K-HOMOLOGY AND INDEX THEORY ON CONTACT MANIFOLDS

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K-HOMOLOGY AND INDEX THEORY ON CONTACT MANIFOLDS
or
BEYOND ELLIPTICITY

K-homology is the dual theory to K-theory. The BD (Baum-Douglas) isomorphism of Atiyah-Kasparov K-homology and K-cycle K-homology provides a framework within which the Atiyah-Singer index theorem can be extended to certain differential operators which are hypoelliptic but not elliptic. This talk will consider such a class of differential operators on compact contact manifolds. These operators have been studied by a number of mathematicians. Working within the BD framework, the index problem will be solved for these operators.
This is joint work with Erik van Erp.

REFERENCE

P. Baum and E. van Erp, *K-homology and index theory on contact manifolds* Acta. Math. 213 (2014) 1-48.

K-homology is the dual theory to K-theory. There are three ways in which K-homology has been defined:

Homotopy Theory K-theory is the cohomology theory and K-homology is the homology theory determined by the Bott (i.e. K-theory) spectrum.
This is the spectrum $\ldots, \mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U, \ldots$

K-Cycles K-homology is the group of K-cycles.

C^*-algebras K-homology is the Atiyah-BDF-Kasparov group $KK^*(A, \mathbb{C})$.

Let X be a finite CW complex.
The three versions of K-homology are isomorphic.

$$K_j^{homotopy}(X) \underset{\longleftarrow}{\overset{\longrightarrow}{}} K_j(X) \longrightarrow KK^j(C(X),\mathbb{C})$$

homotopy theory $\qquad$ K-cycles $\qquad$ Atiyah-BDF-Kasparov

$$j = 0, 1$$

Cycles for K-homology

Let X be a finite CW complex.

Definition

A K-cycle on X is a triple (M, E, φ) such that :

1. M is a compact Spin^c manifold without boundary.
2. E is a $\mathbb{C}$ vector bundle on M.
3. $\varphi\colon M \to X$ is a continuous map from M to X.

What is a Spin^c vector bundle?

Let X be a paracompact Hausdorff topological space.
On X let E be an $\mathbb{R}$ vector bundle which has been oriented.
i.e. the structure group of E has been reduced from
$GL(n,\mathbb{R})$ to $GL^+(n,\mathbb{R})$

$$GL^+(n,\mathbb{R}) = \{[a_{ij}] \in GL(n,\mathbb{R}) \mid \det[a_{ij}] > 0\}$$

$n=$ fiber dimension (E)

Assume $n \geq 3$ and recall that for $n \geq 3$

$$H^2(GL^+(n,\mathbb{R}); \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

Denote by $\mathcal{F}^+(E)$ the principal $GL^+(n,\mathbb{R})$ bundle on X consisting of all positively oriented frames.

A point of $\mathcal{F}^+(E)$ is a pair $\big(x, (v_1, v_2, \ldots, v_n)\big)$ where $x \in X$ and $(v_1, v_2, \ldots, v_n)$ is a positively oriented basis of E_x. The projection $\mathcal{F}^+(E) \to X$ is

$$\big(x, (v_1, v_2, \ldots, v_n)\big) \mapsto x$$

For $x \in X$, denote by

$$\iota_x \colon \mathcal{F}^+_x(E) \hookrightarrow \mathcal{F}^+(E)$$

the inclusion of the fiber at x into $\mathcal{F}^+(E)$.

Note that (with $n \geq 3$)

$$H^2(\mathcal{F}^+_x(E); \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

A Spin^c vector bundle on X is an $\mathbb{R}$ vector bundle E on X (fiber dimension $E \geq 3$) with

1. E has been oriented.
2. $\alpha \in H^2(\mathcal{F}^+(E); \mathbb{Z})$ has been chosen such that $\forall x \in X$

$$\iota_x^*(\alpha) \in H^2(\mathcal{F}_x^+(E); \mathbb{Z}) \text{ is non-zero.}$$

Remarks

1.For $n = 1, 2$ "E is a Spin^c vector bundle" $=$"E has been oriented and an element $\alpha \in H^2(X;\mathbb{Z})$" has been chosen. ($\alpha$ can be zero.)

2. For all values of $n =$ fiber dimension(E), E is a Spin^c vector bundle iff the structure group of E has been changed from $GL(n, \mathbb{R})$ to Spin$^c(n)$. i.e. Such a change of structure group is equivalent to the above definition of Spin^c vector bundle.

Topological obstruction to Spin^c-able

E an $\mathbb{R}$ vector bundle on X.
$w_1(E), w_2(E), \ldots, w_n(E)$ Stiefel-Whitney classes of E
$w_j(E) \in H^j(X; \mathbb{Z}/2\mathbb{Z})$
E is Spin^c-able iff:
(i) $w_1(E) = 0$ (i.e. E is orientable).
and
(ii) $w_2(E)$ is in the image of the mod 2 reduction map

$$H^2(X; \mathbb{Z}) \longrightarrow H^2(X; \mathbb{Z}/2\mathbb{Z})$$

By forgetting some structure a complex vector bundle or a Spin vector bundle canonically becomes a Spin^c vector bundle

$$\begin{array}{ccc} & & \text{complex} \\ & & \Downarrow \\ \text{Spin} & \Rightarrow & \text{Spin}^c \\ & & \Downarrow \\ & & \text{oriented} \end{array}$$

A Spin^c structure for an $\mathbb{R}$ vector bundle E can be thought of as an orientation for E plus a slight extra bit of structure. Spin^c structures behave very much like orientations. For example, an orientation on two out of three $\mathbb{R}$ vector bundles in a short exact sequence determines an orientation on the third vector bundle. An analogous assertion is true for Spin^c structures.

Two Out Of Three Lemma

Lemma

Let

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be a short exact sequence of $\mathbb{R}$*-vector bundles on* X*. If two out of three are Spin*^{c} *vector bundles, then so is the third.*

Definition

Let M be a C^∞ manifold (with or without boundary). M is a Spin^c manifold iff the tangent bundle TM of M is a Spin^c vector bundle on M.

