection[18] $\nabla : X \to X \otimes_{\mathcal{A}_1} \Omega^1_{D_1}(\mathcal{A}_1)$ on the bimodule X such that the Connes' "transfer" formula holds

$$H_2 = X \otimes_{\mathcal{A}_1} H_1 \quad \text{and for} \quad h \in \mathcal{H}_1,\ \xi \in X,$$
$$D_2(\xi \otimes h) = (\mathrm{Id}_X \otimes_\nabla D_1)(\xi \otimes h) := \xi \otimes D_1(h) + (\nabla\xi)(h).$$

Composition consists of the bimodule $X^3 := X^2 \otimes_{\mathcal{A}_2} X^1$ equipped with the connection given, for $\xi_1 \in X^1$, $\xi_2 \in X^2$, $h \in \mathcal{H}_1$, by:

$$(\mathrm{Id}_{X^2} \otimes_{\nabla^2} \nabla^1)(\xi_2 \otimes \xi_1)(h) := \xi_2 \otimes (\nabla^1\xi_1)(h) + (\nabla^2\xi_2)(\xi_1 \otimes h).$$

[18] Here $\Omega^1_D(\mathcal{A})$ denotes the $\mathcal{A}$-bimodule inside the algebra $\Omega_D(\mathcal{A})$ spanned by the commutators $[D, \pi(x)]$, $x \in \mathcal{A}$.

Mesland KK-morphisms of Naive Spectral Triples

A **Mesland transport bimodule**[19] ${}_{\mathcal{A}_2}X_{\mathcal{A}_1}$, with $\Omega_{D_1}(\mathcal{A}_1)$-connection ∇_X, is further equipped with its own regular operator D_X.[20] Connes' transfer formula becomes:

$$D_2(\xi \otimes h) = (D_X \otimes \mathrm{Id} + \mathrm{Id} \otimes_\nabla D_1)(\xi \otimes h), \quad \xi \in X, \ h \in H_1,$$

and the composition of morphisms $(X^2, \nabla^2, D_2) \circ (X^1, \nabla^1, D_1)$ is:

$$(X^2 \otimes_{\mathcal{A}_2} X^1, \mathrm{Id}_{X^2} \otimes_{\nabla^2} \nabla^1, D_{X^2} \otimes \mathrm{Id}_{X^1} + \mathrm{Id}_{X^2} \otimes_{\nabla^2} D_{X^1}).$$

[19] B.Mesland (2012) *Clay Mathematics Proceedings* 16:197-212
B.Mesland (2014) *J Reine Angew Math* 69:101-172

[20] Further conditions of "smoothness" need to be imposed on the bimodules, operators and connections and the Haagerup tensor product has to be used.

Problems ⋆

All the previous definitions of morphisms/transports between naive spectral triples seem restrictive:

- Mesland transport morphisms still describe "metrically rigid" situations: intuitively this is because in general the pull-back of a connection is not in block diagonal form, because the off-diagonal Gauss extrinsic terms do not vanish.
- The most naive suggestion is to couple a Mesland transport bimodule X with a Riemannian morphism $\Phi : H_2 \to X \otimes_{\mathcal{A}_1} H_1$ of spectral triples (intertwining the Clifford algebras and not the Dirac operators). Other notions of morphism might exist.
- The existence of injective limits for categories of such morphisms of (non-unital) naive spectral triples associated to each diffeological plot, must be investigated.

Non-metric Spectral Geometry?

- Non-commutative geometry (via spectral triples) always describes "metric" geometries (Connes' distance).
- Since any paracompact manifold can be equipped with a Riemannian metric, one can always introduce "metrics" as gauge data and discuss smoothness via the Dirac operator.
- Diffeological bundles do not always admit Riemannian metrics. In this respect, commutative spectral triples appear to have a more limited scope, compared to diffeologies.

In principle one can always try to define abstract "spectral data" $(\mathcal{A}, H, S)$ starting from Hilbert spaces H, that are modules over the Grassmann algebra $\Omega_\bullet(M)$ of M, equipped with a connection $\nabla^H : H^\infty \to H^\infty \otimes_{C^\infty(M)} \Omega_1(M)$ and obtain S contracting the connection via the Grassmann action $H \otimes_{C(M)} \Omega_1(M) \to H$. But metric information is still required to obtain a Hilbert space H.

Motivations from Mathematical Physics 1

- It is an open problem to understand how geometrical (space-time) structures might be extracted from observables and their (state-induced) correlations in covariant quantum physics.[21]
- Our study of naive categories of spectral triples, was initially motivated by this problem of reconstruction "a-posteriori" of non-commutative spaces from categories of observables and states in quantum physics.

[21] B P, Conti R, Lewkeeratiyutkul W (2010) **Modular Theory, Non-commutative Geometry and Quantum Gravity** arXiv:1007.4094v2

Motivations from Mathematical Physics 2

- Geometries (generalized spaces) are mostly discussed in the framework of (higher) topoi of (stacks) sheaves on sites.
  - Either a "space" is defined via the topos of sheaves living on it,
  - or (as in diffeologies) a "space" is "constructed" from a sheaf over a specific site (differentiable maps).

