Unbounded KK-theory and KK-bordisms

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Introduction

In NCG, a "noncommutative manifold" is a spectral triple $(\mathcal{A}, \mathcal{H}, D)$:

$$\begin{cases} \mathcal{H} & \text{is a Hilbert space;} \\ \mathcal{A} \subseteq \mathcal{B}(\mathcal{H}) & \text{is a } *\text{-subalgebra;} \\ D = D^* : \mathcal{H} \dashrightarrow \mathcal{H}, & \text{compatible with } \mathcal{A} \text{ through:} \end{cases}$$

$$\mathcal{A} \subseteq \mathrm{Lip}_0(D) := \{T \in \mathcal{B}(\mathcal{H}) : [D, T] \text{ bounded and } (T \pm T^*)(1 + D^* D)^{-1} \in \mathbb{K}(\mathcal{H})\}.$$

Prototypical example

For a Riemannian manifold M, a Clifford bundle E and the Dirac operator

$$\not{D} : C_c^\infty(M, E) \to C_c^\infty(M, E),$$

we form $D := \overline{\not{D}}$. If M is complete, $(C_c^\infty(M), L^2(M, E), D)$ is a spectral triple.

Noncommutative topology

The noncommutative topology of $(\mathcal{A}, \mathcal{H}, D)$ is described by a class $[\mathcal{A}, \mathcal{H}, D] \in K^*(A)$, where $A = \overline{\mathcal{A}}$ and $K^*(A) := KK_*(A, \mathbb{C})$ its K-homology.

Unbounded Kasparov cycles

An unbounded Kasparov cycle for $(\mathcal{A}, B)$ is a pair $(\mathcal{E}, D)$ where:

- $\mathcal{E}$ is a (graded) (A, B)-Hilbert C^*-module;
- $D : \mathcal{E} \dashrightarrow \mathcal{E}$ is a self-adjoint *regular* operator such that $\mathcal{A} \subseteq \mathrm{Lip}_0(D)$.

Let $Z_*(\mathcal{A}, B)$ denote the semigroup of $(\mathcal{A}, B)$-Kasparov cycles.

Subtleties in fixing $\mathcal{A}$

We can define the **set** of unbounded (A, B)-Kasparov modules:

$$\Psi_*(A, B) := \{(\mathcal{A}, \mathcal{E}_B, D) : \ \mathcal{A} \subseteq A \text{ dense and } (\mathcal{E}_B, D) \text{ an } (\mathcal{A}, B)\text{-cycle}\}.$$

The natural mapping $Z_*(\mathcal{A}, B) \to \Psi_*(A, B)$ is rarely surjective!

Prototypical example

For a complete Riemannian manifold M, a B-bundle $\mathcal{E}_B \to M$ and a Dirac type operator

$$\not{D}_{\mathcal{E}} : C_c^\infty(M, \mathcal{E}_B) \to C_c^\infty(M, \mathcal{E}_B),$$

$(L^2(M, \mathcal{E}_B), D_{\mathcal{E}})$ is an unbounded Kasparov cycle for $(C_c^\infty(M), B)$.

Example of B-bundle: If $\tilde{M}$ is the universal cover of M, its Mishchenko bundle

$$\mathcal{L}_M := \tilde{M} \times_{\pi_1(M)} C^*(\pi_1(M)) \to M,$$

is a flat $C^*(\pi_1(M))$-bundle of "rank" 1. Any Dirac type operator $\not{D}$ on a vector bundle $E \to M$ lifts to a Dirac operator

$$\not{D}_{\mathcal{L}} : C_c^\infty(M, E \otimes \mathcal{L}) \to C_c^\infty(M, E \otimes \mathcal{L})$$

KK-theory

The noncommutative topology is in this case encoded by KK-cycles, a pair $(\mathcal{E}, F)$ where $F \in \mathrm{End}^*_B(\mathcal{E})$ satisfies:

$$a(F^2 - 1),\ a(F^* - F),\ [F, a] \in \mathbb{K}_B(\mathcal{E}),\ \forall a \in A. \tag{1}$$

The bounded transform of an unbounded Kasparov cycle is a KK-cycle:

$$\beta(\mathcal{E}, D) := (\mathcal{E}, D(1 + D^2)^{-1/2}).$$

Relations

1. If all terms in (1) are 0, we say that $(\mathcal{E}, F)$ is *degenerate*.
2. If $F : [0, 1] \to \mathrm{End}^*_B(\mathcal{E})$ is norm-continuous and $(\mathcal{E}, F(t))$ is a KK-cycle for all t, we say that $(\mathcal{E}, F(0))$ and $(\mathcal{E}, F(1))$ are *operator homotopic*.
3. $KK_*(A, B)$ is the abelian group of KK-cycles modulo degenerate cycles and operator homotopy.