The Two Out Of Three Lemma implies that the boundary ∂M of a Spin^c manifold M with boundary is again a Spin^c manifold.

Various well-known structures on a manifold M make M into a Spin^c manifold.

$$
\begin{array}{ccc}
 & & \text{(complex-analytic)} \\
 & & \Downarrow \\
\text{(symplectic)} & \Rightarrow & \text{(almost complex)} \\
 & & \Downarrow \\
\text{(contact)} & \Rightarrow & \text{(stably almost complex)} \\
 & & \Downarrow \\
\text{Spin} & \Rightarrow & \text{Spin}^c \\
 & & \Downarrow \\
 & & \text{(oriented)}
\end{array}
$$

A Spin^c manifold can be thought of as an oriented manifold with a slight extra bit of structure. Most of the oriented manifolds which occur in practice are Spin^c manifolds.

A Spin^c manifold comes equipped with a first-order elliptic differential operator known as its Dirac operator. This operator is locally isomorphic (at the symbol level) to the Dirac operator of $\mathbb{R}^n$.

If M is a Spin^c manifold, then $Td(M)$ is

$$Td(M) := \exp^{c_1(M)/2}\widehat{A}(M) \qquad\qquad Td(M) \in H^*(M;\mathbb{Q})$$

If M is a complex-analyic manifold, then M has Chern classes $c_1, c_2, \ldots, c_n$ and

$$\exp^{c_1(M)/2}\widehat{A}(M) = P_{Todd}(c_1, c_2, \ldots, c_n)$$

EXAMPLE. Let M be a compact complex-analytic manifold.
Set $\Omega^{p,q} = C^\infty(M, \Lambda^{p,q} T^*_{\mathbb{C}} M)$
$\Omega^{p,q}$ is the $\mathbb{C}$ vector space of all C^∞ differential forms of type (p,q)
Dolbeault complex

$$0 \longrightarrow \Omega^{0,0} \longrightarrow \Omega^{0,1} \longrightarrow \Omega^{0,2} \longrightarrow \cdots \longrightarrow \Omega^{0,n} \longrightarrow 0$$

The Dirac operator (of the underlying Spin^c manifold) is the assembled Dolbeault complex

$$\bar\partial + \bar\partial^* \colon \bigoplus_j \Omega^{0,\,2j} \longrightarrow \bigoplus_j \Omega^{0,\,2j+1}$$

The index of this operator is the arithmetic genus of M — i.e. is the Euler number of the Dolbeault complex.

TWO POINTS OF VIEW ON SPIN^c MANIFOLDS

1. Spin^c is a slight strengthening of oriented. Most of the oriented manifolds that occur in practice are Spin^c.

2. Spin^c is much weaker than complex-analytic. BUT the assempled Dolbeault complex survives (as the Dirac operator). AND the Todd class survives.

$$M \quad \mathrm{Spin}^c \Longrightarrow \quad \exists \quad Td(M) \in H^*(M;\mathbb{Q})$$

If M is a Spin^c manifold, then $Td(M)$ is

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If M is a complex-analyic manifold, then M has Chern classes $c_1, c_2, \ldots, c_n$ and

$$\exp^{c_1(M)/2}\widehat{A}(M) = P_{Todd}(c_1, c_2, \ldots, c_n)$$

WARNING!!!

The Todd class of a Spin^c manifold is not obtained by complexifying the tangent bundle TM of M and then
applying the Todd polynomial to the Chern classes of $T_{\mathbb{C}}M$.

$$Td(T_{\mathbb{C}}M) = \widehat{A}(M)^2 = \widehat{A}(M) \cup \widehat{A}(M)$$

Correct formula for the Todd class of a Spin^c manifold M is:

$$Td(M) := \exp^{c_1(M)/2}\widehat{A}(M) \qquad\qquad Td(M) \in H^*(M;\mathbb{Q})$$

SPECIAL CASE OF ATIYAH-SINGER

Let M be a compact even-dimensional Spin^c manifold without boundary. Let E be a $\mathbb{C}$ vector bundle on M.

D_E denotes the Dirac operator of M tensored with E.

$$D_E\colon C^\infty(M, \mathcal{S}^+ \otimes E) \longrightarrow C^\infty(M, \mathcal{S}^- \otimes E)$$

$\mathcal{S}^+, (\mathcal{S}^-)$ are the positive (negative) spinor bundles on M.

<u>THEOREM</u> Index$(D_E) = (ch(E) \cup Td(M))[M]$.

SPECIAL CASE OF ATIYAH-SINGER

Let M be a compact even-dimensional Spin^c manifold without boundary. Let E be a $\mathbb{C}$ vector bundle on M.
D_E denotes the Dirac operator of M tensored with E.

<u>THEOREM</u> Index$(D_E) = (ch(E) \cup Td(M))[M]$.

This theorem can be proved as a corollary of Bott periodicity.
See Baum-van Erp expository paper.
K-homology and Fredholm operators I : Dirac Operators
arXiv 1604.03502

In particular, this proves the Hirzebruch-Riemann-Roch theorem.

Also, this proves (for closed even-dimensional Spin^c manifolds) the Hirzebruch signature theorem.

Cycles for K-homology

Let X be a finite CW complex.

Definition

A K-cycle on X is a triple (M, E, φ) such that :

1. M is a compact Spin^c manifold without boundary.
2. E is a $\mathbb{C}$ vector bundle on M.
3. $\varphi\colon M \to X$ is a continuous map from M to X.

