Heat trace for Laplace operators with non-scalar symbols

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Noncommutative index theory
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Asymptotics of heat kernel

(M, g) compact Riemannian manifold
$\dim(M) = d, \qquad \partial M = \emptyset$

P diff. operator order 2, elliptic, on vector bundle V over M
$\sigma(P)$ is definite positive

Then

$$\mathrm{Tr}_{L^2}\, e^{-t\,P} \underset{t\downarrow 0^+}{\sim} \sum_{r=0}^{\infty} a_r(P)\, t^{r-d/2}$$

Example: Laplace-type operators

$$-[\mathbb{1}_V\, g^{\mu\nu}(x)\, \partial_\mu \partial_\nu + v^\nu(x)\, \partial_\nu + w(x)]$$

Importance of coefficients

Just to quote a few

Physics: $d = 4$,

$a_1(P),\ a_2(P)$ give one-loop renormalization, …

If $\mathcal{D}$ is a Dirac operator, $a_2(\mathcal{D}^2) \sim$ Einstein–Hilbert action

Mathematics:

$\zeta_P(0) \sim a_{d/2}, \quad \zeta_P(s) := \mathrm{Tr}\, P^{-s}$ for $\Re(s)$ large with $P > 0$

$V = V^+ \oplus V^-, \quad V^\pm$ complex hermitian bundles,

$D : C^\infty(V^+) \to C^\infty(V^-)$ Dirac-type op., $\quad D_V := \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}$

$$\mathrm{Index}(D_V) = \lim_{t\to 0} \mathrm{Tr}[e^{-t\,DD^*} - e^{-t\,D^*D}] = \lim_{t\to 0} \mathrm{STr}\, e^{-t\,D_V^2}$$

The different methods

Analytical: expansion of integral kernel $K(t,x,x) \underset{t\downarrow 0^+}{\sim} \sum_{r=0}^{\infty} a_r(P)\, t^{r-d/2}$

Variant in spectral geometry: search for invariants or conformal & gauge covariance

Pseudodifferential operators: symbols, parametrics etc

$$e^{-t\,P} = \tfrac{i}{2\pi} \int_{\mathcal{C}} d\lambda\, e^{-\lambda t}\, (P-\lambda)^{-1}$$

$\mathcal{C}$ oriented curve around $\mathbb{R}^+$.

Where are the difficulties if principal symbol is not scalar?

Computation of heat-coefficients

General non-minimal second order differential operator:

$$P = -[u^{\mu\nu}(x)\,\partial_\mu\partial_\nu + v^\nu(x)\,\partial_\nu + w(x)]$$

$u^{\mu\nu},\ v^\mu,\ w$ are $N \times N$ matrices

One difficulty apart complexity

$\sigma_2(x,\xi) = u^{\mu\nu}(x)\,\xi_\mu\xi_\nu$ assume strictly positive eigenvalues, $\forall(x,\xi) \in T^*M$

$\sigma_1(x,\xi) = -iv^\mu(x)\,\xi_\mu$

$\sigma_0(x,\xi) = -w(x)$

$$\sigma_i \in M_N$$

Computation of heat-coefficients

$$a_r(P) = \frac{1}{(2\pi)^d}\,\frac{i}{2\pi}\int_{\lambda\in\mathcal{C}} d\lambda\, dx\, d\xi\, e^{-\lambda}\,\mathrm{tr}\,[b_{2r}(x,\xi,\lambda)]$$

$\lambda \in \mathcal{C}$ and $(x,\,\xi) \in T^*(M)$

$$b_0(x,\xi,\lambda) := (\sigma_2(x,\xi) - \lambda)^{-1} \ \in M_N$$
$$b_r(x,\xi,\lambda) := -\sum_{\substack{r=j+|\alpha|+2-k\\ j<r}} \frac{(-i)^{|\alpha|}}{\alpha!}\,(\partial_\xi^\alpha b_j)\,(\partial_x^\alpha \sigma_k)\, b_0$$