Building *KK* from unbounded cycles

Baaj-Julg '83

If A is separable, there is a dense $\mathcal{A} \subseteq A$ such that

$$\beta : Z_*(\mathcal{A}, B) \to KK_*(A, B), \quad \text{is surjective for any } B.$$

The bounded transform defines a surjection $\Psi_*(A, B) \to KK_*(A, B)$.

Motivating question

Can we do *KK*-theory only using unbounded Kasparov theory?

The Answer?

Blackadar writes in his *K*-theory book on page 165:

> "We leave to the reader the task of appropriately formulating the equivalence relations on $\Psi_*(A, B)$ corresponding to the standard relations..."

Some notions of cycles

A symmetric chain...

...is a pair $(\mathcal{E}, D)$ satisfying that D is a symmetric regular operator with

$$\mathcal{A} \subseteq \mathrm{Lip}(D) := \{T \in \mathrm{End}_B^*(\mathcal{E}) : [D, T] \text{ bounded}\}.$$

A half-closed chain...

...is a symmetric chain $(\mathcal{E}, D)$ such that for any $a \in \mathcal{A}$

$$a\mathrm{Dom}(D^*) \subseteq \mathrm{Dom}(D) \quad \text{and} \quad a(1 + D^*D)^{-1} \in \mathbb{K}_B(\mathcal{E}).$$

If W is a compact Riemannian manifold with boundary and D a Dirac type operator on $\mathcal{E}_B \to W$, D defines a symmetric regular operator D_{min} with

$$\mathrm{Dom}(D_{min}) := H_0^1(W, \mathcal{E}_B).$$

The pair $(L^2(W, \mathcal{E}_B, D_{min})$ is a symmetric $(C^\infty(\overline{W}), B)$-chain and a half-closed $(C_c^\infty(W^\circ), B)$-chain.

Hilsum bordisms

Let $(\mathcal{F}, Q)$ be a symmetric chain (odd) and $(\mathcal{E}, D)$ a cycle (even) for $(\mathcal{A}, B)$.

Boundary data

Boundary data for $(\mathcal{E}, D)$ relative to $(\mathcal{F}, Q)$ is a pair (θ, p) where

- $p \in \mathrm{End}^*_B(\mathcal{F})$ is a projection commuting with A;
- $\theta : p\mathcal{F} \to \mathcal{E} \otimes L^2[0,1]$ is an isomorphism.

- We define $b : C[0,1] \otimes A \to \mathrm{End}^*_B(\mathcal{F})$ by

$$b(\varphi \otimes a) := \theta^{-1}(\varphi \otimes a)\theta p + \varphi(1)a(1-p).$$

- We define $\Psi(D) : \mathcal{E} \otimes L^2[0,1] \dashrightarrow \mathcal{E} \otimes L^2[0,1]$ as the closure of

$$i\gamma_{\mathcal{E}}\partial_x + D : C_c^\infty((0,1), \mathrm{Dom}(D)) \to \mathcal{E} \otimes L^2[0,1].$$

Hilsum bordisms (almost there)

Cycles with boundary

We say that $(\mathcal{E}, D)$ is a boundary of $(\mathcal{F}, Q)$ with boundary data (θ, p) if

1. For $\varphi \in C_c^\infty(0,1]$ it holds that
$$b(\varphi)\mathrm{Dom}\,Q^* \subseteq \mathrm{Dom}\,Q \quad \text{and} \quad Q^* b(\varphi) = Qb(\varphi) \quad \text{on} \quad \mathrm{Dom}\,Q^*.$$
2. For $\varphi \in C_c^\infty(0,1)$ it holds that
$$\varphi \mathrm{Dom}\Psi(D) = \theta b(\varphi)\mathrm{Dom}\,Q \quad \text{and} \quad Q = \theta^{-1}\Psi(D)\theta \quad \text{on} \quad b(\varphi)\mathrm{Dom}\,Q.$$
3. For $\varphi_1, \varphi_2 \in C^\infty[0,1]$ satisfying $\varphi_1\varphi_2 = 0$ it holds that
$$b(\varphi_1)Qb(\varphi_2) = 0.$$

Hilsum bordisms (for real this time)

Proposition (Hilsum)

If $(\mathcal{E}, D)$ is the boundary of $(\mathcal{F}, Q, \theta, p)$, $(\mathcal{F}, Q)$ becomes a symmetric $(C_c^\infty(0,1] \otimes \mathcal{A}, B)$-chain under $b : C[0,1] \otimes A \to \mathrm{End}_B^*(\mathcal{F})$ restricting to a half-closed $(C_c^\infty(0,1) \otimes \mathcal{A}, B)$-chain.