Set $K_*(X) = \{(M, E, \varphi)\}/\sim$ where the equivalence relation $\sim$ is generated by the three elementary steps

- Bordism
- Direct sum - disjoint union
- Vector bundle modification

Isomorphism (M, E, φ) is isomorphic to (M', E', φ') iff $\exists$ a diffeomorphism

$$\psi\colon M \to M'$$

preserving the Spin^c-structures on M, M' and with

$$\psi^*(E') \cong E$$

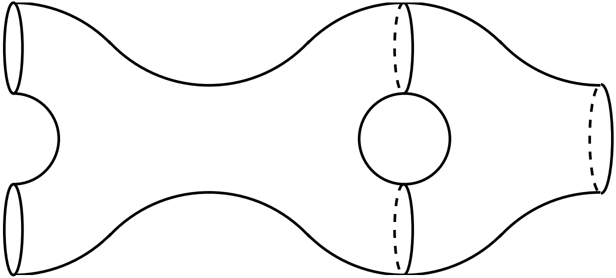
and commutativity in the diagram

$$\begin{tikzcd} M \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & M' \arrow[dl, "\varphi'"] \\ & X & \end{tikzcd}$$

Bordism (M_0, E_0, φ_0) is **bordant** to (M_1, E_1, φ_1) iff $\exists$ (Ω, E, φ) such that:

1. Ω is a compact Spin^c manifold with boundary.
2. E is $\mathbb{C}$ vector bundle on Ω.
3. $(\partial\Omega, E|_{\partial\Omega}, \varphi|_{\partial\Omega}) \cong (M_0, E_0, \varphi_0) \sqcup (-M_1, E_1, \varphi_1)$

$-M_1$ is M_1 with the Spin^c structure reversed.



(M_0, E_0, φ_0) $\qquad \downarrow \qquad$ $(-M_1, E_1, \varphi_1)$

X

Direct sum - disjoint union

Let E, E' be two $\mathbb{C}$ vector bundles on M

$$(M, E, \varphi) \sqcup (M, E', \varphi) \sim (M, E \oplus E', \varphi)$$

Vector bundle modification

Let (M, E, φ) be any K-cycle on X

and let F be any Spin^c vector bundle on M with even-dimensional fibers.

i.e. assume that

$$\dim_{\mathbb{R}}(F_p) \equiv 0 \mod 2 \quad p \in M$$

for every fiber F_p of F

$$\mathbf{1}_{\mathbb{R}} = M \times \mathbb{R}$$

$$S(F \oplus \mathbf{1}_{\mathbb{R}}) := \text{unit sphere bundle of } F \oplus \mathbf{1}_{\mathbb{R}}$$

$$(M, E, \varphi) \sim (S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^* E, \varphi \circ \pi)$$

$$\begin{tikzcd} S(F \oplus \mathbf{1}_{\mathbb{R}}) \arrow[d, "\pi"] \\ M \end{tikzcd}$$

This is a fibration with even-dimensional spheres as fibers.

$F \oplus \mathbf{1}_{\mathbb{R}}$ is a Spin^c vector bundle on M with odd-dimensional fibers. Let Σ be the spinor bundle for $F \oplus \mathbf{1}_{\mathbb{R}}$

$$Cliff_{\mathbb{C}}(F_p \oplus \mathbb{R}) \otimes \Sigma_p \to \Sigma_p$$

$$\pi^*\Sigma = \beta^* \oplus \beta^*_-$$

$$(M, E, \varphi) \sim (S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^* E, \varphi \circ \pi)$$

$$\{(M, E, \varphi)\}/\sim = K_0(X) \oplus K_1(X)$$

$$K_j(X) = \begin{array}{l}\text{subgroup of } \{(M, E, \varphi)\}/\sim \\ \text{consisting of all } (M, E, \varphi) \text{ such that} \\ \text{every connected component of } M \\ \text{has dimension } \equiv j \mod 2,\ j = 0, 1\end{array}$$

Addition in $K_j(X)$ is disjoint union.

$$(M, E, \varphi) + (M', E', \varphi') = (M \sqcup M', E \sqcup E', \varphi \sqcup \varphi')$$

Additive inverse of (M, E, φ) is obtained by reversing the Spin^c structure of M.

$$-(M, E, \varphi) = (-M, E, \varphi)$$

<u>DEFINITION.</u> (M, E, φ) *bounds* $\Longleftrightarrow \exists\, (\Omega, \widetilde{E}, \widetilde{\varphi})$ with :

1. Ω is a compact Spin^c manifold with boundary.
2. $\widetilde{E}$ is a $\mathbb{C}$ vector bundle on Ω.
3. $\widetilde{\varphi}\colon \Omega \to X$ is a continuous map.
4. $(\partial\Omega, \widetilde{E}|_{\partial\Omega}, \widetilde{\varphi}|_{\partial\Omega}) \cong (M, E, \varphi)$

<u>REMARK.</u> $(M, E, \varphi) = 0$ in $K_*(X) \Longleftrightarrow (M, E, \varphi) \sim (M', E', \varphi')$ where (M', E', φ') bounds.

Let X, Y be finite CW complexes and let $f\colon X \to Y$ be a continuous map.

Then $f_*\colon K_j(X) \to K_j(Y)$ is

$$f_*(M, E, \varphi) := (M, E, f \circ \varphi)$$

Atiyah-BDF-Kasparov K-homology

M.F. Atiyah   Brown-Douglas-Fillmore   G.Kasparov

Let X be a finite CW complex.

$C(X) = \{\alpha : X \to \mathbb{C} \quad | \quad \alpha \text{ is continuous}\}$

$\mathcal{L}(\mathcal{H}) = \{\text{bounded operators } T : \mathcal{H} \to \mathcal{H}\}$

Any element in the Atiyah-BDF-Kasparov K-homology group $KK^0(C(X), \mathbb{C})$

is given by a 5-tuple $(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T)$ such that :

- $\mathcal{H}_0$ and $\mathcal{H}_1$ are separable Hilbert spaces.
- $\psi_0 \colon C(X) \longrightarrow \mathcal{L}(\mathcal{H}_0)$ and $\psi_1 \colon C(X) \longrightarrow \mathcal{L}(\mathcal{H}_1)$ are unital $*$-homomorphisms.
- $T : \mathcal{H}_0 \longrightarrow \mathcal{H}_1$ is a (bounded) Fredholm operator.
- For every $\alpha \in C(X)$ the commutator $T \circ \psi_0(\alpha) - \psi_1(\alpha) \circ T \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is compact.