Functions b_{2r}, even for $r = 1$, generate terms like

$$\mathrm{tr}\,[A_1(\lambda)\, B_1\, A_2(\lambda)\, B_2\, A_3(\lambda)\cdots]$$

$A_i(\lambda) = (\sigma_2(x,\xi) - \lambda)^{-n_i}$ do not commute with B_i

Integral in λ is difficult

Computation of heat-coefficients

If one uses the spectral decomposition $\sigma_2 = \sum_i \lambda_i E_i$

$$\operatorname{tr}[A_1(\lambda)\, B_1\, A_2(\lambda)\, B_2\, A_3(\lambda)\cdots]$$

$$= \sum_{i_1,i_2,i_3,\ldots} \Big[\int_{\lambda\in\mathcal{C}} d\lambda\, e^{-\lambda}\, (\lambda_{i_1}-\lambda)^{-n_{i_1}} (\lambda_{i_2}-\lambda)^{-n_{i_2}} (\lambda_{i_3}-\lambda)^{-n_{i_3}} \cdots \Big]$$

$$\operatorname{tr}(E_{i_1}\, B_1\, E_{i_2}\, B_2\, E_{i_3}\cdots)$$

λ-integral is easy

How to recombine the sum?

All coefficients are known
but not explicitly in terms of $u^{\mu\nu}$, v^{μ}, w

References

Only few previous works

Differential forms: $P = c_1 d\delta + c_2 \delta d + E$ with $c_1 \neq c_2$
Gilkey–Branson–Fulling (1991)
Branson–Gilkey–Pierzchalski (1994)
Alexandrov–Vassilevich (1996) (computation of all a_r)

General result for a_1:
Avramidi–Branson (2001)

Special cases: $u^{\mu\nu} = g^{\mu\nu}\mathbb{1} + X^{\mu\nu}$
Gusynin–Gorbar–Korniyak–Romankov (1991-2000)
Guendelman–Leonidov–Nechitailo–Owen (1994)
Ananthanarayan (2008)
Moss–Toms (2014) (computation of a_1, a_2 for particular $X^{\mu\nu}$)

Another approach

Functional approach for kernel of $e^{-t\,P}$ based on the Volterra series

$K(t,x,x')$ = kernel of $e^{-t\,P}$

$$\begin{aligned}
\mathrm{Tr}[e^{-t\,P}] &= \int dx\ \mathrm{tr}[K(t,x,x)],\\
K(t,x,x) &\sim \int d\xi\, e^{-t\,(H+K+P)}\,\mathbb{1}\\
&= \tfrac{1}{t^{d/2}}\int d\xi\, e^{-H-\sqrt{t}\,K-t\,P}\mathbb{1}, \qquad \xi \to t^{1/2}\,\xi
\end{aligned}$$

where

$$\begin{aligned}
H(x,\xi) &:= u^{\mu\nu}(x)\,\xi_\mu\xi_\nu\\
K(x,\xi) &:= -i\,\xi_\mu\,[v^\mu(x) + 2u^{\mu\nu}(x)\,\partial_\nu]
\end{aligned}$$

Another approach

Duhamel formula

$$e^{A+B} = e^{A} + \sum_{k=1}^{\infty} \int_0^1 ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{k-1}} ds_k$$
$$e^{(1-s_1)A}\, B\, e^{(s_1-s_2)A} \cdots e^{(s_{k-1}-s_k)A}\, B\, e^{s_k A}$$

After integration in ξ

$$\mathrm{tr}\, K(t,x,x) \underset{t\downarrow 0}{\simeq} \tfrac{1}{t^{d/2}}[a_0(x) + t\, a_1(x) + t^2\, a_2(x) + \mathcal{O}(t^4)]$$

with

$$a_0(x) = \mathrm{tr}\, \tfrac{1}{(2\pi)^d} \int d\xi\, e^{-H(x,\xi)}$$

$$a_1(x) = \mathrm{tr}\, \tfrac{1}{(2\pi)^d} \int d\xi\, \Big[\int_0^1 ds_1 \int_0^{s_1} ds_2\, e^{(s_1-1)H}\, K\, e^{(s_2-s_1)H}\, K\, e^{-s_2 H} \Big]$$
$$- \mathrm{tr}\, \tfrac{1}{(2\pi)^d} \int d\xi\, \Big[\int_0^1 ds_1\, e^{(s_1-1)H}\, P\, e^{-s_1 H} \Big]$$

