Topology and K-theory of noncommutative weighted projective spaces

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Reference:

- TB & W Szymański, The C^*-algebras of quantum lens and weighted projective spaces, arXiv:1603:04678

Main objectives

- To identify the algebras of continuous functions on quantum weighted projective and lens spaces as graph C^*-algebras.
- To use the above identification to compute the K-theory of these spaces in cases that have not been computed yet.

Graph C^*-algebras and their K-theory

- A directed graph $G = (G^0, G^1, \varrho, \sigma)$ consists of two sets G^0 and G^1 and two mappings $\varrho, \sigma : G^1 \to G^0$.
- $C^*(G)$ is the universal C^*-algebra generated by
  - mutually orthogonal projections P_v, $v \in G^0$,
  - partial isometries S_e, $e \in G^1$,

  subject to the following relations, for all $e \neq f \in G^1$ and all $v \in G^0$ emitting a finite number of edges,

$$S_e^* S_f = 0, \qquad S_e^* S_e = P_{\varrho(e)}, \qquad S_e S_e^* \leq P_{\sigma(e)},$$

$$P_v = \sum_{e \in G^1 \,:\, \sigma(e) = v} S_e S_e^*.$$

Graph C^*-algebras and their K-theory

- Set:

$$V_G := \{v \in G^0 \mid v \text{ emits a finite non-zero } \# \text{ of edges}\}.$$

- Define:

$$\Phi : \mathbb{Z}V_G \longrightarrow \mathbb{Z}G^0, \quad v \longmapsto \sum_{e \in G^1,\, \sigma(e)=v} \varrho(e) - v.$$

- Then:

$$K^0(C^*(G)) = \operatorname{coker}\Phi, \qquad K^1(C^*(G)) = \ker \Phi.$$

The circle group actions on quantum spheres

- $C(S_q^{2n+1})$ is the universal C^*-algebra with generators $z_0, z_1, \ldots, z_n$, subject to the following relations:

$$z_i z_j = q z_j z_i \quad \text{for } i < j, \qquad z_i z_j^* = q z_j^* z_i \quad \text{for } i \neq j,$$

$$z_i z_i^* = z_i^* z_i + (q^{-2} - 1) \sum_{j=i+1}^{n} z_j z_j^*, \qquad \sum_{j=0}^{n} z_j z_j^* = 1,$$

  where $q \in (0, 1)$.
- Fix a sequence of positive integers $\mathrm{m} := m_0, \ldots, m_n$.
- $C(S_q^{2n+1})$ admits the circle group action ϱ_{m},

$$\varrho_{\mathrm{m}} : z_i \mapsto \xi^{m_i} z_i, \qquad i = 0, \ldots, n,$$

  where ξ is the unitary generator of $\mathbb{T}$ (of infinite order).

Quantum weighted projective and lens spaces

- Continuous functions on the q weighted projective space $C(\mathbb{WP}_q^n(\mathrm{m})) \equiv$ fixed points of the $\mathbb{T}$-action ϱ_{m}.
- Continuous functions on the quantum lens space $C(L_q^{2n+1}(N;\mathrm{m})) \equiv$ all the elements $\sum_i x_i$ of $C(S_q^{2n+1})$ that transform according to the rule

$$\sum_i x_i \mapsto \sum_i \xi^{r_i N} x_i, \qquad r_i \in \mathbb{Z},$$

- Equivalently, $C(L_q^{2n+1}(N;\mathrm{m})) \equiv$ fixed points of the $\mathbb{Z}_N$-action on $C(L_q^{2n+1}(N;\mathrm{m}))$,

$$\varrho_{\mathrm{m}}^N : z_i \mapsto \zeta^{m_i} z_i,$$

  where ζ is a generator of $\mathbb{Z}_N$.

Quantum weighted projective vs lens spaces

- The action ϱ_{m} gives rise to the $\mathbb{T}$-action $\hat{\varrho}_{\mathrm{m}}$ on $C(L_q^{2n+1}(N;\mathrm{m}))$ with fixed points being again $C(\mathbb{WP}_q^n(\mathrm{m}))$: an element $x \in C(L_q^{2n+1}(N;\mathrm{m}))$ transforms under $\hat{\varrho}_{\mathrm{m}}$ as $x \mapsto \xi^r x$ provided it transforms as $x \mapsto \xi^{rN} x$ under ϱ_{m}.