$$KK^0(C(X), \mathbb{C}) := \{(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T)\}/\sim$$

$$KK^0(C(X),\mathbb{C}) := \{(\mathcal{H}_0,\psi_0,\mathcal{H}_1,\psi_1,T)\}/\sim$$

$$(\mathcal{H}_0,\psi_0,\mathcal{H}_1,\psi_1,T) + (\mathcal{H}_0',\psi_0',\mathcal{H}_1',\psi_1',T') = \\ (\mathcal{H}_0\oplus\mathcal{H}_0',\psi_0\oplus\psi_0',\mathcal{H}_1\oplus\mathcal{H}_1',\psi_1\oplus\psi_1',T\oplus T')$$

$$-(\mathcal{H}_0,\psi_0,\mathcal{H}_1,\psi_1,T) = (\mathcal{H}_1,\psi_1,\mathcal{H}_0,\psi_0,T^*)$$

Let X be a finite CW complex.
Any element in the Atiyah-BDF-Kasparov K-homology group $KK^1(C(X), \mathbb{C})$
is given by a 3-tuple $(\mathcal{H}, \psi, T)$ such that :

- $\mathcal{H}$ is a separable Hilbert space.
- $\psi\colon C(X) \longrightarrow \mathcal{L}(\mathcal{H})$ is a unital $*$-homomorphism.
- $T : \mathcal{H} \longrightarrow \mathcal{H}$ is a (bounded) self-adjoint Fredholm operator.
- For every $\alpha \in C(X)$ the commutator $T \circ \psi(\alpha) - \psi(\alpha) \circ T \in \mathcal{L}(\mathcal{H})$ is compact.

$$KK^1(C(X),\mathbb{C}) := \{(\mathcal{H},\psi,T)\}/\sim$$

$$(\mathcal{H},\psi,T)+(\mathcal{H}',\psi',T') = (\mathcal{H}\oplus\mathcal{H}',\psi\oplus\psi',T\oplus T')$$

$$-(\mathcal{H},\psi,T) = (\mathcal{H},\psi,-T)$$

Let X, Y be finite CW complexes and let $f\colon X \to Y$ be a continuous map.

Denote by $f^\natural\colon C(X) \leftarrow C(Y)$ the $*$-homomorphism

$$f^\natural(\alpha) := \alpha \circ f \qquad\qquad \alpha \in C(Y)$$

Then $f_*\colon KK^j(C(X), \mathbb{C}) \to KK^j(C(Y), \mathbb{C})$ is

$$f_*(\mathcal{H}, \psi, T) := (\mathcal{H}, \psi \circ f^\natural, T) \qquad\qquad j = 1$$

$$f_*(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) := (\mathcal{H}_0, \psi_0 \circ f^\natural, \mathcal{H}_1, \psi_1 \circ f^\natural, T) \qquad\qquad j = 0$$

Chern character in K-theory

X finite CW complex

$$ch\colon K^j(X) \longrightarrow \bigoplus_l H^{j+2l}(X;\mathbb{Q})$$

$$j = 0, 1$$

$$\mathbb{Q} \otimes_{\mathbb{Z}} K^j(X) \longrightarrow \bigoplus_l H^{j+2l}(X;\mathbb{Q})$$

is an isomorphism of $\mathbb{Q}$ vector spaces.

Chern character in K-cycle K-homology

X finite CW complex $\qquad (M, E, \varphi) \mapsto \varphi_*(ch(E) \cup Td(M) \cap [M])$

$$ch\colon K_j(X) \longrightarrow \bigoplus_l H_{j+2l}(X; \mathbb{Q})$$

$$j = 0, 1$$

$$\mathbb{Q} \otimes_{\mathbb{Z}} K_j(X) \longrightarrow \bigoplus_l H_{j+2l}(X; \mathbb{Q})$$

is an isomorphism of $\mathbb{Q}$ vector spaces.

Chern character and module structure

X a finite CW complex.

$K^*(X)$ is a ring and $K_*(X)$ is a module over this ring.
Chern character respects the ring and module structure.

Theorem (PB and R.Douglas and M.Taylor, PB and N. Higson and T. Schick)

Let X be a finite CW complex.

Then for $j = 0, 1$ the natural map of abelian groups

$$K_j(X) \to KK^j(C(X), \mathbb{C})$$

is an isomorphism.

For $j = 0, 1$ the natural map of abelian groups

$$K_j(X) \to KK^j(C(X), \mathbb{C})$$

is $(M, E, \varphi) \mapsto \varphi_*[D_E]$

where

1. D_E is the Dirac operator of M tensored with E.
2. $[D_E] \in KK^j(C(M), \mathbb{C})$ is the element in the Kasparov K-homology of M determined by D_E.
3. $\varphi_*\colon KK^j(C(M), \mathbb{C}) \to KK^j(C(X), \mathbb{C})$ is the homomorphism of abelian groups determined by $\varphi\colon M \to X$.

Let (M, E, φ) be a K-cycle on X, with M even-dimensional.

$$D_E \colon C^\infty(M, \mathcal{S}^+ \otimes E) \longrightarrow C^\infty(M, \mathcal{S}^- \otimes E)$$

Set $\mathcal{H}_0 = L^2(M, \mathcal{S}^+ \otimes E) \qquad \mathcal{H}_1 = L^2(M, \mathcal{S}^- \otimes E)$

For $j = 0, 1$ define $\psi_j \colon C(M) \to \mathcal{L}(\mathcal{H}_j)$ by :

$$\alpha \mapsto \mathcal{M}_\alpha \qquad \alpha \in C(M)$$

where $\mathcal{M}_\alpha$ is the multiplication operator

$$\mathcal{M}_\alpha(u) = \alpha u \qquad (\alpha u)(p) = \alpha(p)u(p) \qquad \alpha \in C(M), u \in \mathcal{H}_j, p \in M$$

Set $T = D_E(I + D_E^* D_E)^{-1/2}$ Then

$$(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) \in KK^0(C(M), \mathbb{C})$$

EXAMPLE. $S^1 \subset \mathbb{R}^2$

S^1 with its usual Spin^c structure has $\mathcal{S} = S^1 \times \mathbb{C}$.