Another approach

Simplex

$$\Delta_k := \{ s = (s_0, \cdots s_k) \in \mathbb{R}_+^{k+1} \mid 0 \leq s_k \leq s_{k-1} \leq \cdots \leq s_2 \leq s_1 \leq s_0 = 1 \}$$

Expanding K and P

$$\int d\xi \ \int_{\Delta_k} ds \ \ e^{(s_1 - 1)H} \, B_1 \, e^{(s_2 - s_1)H} \, B_2 \cdots B_k \, e^{-s_k H}$$

where $B_i = u^{\mu\nu}, \ v^{\mu}, \ w$ or their derivatives (order 2 at most)
ξ-dependance:

H = polynomial order 2

B_i = polyno.

Algebraic method

Rearrangement lemma (Connes–Moscovici, Lesch)

Idea

First step: View

$$\int_{\Delta_k} ds \;\; e^{(s_1-1)H}\, B_1\, e^{(s_2-s_1)H}\, B_2 \cdots B_k\, e^{-s_k H}$$

as an operator f_k acting on $M_N^{\otimes^k}$

$$f_k(\xi):\; B_1 \otimes \cdots \otimes B_k \to \int_{\Delta_k} ds\, e^{(s_1-1)H}\, B_1\, e^{(s_2-s_1)H}\, B_2 \cdots B_k\, e^{-s_k H}$$

Thus

$$a_0(x) \sim \mathrm{tr} \int d\xi\, f_0[1]$$

$$a_1(x) \sim \mathrm{tr} \int d\xi\, (f_2[K \otimes K] - f_1[P])$$

$$a_2(x) \sim \mathrm{tr} \int d\xi (f_2[P \otimes P] - f_3[K \otimes K \otimes P] - f_3[K \otimes P \otimes K] - f_3[P \otimes K \otimes K])$$

Idea

Second step: erase ξ using

$$\partial e^{-sH} = -\int_0^s ds_1\, e^{(s_1-s)H}\,(\partial H)\,e^{-s_1 H}$$

and Leibniz rule $\quad (K \sim \xi_\mu\,[v^\mu + 2u^{\mu\nu}\,\partial_\nu])$

$$f_k(\xi)[B_1 \otimes \cdots \otimes B_i\partial \otimes \cdots \otimes B_k] = \sum_{j=i+1}^{k} f_k(\xi)[B_1 \otimes \cdots \otimes (\partial B_j) \otimes \cdots \otimes B_k]$$

$$- \sum_{j=i}^{k} f_{k+1}(\xi)[B_1 \otimes \cdots \otimes B_j \otimes (\partial H) \otimes B_{j+1} \otimes \cdots \otimes B_k]$$

Conclusion

$$\int d\xi\, \xi_{\mu_1} \cdots \xi_{\mu_\ell}\, f_k(\xi)[\,\mathbb{B}_k^{\mu_1\ldots\mu_\ell}] \in M_N$$

with $\mathbb{B}_k^{\mu_1\ldots\mu_\ell} \in M_N^{\otimes^k}$ independent of ξ

Idea

Next step: use algebra to rewrite operators $f_k : M_N^{\otimes^k} \to M_N$
Framework : Bounded operators on Hilbert spaces
Define Hilbert space

$$M_N^{\otimes^k} \quad \text{with Hilbert–Schmidt norm}$$

$$\mathbf{m}\,(B_0 \otimes \cdots \otimes B_k) := B_0 \cdots B_k$$

$$\kappa\,(B_1 \otimes \cdots \otimes B_k) := \mathbb{1} \otimes B_1 \otimes \cdots \otimes B_k$$

$$\iota\,(A_0 \otimes \cdots \otimes A_k)[B_1 \otimes \cdots \otimes B_k] := A_0 B_1 A_1 \cdots B_k A_k$$