What has been known?

Special cases:

- [Hong-Szymański '03] $\gcd(m_i, N) = 1$:
  $C(L_q^{2n+1}(N; \mathrm{m}))$ are graph C^*-algebras,

  $$K_1(C(L_q^{2n+1}(N; \mathrm{m}))) = \mathbb{Z};$$

  Some examples of K_0 have been calculated.
- [Hong-Szymański '02] $m_0 = \ldots = m_n = 1$:
  $C(\mathbb{WP}_q^n(\mathrm{m}))$ are graph C^*-algebras,

  $$K_0(C(\mathbb{WP}_q^n(\mathrm{m}))) = \mathbb{Z}^{n+1}, \quad K_1(C(\mathbb{WP}_q^1(\mathrm{m}))) = 0.$$
- [Brzeziński-Fairfax '12] $\gcd(m_0, m_1) = 1$:

  $$C(\mathbb{WP}_q^1(\mathrm{m})) \cong \mathbb{C} \oplus \mathcal{K}^{m_1}, \quad \text{hence}$$

  $$K_0(C(\mathbb{WP}_q^1(\mathrm{m}))) = \mathbb{Z}^{m_1+1}, \quad K_1(C(\mathbb{WP}_q^1(\mathrm{m}))) = 0.$$

What has been known?

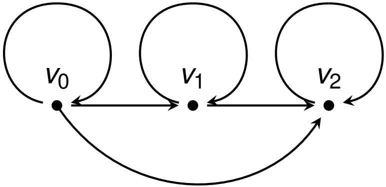
- [D'Andrea-Landi '15] $m_i = \prod_{j \neq i} p_j$, for some pairwise coprime $p_0, \ldots, p_n$:
  The generators of the coordinate algebra of $\mathbb{WP}_q^n(\mathrm{m})$ and irreducible representations are known, Fredholm modules have been constructed. Also:

$$\mathbb{Z}^{1+\sum_{k=1}^{n} p_0 \cdots p_{k-1}} \subseteq K_0(C(\mathbb{WP}_q^n(\mathrm{m}))).$$

- [Arici-Brain-Landi '15] $m_0 = \ldots = m_n = 1$:
  K_0-groups of $C(L_q^{2n+1}(N; \mathrm{m}))$ for $n = 1, 2, 3, 4$ have been calculated.

Quantum spheres as graph C^*-algebras

- Hong and Szymański have shown that $C(S_q^{2n+1})$ is the C^*-algebra associated to a graph L_{2n+1} defined as follows.
- L_{2n+1} has $n+1$ vertices $v_0, v_1, \ldots, v_n$, and $(n+1)(n+2)/2$ edges e_{ij}, $i = 0, \ldots, n$, $j = i, \ldots, n$, with v_i the source and v_j the range of e_{ij}.
- L_5:



Actions on graph C^*-algebras

- The actions of $\mathbb{Z}_N$ and $\mathbb{T}$ translate to the actions on the graph C^*-algebra $C^*(L_{2n+1})$:
- The (lens space yielding) $\mathbb{Z}_N$-action

$$\varrho^N_{\mathrm{m}} : S_{e_{ij}} \mapsto \zeta^{m_i} S_{e_{ij}}, \quad \varrho^N_{\mathrm{m}} : P_{v_i} \mapsto P_{v_i}.$$

- The (wieghted projective space yielding) $\mathbb{T}$-action

$$\varrho_{\mathrm{m}} : S_{e_{ij}} \mapsto \xi^{m_i} S_{e_{ij}}, \quad \varrho_{\mathrm{m}} : P_{v_i} \mapsto P_{v_i}.$$