The Dirac operator $D\colon L^2(S^1) \to L^2(S^1)$ is:

$$D = -i\frac{\partial}{\partial\theta}$$

The functions $e^{in\theta}$ are an orthonormal basis for $L^2(S^1)$.

Each $e^{in\theta}$ is an eigenvector of D :

$$-i\frac{\partial}{\partial\theta}(e^{in\theta}) = ne^{in\theta} \qquad\qquad n \in \mathbb{Z}$$

D is an unbounded self-adjoint operaror. $D^* = D$.

The bounded operator $T := D(I + D^*D)^{-1/2}$ is

$$T(e^{in\theta}) = \frac{n}{\sqrt{1+n^2}}e^{in\theta} \qquad\qquad n \in \mathbb{Z}$$

K-cycles and string theory

K-cycles are very closely connected to the D-branes of string theory. A D-brane is a K-cycle for the twisted K-homology of space-time.

In some models, the D-branes are allowed to evolve with time. This evolution is achieved by permitting the D-branes to change by the three elementary steps. Thus the underlying *charge* of a D-brane (i.e. the element in the twisted K-homology of space-time determined by the D-brane) remains unchanged as the D-brane evolves.

For more, see Jonathan Rosenberg's CBMS string theory lectures. Also, see Baum-Carey-Wang paper *K-cycles for twisted K-homology* Journal of K-theory 12, 69-98, 2013.

Comparison of $K_*(X)$ and $KK^*(C(X), \mathbb{C})$

Given some analytic data on X (i.e. an index problem) it is usually easy to construct an element in $KK^*(C(X), \mathbb{C})$. This does not solve the given index problem. $KK^*(C(X), \mathbb{C})$ does not have a simple explicitly defined chern character mapping it to $H_*(X; \mathbb{Q})$.

$K_*(X)$ does have a simple explicitly defined chern character mapping it to $H_*(X; \mathbb{Q})$.

$$ch\colon K_j(X) \longrightarrow \bigoplus_l H_{j+2l}(X; \mathbb{Q}) \qquad j = 0, 1$$

$$(M, E, \varphi) \mapsto \varphi_*(\mathrm{ch}(E) \cup Td(M) \cap [M])$$

With X a finite CW complex, suppose a datum (i.e. some analytical information) is given which then determines an element $\xi \in KK^j(C(X), \mathbb{C})$.

QUESTION : What does it mean to solve the index problem for ξ?

ANSWER : It means to explicitly construct the K-cycle (M, E, φ) such that

$$\mu(M, E, \varphi) = \xi$$

where $\mu \colon K_j(X) \to KK^j(C(X), \mathbb{C})$ is the natural map of abelian groups.

Suppose that $j = 0$ and that a K-cycle (M, E, φ) with

$$\mu(M, E, \varphi) = \xi$$

has been constructed. It then follows that for any $\mathbb{C}$ vector bundle F on X

$$\mathrm{Index}(F \otimes \xi) = \epsilon_*(ch(F) \cap ch(M, E, \varphi))$$

$\epsilon\colon X \longrightarrow \cdot$ $\qquad$ ϵ is the map of X to a point.

$$ch(M, E, \varphi) := \varphi_*(\mathrm{ch}(E) \cup Td(M) \cap [M])$$

EQUIVALENTLY Suppose that $j = 0$ and that a K-cycle (M, E, φ) with

$$\mu(M, E, \varphi) = \xi$$

has been constructed. It then follows that

$$\mathcal{I}(\xi) = \varphi_*(ch(E) \cup Td(M) \cap [M])$$

where $\mathcal{I}(\xi)$ is (by definition) the unique element of $H_{even}(X; \mathbb{Q}) = \bigoplus_l H_{2l}(X; \mathbb{Q})$ such that for any $\mathbb{C}$ vector bundle F on X

$$\mathrm{Index}(F \otimes \xi) = \epsilon_*(ch(F) \cap \mathcal{I}(\xi))$$

$\epsilon \colon X \longrightarrow \cdot$ $\qquad$ ϵ is the map of X to a point.

REMARK. If the construction of the K-cycle (M, E, φ) with

$$\mu(M, E, \varphi) = \xi$$

has been done correctly, then it will work in the equivariant case and in the case of families of operators.

Example

General case of the Atiyah-Singer index theorem

Let X be a compact C^∞ manifold without boundary.
X is not required to be oriented.
X is not required to be even dimensional.
On X let

$$\delta : C^\infty(X, E_0) \longrightarrow C^\infty(X, E_1)$$

be an elliptic differential (or pseudo-differential) operator.
Then δ determines an element

$$[\delta] \in KK^0(C(X), \mathbb{C})$$

The K-cycle on X – which solves the index problem for δ – is

$$(S(TX \oplus 1_{\mathbb{R}}), E_\sigma, \pi).$$

$$(S(TX \oplus 1_{\mathbb{R}}), E_\sigma, \pi)$$

$S(TX \oplus 1_{\mathbb{R}})$ is the unit sphere bundle of $TX \oplus 1_{\mathbb{R}}$.

$\pi\colon S(TX \oplus 1_{\mathbb{R}}) \longrightarrow X$ is the projection of $S(TX \oplus 1_{\mathbb{R}})$ onto X.

$S(TX \oplus 1_{\mathbb{R}})$ is even-dimensional and is a Spin^c manifold.

E_σ is the $\mathbb{C}$ vector bundle on $S(TX \oplus 1_{\mathbb{R}})$ obtained by doing a clutching construction using the symbol σ of δ.

$$\mu((S(TX \oplus 1_{\mathbb{R}}), E_\sigma, \pi)) = [\delta]$$

$$\Downarrow$$

$$\text{Index}(\delta) = (ch(E_\sigma) \cup Td(S(TX \oplus 1_{\mathbb{R}})))[(S(TX \oplus 1_{\mathbb{R}})]$$

which is the general Atiyah-Singer formula.