For any matrix $A \in M_N$, define

$$R_i(A)\,[B_0 \otimes \cdots \otimes B_k] := B_0 \otimes \cdots \otimes B_i \underset{\uparrow_i}{A} \otimes \cdots \otimes B_k$$

$$\rho\,(A_0 \otimes \cdots \otimes A_k) := R_0(A_0) \cdots R_k(A_k)$$

Idea

Links between $M_N^{\otimes^{k+1}}$, $\mathcal{B}(M_N^{\otimes^{k+1}})$ and $\mathcal{B}(M_N^{\otimes^k}, M_N)$

$$\begin{tikzcd}
 & \mathcal{B}(M_N^{\otimes^{k+1}}) \arrow[d, "m \circ \kappa^*"] \\
M_N^{\otimes k+1} \arrow[ur, "\rho"] \arrow[r, "\simeq", "\iota"'] & \mathcal{B}(M_N^{\otimes^k}, M_N)
\end{tikzcd}$$

$$c_k(s, A) := (1 - s_1)\, A \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \;+\; \mathbb{1} \otimes (s_1 - s_2)\, A \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \\ + \cdots + \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes s_k\, A$$

Conclusion

$$f_k(\xi) = \int_{\Delta_k} ds \;\iota\,[e^{-\xi_\mu \xi_\nu \, c_k(s, u^{\mu\nu})}] \in \mathcal{B}(M_N^{\otimes^k}, M_N)$$

is computable and independent on the variables $B_0 \otimes \cdots \otimes B_k$

Idea

$$C_k(s, A) := \rho\,[c_k(s, A)] \in \mathcal{B}(M_N^{\otimes^{k+1}})$$

$$C_k(s, A) = (1 - s_1)\, R_0(A) + (s_1 - s_2)\, R_1(A) + \cdots + s_k\, R_k(A)$$

In particular

$$C_k(s, A) \geq 0 \;\text{ if }\; A \geq 0,\; s \in \Delta_k$$

justifies previous lift

For $p \in \mathbb{N},\; k \in \mathbb{N}$

$$T_{k,p}(x) := \int_{\Delta_k} ds \int d\xi\, \xi_{\mu_1} \cdots \xi_{\mu_{2p}}\, e^{-\xi_\alpha \xi_\beta\, C_k(s, u^{\alpha\beta}(x))} \;\in \mathcal{B}(M_N^{\otimes^{k+1}})$$

Idea

ξ-integration

spherical coordinates: $\xi = r\,\sigma$ with $r = (g^{\mu\nu}\xi_\mu\xi_\nu)^{1/2}$, $\sigma = r^{-1}\xi \in S_g^{d-1}$

$$u[\sigma] := u^{\mu\nu}\sigma_\mu\sigma_\nu \qquad \text{(positive definite matrix)}$$

$$\begin{aligned} T_{k,p}(x) &= \int_{\Delta_k} ds \int d\xi\, \xi_{\mu_1}\cdots\xi_{\mu_{2p}}\, e^{-\xi_\alpha\xi_\beta\, C_k(s,u^{\alpha\beta}(x))} \in \mathcal{B}(M_N^{\otimes^{k+1}}) \\ &= \int_{\Delta_k} ds \int_{S_g^{d-1}} d\Omega_g(\sigma)\,\sigma_{\mu_1}\cdots\sigma_{\mu_{2p}} \int_0^\infty dr\, r^{d-1+2p}\, e^{-r^2 C_k(s,u[\sigma])} \\ &= \tfrac{\Gamma(d/2+p)}{2} \int_{S_g^{d-1}} d\Omega_g(\sigma)\,\sigma_{\mu_1}\cdots\sigma_{\mu_{2p}} \int_{\Delta_k} ds\;\, C_k(s,u[\sigma])^{-(d/2+p)} \end{aligned}$$

s-integration

$$I_{\alpha,k}(r_0,r_1,\dots,r_k) := \int_{\Delta_k} ds\,[(1-s_1)r_0 + (s_1-s_2)r_1 + \cdots + s_k r_k]^{-\alpha}$$

$\alpha = d/2 + p$, apply functional calculus:

$$I_{\alpha,k}\big(R_0(u[\sigma]),\dots,R_k(u[\sigma])\big) = \sum_{i_0,\dots,i_k} I_{\alpha,k}(\lambda_{i_0},\dots,\lambda_{i_k})\,R_0(E_{i_0}[\sigma])\cdots R_k(E_{i_k}[\sigma])$$