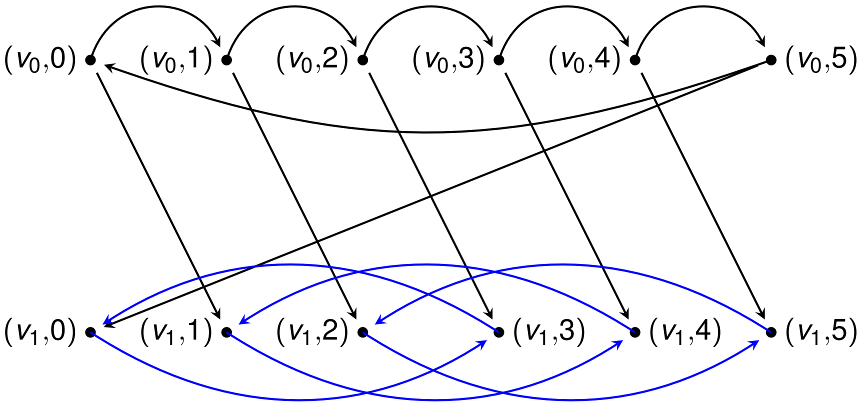
Quantum lens spaces as graph C^*-algebras

- Since $C(L_q^{2n+1}(N; \mathrm{m}))$ equals fixed points of a finite abelian group action on $C^*(L_{2n+1})$, a theorem by Crisp implies

$$C(L_q^{2n+1}(N; \mathrm{m})) \cong \left(\sum_{i=0}^{n} P_{(v_i,0)}\right) C^*(L_{2n+1} \times_c \mathbb{Z}_N) \left(\sum_{i=0}^{n} P_{(v_i,0)}\right).$$

- $L_{2n+1} \times_c \mathbb{Z}_N$ is the skew product graph with labelling c induced by the $\mathbb{Z}_N$-action:
  - **vertices**: (v_i, r), $i = 0, \ldots, n$, $r = 0, \ldots, N-1$;
  - **edges**: (e_{ij}, r), $i, j = 0, \ldots n$, $i \leq j$, $r = 0, \ldots N-1$, with $(v_i, r - m_i \mod N)$ being the source and (v_j, r) being the range of (e_{ij}, r).

Example: $n = 1$, $N = 6$, $m_0 = 1$, $m_1 = 3$



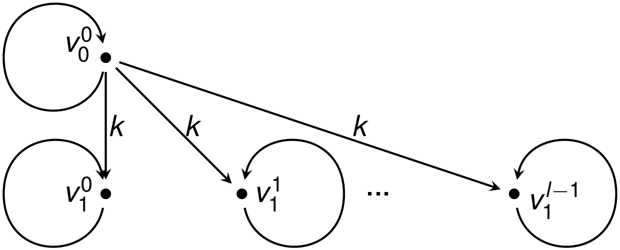
The graphs for quantum lens spaces

- A path in $L_{2n+1} \times_c \mathbb{Z}_N$ from (v_i, r) to (v_j, s) that does not pass through (v_k, t), with $k = i+1, \ldots, j-1$ and $t = 0, \ldots, \gcd(m_k, N) - 1$ is termed **admissible.** The number of such paths is denoted by n_{ij}^{rs}.
- Based on $L_{2n+1} \times_c \mathbb{Z}_N$ we construct a graph $L_{2n+1}^{N;\mathrm{m}}$:
  - **vertices**: v_i^r, $i = 0, \ldots, n$, $r = 0, \ldots, \gcd(N, m_i) - 1$;
  - **edges**: $e_{ij;a}^{rs}$ from v_i^r to v_j^s, with $a = 1, \ldots, n_{ij}^{rs}$.
- Since there are no edges (e_{ij}, r) in $L_{2n+1} \times_c \mathbb{Z}_N$ with $i > j$, one may assume that $i \leq j$ in $e_{ij;a}^{rs}$.
- In v_i^r, we refer to the index i as labelling the *levels*, and to index r as labelling the *loops* in $L_{2n+1}^{N;\mathrm{m}}$.