FACT:
If M is a closed odd-dimensional C^∞ manifold
and D is any elliptic differential operator on M,
then $\operatorname{Index}(D) = 0$.

EXAMPLE:
$M = S^3 = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \mid a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1\}$
x_1, x_2, x_3, x_4 are the usual co-ordinate functions on $\mathbb{R}^4$.

$$x_j(a_1, a_2, a_3, a_4) = a_j \qquad j = 1, 2, 3, 4$$

$$\frac{\partial}{\partial x_j} \text{ usual vector fields on } \mathbb{R}^4 \qquad j = 1, 2, 3, 4$$

On S^3 consider the (tangent) vector fields V_1, V_2, V_3

$$V_1 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4}$$

$$V_2 = x_3\frac{\partial}{\partial x_1} - x_4\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial x_4}$$

$$V_3 = x_4\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial x_4}$$

Let r be any positive integer and let $\gamma\colon S^3 \longrightarrow M(r,\mathbb{C})$ be a C^∞ map.
$M(r,\mathbb{C})$:= {r×r matrices of complex numbers}.
Form the operator $P_\gamma := i\gamma(V_1 \otimes I_r) - V_2^2 \otimes I_r - V_3^2 \otimes I_r$.
$I_r := r \times r$ identity matrix.

$$P_\gamma\colon C^\infty(S^3, S^3 \times \mathbb{C}^r) \longrightarrow C^\infty(S^3, S^3 \times \mathbb{C}^r)$$

$$P_\gamma := i\gamma(V_1 \otimes I_r) - V_2^2 \otimes I_r - V_3^2 \otimes I_r$$

$I_r := r \times r$ identity matrix. $\qquad i = \sqrt{-1}.$

$$P_\gamma \colon C^\infty(S^3, S^3 \times \mathbb{C}^r) \longrightarrow C^\infty(S^3, S^3 \times \mathbb{C}^r)$$

LEMMA.
Assume that for all $p \in S^3, \gamma(p)$ does not have any odd integers among its eigenvalues i.e.

$$\forall p \in S^3,\ \ \forall \lambda \in \{\ldots -3, -1, 1, 3, \ldots\} \Longrightarrow \lambda I_r - \gamma(p) \in GL(r, \mathbb{C})$$

then $\dim_{\mathbb{C}}$ (Kernel P_γ) $< \infty$ and $\dim_{\mathbb{C}}$ (Cokernel P_γ) $< \infty$.

With γ as in the above lemma, for each odd integer n , let

$$\gamma_n \colon S^3 \longrightarrow GL(r, \mathbb{C}) \quad \text{be}$$

$$p \longmapsto nI_r - \gamma(p)$$

By Bott periodicity if $r \geq 2$, then $\pi_3 GL(r, \mathbb{C}) = \mathbb{Z}$.
Hence for each odd integer n have the Bott number $\beta(\gamma_n)$.

PROPOSITION. With γ as above and $r \geq 2$

$$\text{Index}(P_\gamma) = \sum_{n \text{ odd}} \beta(\gamma_n)$$

S^{2n+1} = unit sphere of $\mathbb{R}^{2n+2}$ $\qquad S^{2n+1} \subset \mathbb{R}^{2n+2} \qquad n = 1, 2, 3, \ldots$
On S^{2n+1} there is the nowhere-vanishing vector field V
V=

$$x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4} + \cdots + x_{2n+2} \frac{\partial}{\partial x_{2n+1}} - x_{2n+1} \frac{\partial}{\partial x_{2n+2}}$$

$$V = \sum_{i=1}^{n+1} x_{2i} \frac{\partial}{\partial x_{2i-1}} - x_{2i-1} \frac{\partial}{\partial x_{2i}}$$

Let θ be the 1-form on S^{2n+1}

$$\theta = \sum_{i=1}^{n+1} x_{2i} dx_{2i-1} - x_{2i-1} dx_{2i}$$

Then:

- $\theta(V) = 1$
- $\theta(d\theta)^n$ is a volume form on S^{2n+1} i.e. $\theta(d\theta)^n$ is a nowhere-vanishing C^∞ $2n+1$ form on S^{2n+1}.

Let H be the null-space of θ.

$$H = \{v \in TS^{2n+1} \mid \theta(v) = 0\}$$

H is a C^∞ sub vector bundle of TS^{2n+1} with

$$\text{For all } x \in S^{2n+1}, \dim_{\mathbb{R}}(H_x) = 2n$$

The sub-Laplacian

$$\Delta_H \colon C^\infty(S^{2n+1}) \to C^\infty(S^{2n+1})$$

is locally $-W_1^2 - W_2^2 - \cdots - W_{2n}^2$
where $W_1, W_2, \ldots, W_{2n}$ is a locally defined C^∞ orthonormal frame for H.
These locally defined operators are then patched together using a C^∞ partition of unity to give the sub-Laplacian Δ_H.

Let r be a positive integer and let $\gamma \colon S^{2n+1} \longrightarrow M(r, \mathbb{C})$ be a C^∞ map.
$M(r, \mathbb{C})$:= {r×r matrices of complex numbers}.

Assume: For each $x \in S^{2n+1}$
$\{$Eigenvalues of $\gamma(x)\} \cap \{\ldots, -n-4, -n-2, -n, n, n+2, n+4, \ldots\} = \emptyset$
i.e. $\forall x \in S^{2n+1}$,
$\lambda \in \{\ldots -n-4, -n-2, -n, n, n+2, n+4, \ldots\} \Longrightarrow \lambda I_r - \gamma(x) \in GL(r, \mathbb{C})$

Let

$$\gamma\colon S^{2n+1} \longrightarrow M(r, \mathbb{C})$$

be as above, $P_\gamma\colon C^\infty(S^{2n+1}, S^{2n+1} \times \mathbb{C}^r) \to C^\infty(S^{2n+1}, S^{2n+1} \times \mathbb{C}^r)$ is defined:

$$P_\gamma = i\gamma(V \otimes I_r) + (\Delta_H) \otimes I_r \qquad I_r = r \times r \text{ identity matrix} \quad i = \sqrt{-1}$$

P_γ is a differential operator (of order 2) and is hypoelliptic but not

elliptic. P_γ is Fredholm.