About combinatorics

Theorem

For a d-dimensional manifold M
computation of $a_r(P)$ needs the $3r+1$ integrals
$I_{d/2,r},\ I_{d/2+1,r+1},\ \dots,\ I_{d/2+3r,4r}$

$$I_{\alpha,k}(r_0, r_1, \dots, r_k) = \int_{\Delta_k} ds\,[(1-s_1)\,r_0 + (s_1-s_2)\,r_1 + \cdots + s_k r_k]^{-\alpha}$$

Computations of $I_{\alpha,k}$ for a_r

$a_{d/2}$ is invariant by dilation in $e^{-t\,P}$

Proposition (d even)

$r < d/2$

For $n \in \mathbb{N}$, $n \geq k+1$ and $k \in \mathbb{N}^*$

$$I_{n,k}(r_0,\dots,r_k)$$
$$= \tfrac{(r_0\cdots r_k)^{-(n-k)}}{(n-1)\cdots(n-k)} \sum_{\substack{0\leq l_k \leq l_{k-1}\leq\cdots \\ \cdots\leq l_1\leq n-(k+1)}} r_0^{l_1}\, r_1^{l_2+(n-(k+1))-l_1} \cdots r_{k-1}^{l_k+(n-(k+1))-l_{k-1}}\, r_k^{(n-(k+1))-l_k}$$

$d/2 \leq r$

For any $k \in \mathbb{N}^*$, and $\alpha_0 = d/2 - r \in \{0,1,\cdots,k-1\}$

$$I_{k-d/2+r,k}(r_0,\dots,r_k) = \tfrac{(-1)^{k-\alpha_0-1}}{(k-\alpha_0-1)!\,\alpha_0!} \sum_{i=0}^{k} \Big[\prod_{\substack{j=0\\ j\neq i}}^{k} (r_i - r_j)^{-1}\Big]\, r_i^{\alpha_0} \log r_i$$

Computations of $I_{\alpha,k}$ for a_r

Proposition (d odd)

If $d/2 - r = \ell + 1/2$ *with* $\ell \in \mathbb{N}$

$$I_{\ell+1/2,0}(r_0) = r_0^{-\ell-1/2},$$
$$I_{\ell+3/2,1}(r_0, r_1) = \tfrac{2}{2\ell+1}(\sqrt{r_0}\sqrt{r_1})^{-2\ell-1}(\sqrt{r_0}+\sqrt{r_1})^{-1}\sum_{0\le l_1\le 2\ell}\sqrt{r_0}^{\,l_1}\sqrt{r_1}^{\,2\ell-l_1}$$

If $d/2 - r = -\ell - 1/2$ *with* $\ell \in \mathbb{N}$

$$I_{-\ell-1/2,0}(r_0) = r_0^{\ell+1/2},$$
$$I_{-\ell+1/2,1}(r_0, r_1) = \tfrac{2}{2\ell+1}(\sqrt{r_0}+\sqrt{r_1})^{-1}\sum_{0\le l_1\le 2\ell}\sqrt{r_0}^{\,l_1}\sqrt{r_1}^{\,2\ell-l_1}$$

Example $u^{\mu\nu} = g^{\mu\nu} u$

Hypothesis:

$$u^{\mu\nu}(x) := g^{\mu\nu}(x)\, u(x)$$

implies that

$$H(x,\xi) = u^{\mu\nu}\, \xi_\mu \xi_\nu = |\xi|^2_{g(x)}\, u(x)$$

Gaussian integrals

Example $u^{\mu\nu} = g^{\mu\nu} u$ and $d = 2m$ even

Theorem

If $P = -(u\, g^{\mu\nu}\partial_\mu\partial_\nu + v^\nu\partial_\nu + w)$ *is selfadjoint elliptic acting on* $L^2(M, V)$
(M, g) *: Riemannian manifold and* V*: vector bundle over* M
u, v^μ, w *are local maps on* M *with values in* M_N*,* u *positive and invertible*
Then