Definition: Hilsum bordisms

A bordism is a chain with boundary $(\mathcal{F}, Q, \theta, p)$ defining a half-closed $(C_c^\infty(0,1] \otimes \mathcal{A}, B)$-chain. We write

$$\partial(\mathcal{F}, Q, \theta, p) = (\mathcal{E}, D).$$

If $\partial(\mathcal{F}, Q, \theta, p) = (\mathcal{E}, D) - (\mathcal{E}', D')$ we write $(\mathcal{E}, D) \sim_{bor} (\mathcal{E}', D')$.

Theorem (Hilsum)

If $\partial(\mathcal{F}, Q, \theta, p) = (\mathcal{E}, D)$, then $\beta(\mathcal{E}, D) = 0$ in $KK_*(A, B)$.

The "obvious" example

The spin^c-bordism group

Classically, for a (compact) topological space X, the spin^c-bordism group is:

$$\Omega_*^{\mathrm{spin}^c}(X) := \{(M, f) : M \text{ closed spin}^c\text{-manifold and } f : M \to X\}/\sim_{bor},$$

where $(M, f) \sim_{bor} 0$ if $(M, f) = \partial(W, g)$. Bordism invariance gives a mapping

$$\Omega_*^{\mathrm{spin}^c}(X) \ni (M, f) \mapsto f_*[D^M] = [(L^2(M, S_M), D^M)] \in K_*(X) = KK_*(C(X), \mathbb{C}).$$

The associated Hilsum bordism

For a Riemannian compact manifold X, the algebra of interest is $\mathcal{A} = \mathrm{Lip}(X)$. On a compact Riemannian spin^c-manifold with boundary W, and

$$g_W = \mathrm{d}x^2 + g_{\partial W} \quad \text{in a collar neighborhood} \quad [0,1] \times \partial W \subseteq W \quad \text{of } \partial W.$$

Assume that $f : W \to X$ is Lipschitz; inducing an action of $\mathrm{Lip}(X)$ on $L^2(W, S_W)$ as well as $H^1(W, S_W)$.

The collection $(L^2(W, S_W), D^W, \mathrm{id}, \chi_{[0,1]\times\partial W})$ is a bordism of $(\mathrm{Lip}(X), \mathbb{C})$-cycles with

$$\partial(L^2(W, S_W), D^W, \mathrm{id}, \chi_{[0,1]\times\partial W}) = (L^2(\partial W, S_{\partial W}), D^{\partial W}).$$

Main results

Theorem (Deeley-Goffeng-Mesland '14)

Bordism defines an equivalence relation defining an abelian group:

$$\Omega_*(\mathcal{A}, B) := Z_*(\mathcal{A}, B)/\sim_{bor}.$$

This construction satisfies:

1. The bounded transform induces a homomorphism
$$\beta : \Omega_*(\mathcal{A}, B) \to KK_*(A, B).$$
For any separable C^*-algebra A there is a dense $*$-subalgebra $\mathcal{A} \subseteq A$ making β surjective.
2. If $\mathcal{A} = \mathbb{C}$, $\Omega_*(\mathbb{C}, B) \cong K_*(B)$ via β.
3. If $C^\infty(X) \subseteq \mathcal{A} \subseteq \mathrm{Lip}(X)$ for a closed Riemannian manifold X, the bounded transform $\beta : \Omega_*(\mathcal{A}, B) \to KK_*(C(X), B)$ is split-surjective.
4. If $\mathcal{I} \subseteq \mathcal{K}(\mathcal{H})$ is a regular symmetrically normed ideal of compact operators the subgroup $\Omega^e_*(\mathcal{I}, B)$ generated by essential cycles satisfies $\Omega^e_*(\mathcal{I}, B) \cong K_*(B)$.