  A spectral geometrical description of diffeologies is a very useful exercise in such wider context.
- A more "relational" approach to the definition of "geometries" (without "probes" and "targets") might be appropriate; hence shifts from usual formulations beyond categories and sheaves are not excluded: see for example the recent proposals by Flori-Fritz.[22]

[22] Flori C, Fritz T (2013) **Compositories and Gleaves** arXiv:1308.6548

This research is (small) part of the wider project

Categorical Non-commutative Geometry and Modular Algebraic Quantum Gravity

More specific results already obtained:

- Gel'fand-Naĭmark duality for commutative full C*-categories
  Fell bundles enriched in monoidal $*$-categories
  (with R.Conti - W.Lewkeeratiyutkul)
- Kreĭn C*-modules and Kreĭn C*-categories (with K.Rutamorn)
- definition of strict higher globular C*-categories
  non-commutative exchange property for higher categories
  examples: hypermatrices as convolution hyper-C*-algebras
  applications to relational quantum mechanics
  (with R.Conti - W.Lewkeeratiyutkul - N.Suthichitranont)
- involutive double categories and involutive version of
  Brown-Mosa-Spencer theorem (with R.Conti - R.Martins)

Current ongoing investigations:

- non-commutative Gel'fand-Naĭmark duality (with R.Conti - N. Pitiwan)
- involutive weak higher globular (C*)-categories (with P.Bejrakarboom)
- horizontal categorification of spectral geometries (with R.Conti - C.Putthirungroj)
- reconstruction of non-commutative space-time from states via categorical Tomita-Takesaki theory (with R.Conti - M.Raasakka)
- diffeologies spectral triples and categorical definitions of generalized spaces (with R.Conti - F.Germ)

References

Diffeologies - Basic References

- Souriau J-M (1980) **Groupes Differentielles** *Lecture Notes in Mathematics* 836:91-128
- Hector G (1995) **Géometrie et Topologie des Espaces Difféologiques** *Analysis and Geometry in Foliated Manifolds* 55-80 World Scientific
- Iglesias-Zemmour P (2013) **Diffeology** American Mathematical Society

Smooth Spaces

- Baez J, Hoffnung J (2008) **Convenient Categories of Smooth Spaces** `arXiv:0807.1704`
- Stacey A (2008) **Comparative Smootheology** `arXiv:0802.2225`

Diffeologies - Recent Developments

- Christensen JD, Sinnamon G, Wu E (2013) **The D-topology for Diffeological Spaces** `arXiv:1302.2935`
- Christensen JD, Wu E (2014) **Tangent Spaces and Tangent Bundles for Diffeological Spaces** `arXiv:1411.5425`
- Christensen JD, Wu E (2015) **Tangent Spaces of Bundles and of Filtered Diffeological Spaces** `arXiv:1510.09182`
- Pervova E (2015) **Multilinear Algebra in the Context of Diffeology** `arXiv:1504.08186`
- Pervova E (2015) **Diffeological Clifford Algebras and Pseudo-bundles of Clifford Modules** `arXiv:1505.06894v3`
- Pervova E (2015) **On the Notion of Scalar Product for Finite-dimensional Diffeological Vector Spaces** `arXiv:1507.03787`
- Pervova E (2016) **Diffeological Vector Pseudo-bundles** *Topology and Its Applications* 202:269-300 `arXiv:1509.03023`
- Pervova E (2016) **Diffeological Vector Pseudo-bundles and Diffeological Pseudo-metrics on Them as Substitutes for Riemannian Metrics** `arXiv:1601.00170`

Categorical Non-commutative Geometry and Categories of Spectral Triples

- Mesland B (2012) **Spectral Triples and KK-theory: a Survey of Topics in Noncommutative Geometry** *Clay Mathematics Proceedings* 16:197-212 `arXiv:1304.3802`
- Mesland B (2014) **Unbounded Bivariant K-theory and Correspondences in Noncommutative Geometry** *J Reine Angew Math* 69:101-172 `arXiv:0904.4383`
- Bertozzini P, Conti R, Lewkeetatiyutkul W (2006) **A Category of Spectral Triples and Discrete Groups with Length Function** *Osaka J Math* 43(2):327-350 `arXiv:math.OA/0502583`
- Bertozzini P, Jaffrennou F (2013) **Remarks on Morphisms of Spectral Geometries** *East West Journal of Mathematics* 15(1):15-24 `arXiv:1409.1342`
- Bertozzini P, Conti R, Lewkeeratiyutkul W (2012) **Categorical Non-commutative Geometry** *Journal of Physics: Conference Series* 346:012003 `arXiv:1409.1337`

Thank You for Your Kind Attention!

We acknowledge the partial support from

- IMPAN Simons Semester
  "Noncommutative Geometry: The Next Generation",
- Thammasat University
  Department of Mathematics and Statistics,
- Thammasat University Research Grant n. 2/15/2556:
  "Categorical Non-commutative Geometry".

This file has been realized using the "beamer" LaTeX package of the TeX-live distribution and Winefish editor on Ubuntu Linux.