The formula for the index of P_γ is

$\mathrm{Index}\, P_\gamma =$

$$\sum_{j=0}^{N} \begin{pmatrix} n+j-1 \\ j \end{pmatrix} \left[\beta((n+2j)I_r - \gamma) + (-1)^{n+1}\beta((n+2j)I_r) + \gamma)\right]$$

$\beta((n+2j)I_r - \gamma) :=$ the Bott number of $(n+2j)I_r - \gamma$

$(n+2j)I_r - \gamma \colon S^{2n+1} \to GL(r, \mathbb{C})$

Remark on the S^{2n+1} example

$$V = \sum_{i=1}^{n+1} x_{2i}\frac{\partial}{\partial x_{2i-1}} - x_{2i-1}\frac{\partial}{\partial x_{2i}}$$

θ is the 1-form on S^{2n+1}

$$\theta = \sum_{i=1}^{n+1} x_{2i}dx_{2i-1} - x_{2i-1}dx_{2i}$$

$$\theta(V) = 1$$

V is the vector field along the orbits for the usual action of S^1 on S^{2n+1}.

$$S^1 \times S^{2n+1} \longrightarrow S^{2n+1}$$

The quotient space S^{2n+1}/S^1 is $\mathbb{C}P^n$.
Denote the quotient map by $\pi\colon S^{2n+1} \to \mathbb{C}P^n$.

$$\pi\colon S^{2n+1} \to \mathbb{C}P^n$$

THEN $H :=$ null space of $\theta = \pi^*(T\mathbb{C}P^n)$ is a $\mathbb{C}$ vector bundle on S^{2n+1}.

Contact Manifolds

A contact manifold is an odd dimensional C^∞ manifold X
dimension$(X) = 2n+1$
with a given C^∞ 1-form θ such that

$$\theta(d\theta)^n \text{ is non zero at every } x \in X - i.e. \; \theta(d\theta)^n \text{ is a volume form for } X.$$

Let X be a compact connected contact manifold without boundary $(\partial X = \emptyset)$.
Set dimension$(X) = 2n+1$.
Let r be a positive integer and let $\gamma\colon X \longrightarrow M(r,\mathbb{C})$ be a C^∞ map.
$M(r,\mathbb{C}) :=$ {r×r matrices of complex numbers}.

Assume: For each $x \in X$,
$\{\text{Eigenvalues of } \gamma(x)\} \cap \{\ldots, -n-4, -n-2, -n, n, n+2, n+4, \ldots\} = \emptyset$
i.e. $\forall x \in X$,
$\lambda \in \{\ldots -n-4, -n-2, -n, n, n+2, n+4, \ldots\} \Longrightarrow \lambda I_r - \gamma(x) \in GL(r,\mathbb{C})$

$\gamma\colon X \longrightarrow M(r,\mathbb{C})$
Are assuming : $\forall x \in X$,
$\lambda \in \{\ldots -n-4, -n-2, -n, n, n+2, n+4, \ldots\} \Longrightarrow \lambda I_r - \gamma(x) \in GL(r,\mathbb{C})$

Associated to γ is a differential operator P_γ which is hypoelliptic and Fredholm.

$$P_\gamma\colon C^\infty(X, X\times\mathbb{C}^r) \longrightarrow C^\infty(X, X\times\mathbb{C}^r)$$

P_γ is constructed as follows.

The sub-Laplacian Δ_H

Let H be the null-space of θ.

$$H = \{v \in TX \mid \theta(v) = 0\}$$

H is a C^∞ sub vector bundle of TX with

$$\text{For all } x \in X, \dim_{\mathbb{R}}(H_x) = 2n$$

The sub-Laplacian

$$\Delta_H \colon C^\infty(X) \to C^\infty(X)$$

is locally $-W_1^2 - W_2^2 - \cdots - W_{2n}^2$
where $W_1, W_2, \ldots, W_{2n}$ is a locally defined C^∞ orthonormal frame for H.
These locally defined operators are then patched together using a C^∞ partition of unity to give the sub-Laplacian Δ_H.

The Reeb vector field

The Reeb vector field is the unique C^∞ vector field W on X with :

$$\theta(W) = 1 \text{ and } \forall v \in TX,\ d\theta(W, v) = 0$$

Let

$$\gamma\colon X \longrightarrow M(r, \mathbb{C})$$

be as above, $P_\gamma\colon C^\infty(X, X \times \mathbb{C}^r) \to C^\infty(X, X \times \mathbb{C}^r)$ is defined:

$$P_\gamma = i\gamma(W \otimes I_r) + (\Delta_H) \otimes I_r \qquad I_r = r \times r \text{ identity matrix} \qquad i = \sqrt{-1}$$

P_γ is a differential operator (of order 2) and is hypoelliptic but not elliptic.

These operators P_γ have been studied by :

- R.Beals and P.Greiner *Calculus on Heisenberg Manifolds* Annals of Math. Studies 119 (1988).
- C.Epstein and R.Melrose.
- E. van Erp *The Atiyah-Singer index formula for subelliptic operators on contact manifolds. Part 1 and Part 2* Annals of Math. 171(2010).

A class of operators with somewhat similar analytic and topological properties has been studied by A. Connes and H. Moscovici.
M. Hilsum and G. Skandalis.

Set $T_\gamma = P_\gamma(I + P_\gamma^* P_\gamma)^{-1/2}$.