The bread and butter of bordisms

The homotopy lemma

Let $(\mathcal{E}, D_0)$ and $(\mathcal{E}, D_1)$ be cycles. Assume that there is a dense submodule $\mathcal{W} \subseteq \mathcal{E}$ and $\hat{\not{D}} : [0,1] \to \mathrm{Hom}_B(\mathcal{W}, \mathcal{E})$ such that

1. $\hat{\not{D}}$ defines an essentially self-adjoint regular $C[0,1] \otimes B$-linear operator on $C[0,1] \otimes \mathcal{E}$ with closure $\hat{D}$ such that
   - the pointwise defined derivative $\hat{\not{D}}' : [0,1] \to \mathrm{Hom}_B(\mathcal{W}, \mathcal{E})$ exists and
   - $\hat{\not{D}}'(1 + \hat{D}^2)^{-\frac{1}{2}}$ extends to a bounded operator on $C[0,1] \otimes \mathcal{E}$.
2. $(\mathcal{E}, \hat{D}(t))$ defines a cycle for any t and $\sup_t \|[\hat{D}(t), a]\|_{\mathrm{End}^*_B(\mathcal{E})} < \infty$, $\forall a \in \mathcal{A}$;
3. $\hat{D}(j) = D_j$ in the endpoints $j = 0, 1$.

Then
$$(\mathcal{E}, D_0) \sim_{bor} (\mathcal{E}, D_1).$$

Corollary: Bounded perturbations

The class of $(\mathcal{E}, D)$ in $\Omega_*(\mathcal{A}, B)$ is not changed by ε-bounded perturbations of D.

Degenerate cycles

Definition: Degeneracy of cycles

Let $(\mathcal{E}, D)$ be an $(\mathcal{A}, B)$-cycle. If $D = D_0 + S$ where

1. D_0 and S are self-adjoint regular operators;
2. S admits a bounded inverse and commutes with the A-action;
3. S and D_0 preserve each others domains and

$$[D_0, S]_s = 0 \quad \text{on} \quad \mathrm{Dom} D_0 S = \mathrm{Dom} S D_0.$$

4. $\pi(\mathcal{A}) \subseteq \mathrm{Lip}(D_0)$,

we say that $(\mathcal{E}, D)$ is weakly degenerate.

When $D_0 = 0$, the terminology degenerate cycle was coined by Kucerovsky motivated by that if $(\mathcal{E}, D)$ is degenerate, the bounded Kasparov cycle $(\mathcal{E}, D|D|^{-1})$ is degenerate.

Degenerate cycles, continued

Theorem: Weak degeneracies and bordisms

If $(\mathcal{E}, D)$ is weakly degenerate it is null-bordant.

Sketch of proof.

Take $\mathcal{F} = \mathcal{E} \otimes L^2(\mathbb{R}_+)$ and Q as the closure of

$$i\gamma\partial_x + D_0 + XS : C_c^\infty((0,\infty), \mathrm{Dom}(D)) \to \mathcal{F},$$

where $X \in C^\infty(\mathbb{R}_+)$ is given by

$$X(x) = \begin{cases} 1, & \text{near } x = 0 \\ \sqrt{1+x^2}, & \text{for large } x \end{cases}.$$

Then $Q^*Q \approx -\partial_x^2 + x^2 S^2 + (S^2 + D^2)$. $\square$

Weakly degenerate versus degenerate

We say that a bounded cycle $(\mathcal{E}, F)$ is weakly degenerate if F is invertible and $(\mathcal{E}, F|F|^{-1})$ is degenerate.

Proposition: Lifting weak degeneracies

If $(\mathcal{E}, D)$ satisfies that $b(\mathcal{E}, D) = (\mathcal{E}, D(1 + D^2)^{-1/2})$ is weakly degenerate, then $(\mathcal{E}, D)$ is null bordant.

Proof.

$$(\mathcal{E}, D) \sim_{bor} (\mathcal{E}, D + \gamma_{\mathcal{E}} D|D|^{-1}).$$

The decomposition $D + \gamma_{\mathcal{E}} D|D|^{-1} = D_0 + S$, where $D_0 = D$ and $S = \gamma_{\mathcal{E}} D|D|^{-1}$, shows that $D + \gamma_{\mathcal{E}} D|D|^{-1}$ is weakly degenerate □

Question

If A and B are separable C^*-algebras, does there exist a dense $*$-subalgebra $\mathcal{A} \subseteq A$ for which the bounded transform

$$\beta : \Omega_*(\mathcal{A}, B) \to KK_*(A, B) \quad \text{is an isomorphism?}$$

Thanks

Thanks for your attention!