Observations on $L_{2n+1}^{N;\mathrm{m}}$

- There is a single loop at each vertex.
- Except for loops, edges always point to higher loop or higher level.
- The number of edges depends on the relative primness of the m_i and N.
- If all the m_i divide N, then the graph $L_{2n+1}^{N;\mathrm{m}}$ consists of $n+1$ levels of interconnected loops with m_i mutually disconnected loops at the i-th level.
- If all the m_i are coprime with N, the graph $L_{2n+1}^{N;\mathrm{m}}$ coincides with the graph described earlier by Hong and Szymański

Example: $L_3^{kl;1,l}$



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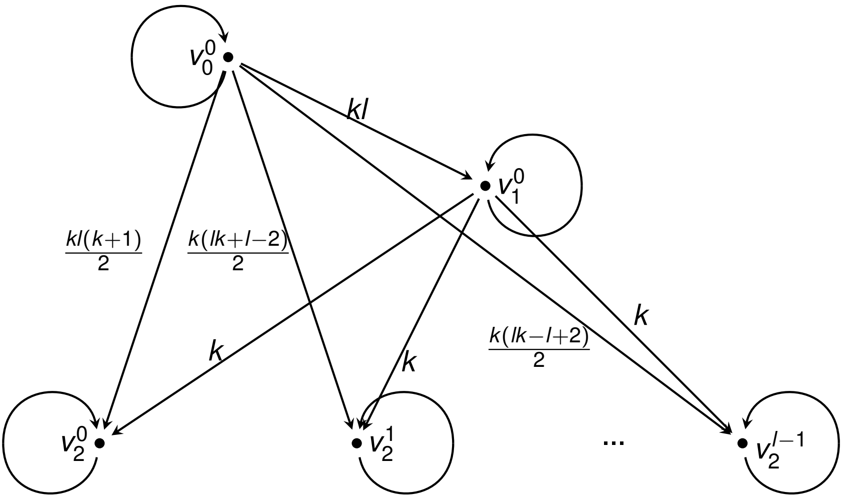
Example: $L_5^{kl;1,1,l}$



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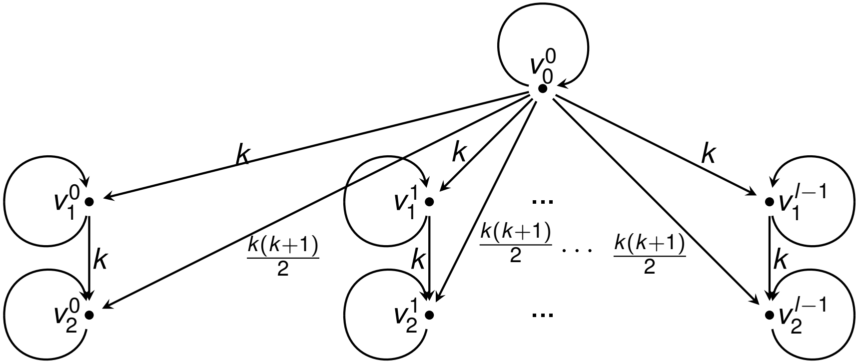

Quantum lens spaces as graph algebras

Theorem

$$C^*(L^{N;\underline{m}}_{2n+1}) \cong C(L_q^{2n+1}(N;\underline{m})).$$

Corollary

The following sequence of C^*-algebras

$$0 \longrightarrow (\mathcal{K}\otimes C(\mathbb{T}))^{\oplus c_n} \longrightarrow C(L_q^{2n+1}(N;\underline{m})) \longrightarrow C(L_q^{2n-1}(N;\underline{m})) \longrightarrow 0,$$

where $c_n = \gcd(m_n, N)$, is exact.

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Examples of K_0

$$K_0(C(L_q^3(kl;1,l))) = \mathbb{Z}^l \oplus \mathbb{Z}_k.$$

$$K_0(C(L_q^5(kl;1,1,l))) = \mathbb{Z}^l \oplus \begin{cases} \mathbb{Z}_k \oplus \mathbb{Z}_k & \text{if } k \text{ is odd or } l \text{ is even,} \\ \mathbb{Z}_{2k} \oplus \mathbb{Z}_{\frac{k}{2}} & \text{if } k \text{ is even and } l \text{ is odd.} \end{cases}$$

$$K_0(C(L_q^5(kl;1,l,l))) = \mathbb{Z}^l \oplus \begin{cases} \mathbb{Z}_k^{l+1} & \text{if } k \text{ is odd,} \\ \mathbb{Z}_{2k} \oplus \mathbb{Z}_{\frac{k}{2}} \oplus \mathbb{Z}_k^{l-1} & \text{if } k \text{ is even.} \end{cases}$$