$$
\begin{aligned}
a_1 = \frac{\sqrt{|g|}}{2^{2m}\,\pi^m}\, \Big(& \alpha \operatorname{tr}(u^{-m+1}) + \operatorname{tr}(u^{-m} w) \\
& + \tfrac{m-2}{6}\big[\tfrac{1}{2} g^{\mu\nu} g_{\rho\sigma}(\partial_\nu g^{\rho\sigma}) - (\partial_\nu g^{\mu\nu})\big] \operatorname{tr}(u^{-m}\partial_\mu u) \\
& - \tfrac{m-2}{6} g^{\mu\nu} \operatorname{tr}(u^{-m}\partial_\mu\partial_\nu u) + \tfrac{1}{2} g_{\mu\nu}(\partial_\rho g^{\rho\nu}) \operatorname{tr}(u^{-m} v^\mu) - \frac{1}{2}\operatorname{tr}(u^{-m}\partial_\mu v^\mu) \\
& - \tfrac{1}{4m}\sum_{\ell=0}^{m-1} g_{\mu\nu} \operatorname{tr}(u^{-\ell-1} v^\mu u^{\ell-m} v^\nu) + \tfrac{1}{2m}\sum_{\ell=0}^{m-1} (m-2\ell)\, \operatorname{tr}[u^{-\ell-1} v^\mu\, u^{\ell-m}(\partial_\mu u)] \\
& + \tfrac{1}{6m}\sum_{\ell=0}^{m-1} [m^2 - 2m - 3\ell(m-\ell-1)]\, g^{\mu\nu} \operatorname{tr}[u^{-\ell-1}(\partial_\mu u)\, u^{\ell-m}(\partial_\nu u)] \Big)
\end{aligned}
$$

Example $u^{\mu\nu} = g^{\mu\nu} u$ and d even

$$
\begin{aligned}
\alpha := {} & \tfrac{1}{3}(\partial_\mu \partial_\nu g^{\mu\nu}) - \tfrac{1}{12} g^{\mu\nu} g_{\rho\sigma}(\partial_\mu \partial_\nu g^{\rho\sigma}) + \tfrac{1}{48} g^{\mu\nu} g_{\rho\sigma} g_{\alpha\beta}(\partial_\mu g^{\rho\sigma})(\partial_\nu g^{\alpha\beta}) \\
& + \tfrac{1}{24} g^{\mu\nu} g_{\rho\sigma} g_{\alpha\beta}(\partial_\mu g^{\rho\alpha})(\partial_\nu g^{\sigma\beta}) - \tfrac{1}{12} g_{\rho\sigma}(\partial_\mu g^{\mu\nu})(\partial_\nu g^{\rho\sigma}) \\
& + \tfrac{1}{12} g_{\rho\sigma}(\partial_\mu g^{\nu\rho})(\partial_\nu g^{\mu\sigma}) - \tfrac{1}{4} g_{\rho\sigma}(\partial_\mu g^{\mu\rho})(\partial_\nu g^{\nu\sigma})
\end{aligned}
$$

a_1 is gauge covariant

Diffeomorphism invariance and gauge covariance

Change of coordinates

Gauge transformation

P is well defined on sections of V if and only if $u^{\mu\nu}$, v^{μ} and w satisfy some relations

∇_μ = (gauge) covariant derivative on V:

$$\nabla_\mu s := \partial_\mu s + A_\mu s \quad \text{for section } s \text{ of } V$$

$$P = -(|g|^{-1/2}\nabla_\mu |g|^{1/2} u^{\mu\nu}\nabla_\nu + p^\mu \nabla_\mu + q)$$

$$p^\mu = v^\mu - \tfrac{1}{2}(\partial_\nu \log|g|)u^{\mu\nu} - \partial_\nu u^{\mu\nu} + u^{\mu\nu}A_\nu - A_\nu u^{\mu\nu}$$

$$q = w - \tfrac{1}{2}(\partial_\mu \log|g|)u^{\mu\nu}A_\nu - (\partial_\mu u^{\mu\nu})A_\nu - u^{\mu\nu}(\partial_\mu A_\nu) - A_\mu u^{\mu\nu}A_\nu - p^\mu A_\mu$$

Diffeomorphism invariance and gauge covariance

$\widehat{\nabla}_\mu$ = connection combining ∇_μ and linear connection induced by g