Let $\psi\colon C(X) \to \mathcal{L}(L^2(X) \otimes_{\mathbb{C}} \mathbb{C}^r)$ be

$$\psi(\alpha)(u_1, u_2, \ldots, u_r) = (\alpha u_1, \alpha u_2, \ldots, \alpha u_r)$$

where for $x \in X$ and $u \in L^2(X), (\alpha u)(x) = \alpha(x)u(x)$

$$\alpha \in C(X) \qquad\qquad u \in L^2(X)$$

Then

$$(L^2(X) \otimes_{\mathbb{C}} \mathbb{C}^r,\ \psi,\ \ L^2(X) \otimes_{\mathbb{C}} \mathbb{C}^r,\ \psi,\ \ T_\gamma) \in KK^0(C(X), \mathbb{C})$$

Denote this element of $KK^0(C(X), \mathbb{C})$ by $[P_\gamma]$.

$$[P_\gamma] \in KK^0(C(X), \mathbb{C})$$

$$[P_\gamma] \in KK^0(C(X), \mathbb{C})$$

QUESTION.What is the K-cycle that solves the index problem for $[P_\gamma]$?

ANSWER. To construct this K-cycle, first recall that the given 1-form θ which makes X a contact manifold also makes X a stably almost complex manifold :

$$\text{(contact)} \Longrightarrow \text{(stably almost complex)}$$

(contact) $\Longrightarrow$ (stably almost complex)

Let θ, H, and W be as above. Then :

- $TX = H \oplus 1_{\mathbb{R}}$ where $1_{\mathbb{R}}$ is the (trivial) $\mathbb{R}$ line bundle spanned by W.
- A morphism of C^∞ $\mathbb{R}$ vector bundles $J : H \to H$ can be chosen with $J^2 = -I$ and $\forall x \in X$ and $u, v \in H_x$

$$d\theta(Ju, Jv) = d\theta(u, v) \qquad d\theta(Ju, u) \geq 0$$

- J is unique up to homotopy.

(contact) $\Longrightarrow$ (stably almost complex)

$J\colon H \to H$ is unique up to homotopy.
Once J has been chosen :

$$H \text{ is a } C^\infty\ \mathbb{C} \text{ vector bundle on X.}$$
$$\Downarrow$$
$$TX \oplus 1_{\mathbb{R}} = H \oplus 1_{\mathbb{R}} \oplus 1_{\mathbb{R}} = H \oplus 1_{\mathbb{C}} \text{ is a } C^\infty\ \mathbb{C} \text{ vector bundle on } X.$$
$$\Downarrow$$
$$X \times S^1 \ \text{ is an almost complex manifold.}$$

REMARK. An almost complex manifold is a $\mathbb{C}^\infty$ manifold Ω with a given morphism $\zeta\colon T\Omega \to T\Omega$ of C^∞ $\mathbb{R}$ vector bundles on Ω such that

$$\zeta \circ \zeta = -I$$

The conjugate almost complex manifold is Ω with ζ replaced by $-\zeta$.

NOTATION. As above $X \times S^1$ is an almost complex manifold, $\overline{X \times S^1}$ denotes the conjugate almost complex manifold.

Since (almost complex)$\Longrightarrow$ (Spin^c), the disjoint union $X \times S^1 \sqcup \overline{X \times S^1}$ can be viewed as a Spin^c manifold.

Let

$$\pi\colon X \times S^1 \sqcup \overline{X \times S^1} \longrightarrow X$$

be the evident projection of $X \times S^1 \sqcup \overline{X \times S^1}$ onto X.
i.e.

$$\pi(x, \lambda) = x \qquad\qquad (x, \lambda) \in X \times S^1 \sqcup \overline{X \times S^1}$$

The solution K-cycle for $[P_\gamma]$ is $(X \times S^1 \sqcup \overline{X \times S^1},\ E_\gamma,\ \pi)$

$$E_\gamma = \Big(\bigoplus_{j=0}^{j=N} L(\gamma, n+2j)\otimes\pi^*\mathrm{Sym}^j(H)\Big)\bigsqcup\Big(\bigoplus_{j=0}^{j=N} L(\gamma, -n-2j)\otimes\pi^*\mathrm{Sym}^j(H^*)\Big)$$

1. "Sym^j" is " j-th symmetric power".
2. H^* is the dual vector bundle of H.
3. N is any positive integer such that : $n + 2N > \sup\{||\gamma(x)||, x \in X\}$.
4. $L(\gamma, n + 2j)$ is the $\mathbb{C}$ vector bundle on $X \times S^1$ obtained by doing a clutching construction using $(n + 2j)I_r - \gamma\colon X \to GL(r, \mathbb{C})$.
5. Similarly, $L(\gamma, -n - 2j)$ is obtained by doing a clutching construction using $(-n - 2j)I_r - \gamma\colon X \to GL(r, \mathbb{C})$.

Restriction of E_γ to $X \times S^1$

Let N be any positive integer such that :

$$n + 2N > \sup\{||\gamma(x)||, x \in X\}$$

The restriction of E_γ to $X \times S^1$ is:

$$E_\gamma \mid X \times S^1 = \bigoplus_{j=0}^{j=N} L(\gamma, n+2j) \otimes \pi^* \mathrm{Sym}^j(H)$$

Restriction of E_γ to $\overline{X \times S^1}$

The restriction of E_γ to $\overline{X \times S^1}$ is:

$$E_\gamma \mid \overline{X \times S^1} = \bigoplus_{j=0}^{j=N} L(\gamma, -n-2j) \otimes \pi^* \mathrm{Sym}^j(H^*)$$

Here H^* is the dual vector bundle of H:

$$H_x^* = \mathrm{Hom}_{\mathbb{C}}(H_x, \mathbb{C}) \qquad\qquad x \in X$$

$$E_\gamma = \Big(\bigoplus_{j=0}^{j=N} L(\gamma, n+2j) \otimes \pi^* \mathrm{Sym}^j(H)\Big) \bigsqcup \Big(\bigoplus_{j=0}^{j=N} L(\gamma, -n-2j) \otimes \pi^* \mathrm{Sym}^j(H^*)\Big)$$

Theorem (PB and Erik van Erp)

$$\mu(X \times S^1 \sqcup \overline{X \times S^1},\ E_\gamma\, ,\ \pi) = [P_\gamma]$$