Examples of K_0

$$K_0(C(L_q^7(kl;1,1,1,l))) = \mathbb{Z}^l \oplus \begin{cases} \mathbb{Z}_k \oplus \mathbb{Z}_k \oplus \mathbb{Z}_k & \text{if } k|\alpha \ \&\ k|\beta, \\ \mathbb{Z}_{\frac{k}{6}} \oplus \mathbb{Z}_k \oplus \mathbb{Z}_{6k} & \text{if } k|\alpha \ \&\ \beta \equiv \frac{k}{6}, \frac{5k}{6} \pmod{k}, \\ \mathbb{Z}_{\frac{k}{3}} \oplus \mathbb{Z}_k \oplus \mathbb{Z}_{3k} & \text{if } k|\alpha \ \&\ \beta \equiv \frac{k}{3}, \frac{2k}{3} \pmod{k}, \\ \mathbb{Z}_{\frac{k}{2}} \oplus \mathbb{Z}_k \oplus \mathbb{Z}_{2k} & \text{if } k|\alpha \ \&\ \beta \equiv \frac{k}{2} \pmod{k}, \\ \mathbb{Z}_{\frac{k}{2}} \oplus \mathbb{Z}_{\frac{k}{2}} \oplus \mathbb{Z}_{4k} & \text{if } k \not| \alpha \ \&\ \frac{k}{2}|\beta, \\ \mathbb{Z}_{\frac{k}{6}} \oplus \mathbb{Z}_{\frac{k}{2}} \oplus \mathbb{Z}_{12k} & \text{if } k \not| \alpha \ \&\ \frac{k}{2} \not| \beta, \end{cases}$$

where

$$\alpha := n_{13}^{00} = \frac{kl(k+1)}{2}, \quad \beta := n_{03}^{00} = \alpha \frac{2kl+l+3}{6}.$$

If $l = 1$ all these cases reduce to [Arici-Brain-Landi'15].

Quantum weighted projective lines as AF graph C^*-algebras

Let $g := \gcd(m_0, m_1)$ and $\tilde{m}_1 := m_1/g$, and define

$$W_1(\mathrm{m}):\quad \begin{tikzcd} & w_0 \arrow[dl, "\infty"] \arrow[d, "\infty"] \arrow[drr, "\infty"] & \ldots & \\ w_1 & w_2 & \cdots & w_{\tilde{m}_1} \end{tikzcd}$$

Theorem

- $C(\mathbb{WP}^1_q(\mathrm{m})) \cong C^*(W_1(\mathrm{m}))$.
- $K_0(C(\mathbb{WP}^1_q(\mathrm{m}))) = \mathbb{Z}^{1+\tilde{m}_1}$, $K_1(C(\mathbb{WP}^1_q(\mathrm{m}))) = 0$.

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The K-theory of quantum weighted projective spaces

Theorem

Let $\mathrm{m} := m_0, \ldots, m_n$. If there exists $j \in \{0, 1, \ldots, n-1\}$ so that m_j is relatively prime with m_n. Then there is an exact sequence

$$0 \longrightarrow \mathcal{K}^{m_n} \longrightarrow C(\mathbb{WP}_q^n(\mathrm{m})) \longrightarrow C(\mathbb{WP}_q^{n-1}(\mathrm{m})) \longrightarrow 0.$$

Corollary

Let $\mathrm{m} := m_0, \ldots, m_n$. If for each $j \geq 1$ there is an $i < j$ so that m_i and m_j are relatively prime. Then

$$K_0(C(\mathbb{WP}_q^n(\mathrm{m}))) = \mathbb{Z}^{1+\sum_{i=1}^n m_i}, \qquad K_1(C(\mathbb{WP}_q^n(\mathrm{m}))) = 0.$$

[Arici '16] shows that in this case $C(\mathbb{WP}_q^n(\mathrm{m}))$ is KK-equivalent to $\mathbb{C}^{1+\sum_{i=1}^n m_i}$.

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$$K_0(C(\mathbb{WP}_q^n(\mathrm{m}))) = \mathbb{Z}^{1+\sum_{i=1}^n m_i}, \qquad K_1(C(\mathbb{WP}_q^n(\mathrm{m}))) = 0.$$

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