Theorem

$P = -(|g|^{-1/2}\nabla_\mu |g|^{1/2} g^{\mu\nu} u \nabla_\nu + p^\mu \nabla_\mu + q)$ is selfadjoint elliptic
u, p^μ, q are sections of endomorphisms on V with u positive and invertible

$$\begin{aligned}
a_1 = \tfrac{\sqrt{|g|}}{2^{2m}\,\pi^m}\Big(&\tfrac{1}{6} R \,\mathrm{tr}[u^{1-m}] + \mathrm{tr}[u^{-m} q] - \tfrac{m+1}{6} g^{\mu\nu}\,\mathrm{tr}[u^{-m} \widehat{\nabla}_\mu \widehat{\nabla}_\nu u] \\
&- \tfrac{1}{2}\,\mathrm{tr}[u^{-m}\widehat{\nabla}_\mu p^\mu] + \sum_{\ell=0}^{m-1} \tfrac{2m(m+1)+6\ell(\ell-m-1)-3}{12m} g^{\mu\nu}\,\mathrm{tr}[u^{-\ell-1}(\widehat{\nabla}_\mu u) u^{\ell-m} (\widehat{\nabla}_\nu u)] \\
&+ \tfrac{1}{2m} \sum_{\ell=0}^{m-1} (m-2\ell-1)\,\mathrm{tr}[u^{-\ell-1} p^\mu u^{\ell-m} (\widehat{\nabla}_\mu u)] \\
&- \tfrac{1}{4m} \sum_{\ell=0}^{m-1} g_{\mu\nu}\,\mathrm{tr}[u^{-\ell-1} p^\mu u^{\ell-m} p^\nu]\Big)
\end{aligned}$$

Usual formula if $u = \mathbb{1}$!

Explicit formulae

This machinery gives all coefficients a_r

(computer is welcome!)

Is the heat coefficients computing solved?

On the method

This machinery gives all coefficients a_r

(computer is welcome!)

Is the heat-coefficients computing problem solved?

Yes, but the formulae are not necessarily explicit!

Explicit formulae?

In odd dimension, a_1 is never explicit

(unless u and v^μ have commutation relations)

$$\frac{1}{R_1(u) + R_2(u)} \; B_0 \otimes B_1 \otimes B_2 \otimes B_3 \qquad \text{is never explicit :}$$

Remark:

$$\frac{1}{x+y} \neq \sum_{\text{finite}} h_{(1)}(x)\, h_{(2)}(y)$$

for any continuous functions $h_{(i)}$

Extension to Physics old and new

Method works with

$$u^{\mu\nu} = g^{\mu\nu}\mathbb{1} + c\,X^{\mu\nu}$$

$X^{\mu\nu}$ is a projection

Navier equations: $v(t,x) : \mathbb{R} \times \Omega \to \mathbb{R}^3$

$$\rho_0 \frac{\partial v}{\partial t^2} = \mu\,\Delta + (\lambda + \mu)\,\nabla\,(\nabla\cdot v) + f, \qquad \lambda, \mu \text{ Lamé constants}$$

Bundle = (co)tangent bundle of M

$$(X^{\mu\nu})^{\beta}{}_{\alpha} = \tfrac{1}{2}\,(g^{\mu\beta}\,\delta^{\nu}_{\alpha} + g^{\nu\beta}\,\delta^{\mu}_{\alpha})$$

Yang–Mills field, quantum field theory of gravity and the like

$$P^{\beta}{}_{\alpha} = -\delta^{\alpha}_{\beta}\,D^2 - c\,D^{\beta}D_{\alpha} + R^{\beta}{}_{\alpha} - F^{\beta}{}_{\alpha}$$

$$D_{\alpha} = \nabla_{\alpha} + A_{\alpha}$$

Conclusion

For computation of heat coefficients of Laplace type operators

- solved the most simple extension with non scalar symbols

- showed that there is no explicit formulae when dim(M) is odd

- showed that the method applies to Physics

Perspective?

Compute more a_r: possible with computer!

??? Replace matrices $u^{\mu\nu}$, v^μ, w by bounded operators

$\to$ adapt the approach for conformal deformation of NCG

Connes–Tretkoff–Moscovici, Fathizadeh–Khalkhali, Liu, Sitarz, ...

Compute indices

$$D_V := \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}$$

McKean–Singer formula

$$\mathrm{Index}(D_V) = \int_{x\in M} dx\, [a_{d/2}^{DD^*}(x) - a_{d/2}^{D^*D}(x)]$$
$$= \cdots$$