Operator $*$-correspondences: Representations and pairings with unbounded KK-theory

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October 26, 2016

Operator spaces

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An **operator space** *is a vector space* $\mathcal{X}$ *equipped with a norm* $\| \cdot \|_{\mathcal{X}} : M_n(\mathcal{X}) \to [0, \infty)$ *for each* $n \in \mathbb{N}$ *such that*

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Theorem (Ruan)

Any operator space $\mathcal{X}$ *is completely isometric to a closed subspace of* $\mathscr{L}(H)$ *for some Hilbert space* H*.*

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Theorem (Blecher, K., Mesland)

Any operator $*$-algebra $\mathcal{A}$ is completely bounded isomorphic to a closed subalgebra of $\mathscr{L}(H)$ for some Hilbert space H. Moreover, we may assume the existence of a selfadjoint unitary operator $U : H \to H$ such that $Ua^*U = a^\dagger$ for all $a \in \mathcal{A}$.

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3. *Then the (canonical matrix norms coming from the) algebra homomorphism*

$$\mathcal{A} \to M_2(A) \qquad a \mapsto \begin{pmatrix} a & 0 \\ \delta(a) & a \end{pmatrix}$$

provides $\mathcal{A}$ with an operator $*$-algebra structure.

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2. *A completely contractive and non-degenerate inner product*

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5. $U\pi(\xi)^* U\pi(\eta) = \pi(\langle \xi, \eta \rangle).$

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4. *Suppose that we have a closed metric δ-connection*
$$\nabla : \mathcal{X} \to X \qquad \langle \nabla(\xi), \eta \rangle + \langle \xi, \nabla(\eta) \rangle = \delta(\langle \xi, \eta \rangle)$$
such that $[\nabla, a] : \overline{\mathcal{X}} \to X$ is bounded in operator norm for all $a \in \mathcal{A}$.

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Provide $\mathcal{X}$ with the structure of an operator $*$-correspondence from $\mathcal{A}_\nabla$ to $\mathcal{B}$.

C^*-completions

Convention

1. *From now on, any operator $*$-algebra $\mathcal{A}$ comes equipped with a fixed C^*-norm $\| \cdot \|$. The C^*-closure A is σ-unital and the inclusion $\mathcal{A} \to A$ is completely bounded.*

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2. *From now on, any operator $*$-correspondence $\mathcal{X}$ from $\mathcal{A}$ to $\mathcal{B}$ is assumed to sit densely inside an essential and countably generated C^*-correspondence X from A to B.*

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Such that the following holds:

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2. *There exists a completely bounded map*

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such that

$$\Gamma^{1/2} \rho_\Gamma(b, \lambda) \Gamma^{1/2}(\xi) = D(b + \lambda)\Gamma(\xi) - \Gamma(b + \lambda) D(\xi)$$

for all $(b, \lambda) \in \widetilde{\mathcal{B}}$ *and all* $\xi \in \mathscr{D}(D)$.

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3. *The sequence* $\{b \cdot \Gamma(\Gamma + 1/n)^{-1}\}$ *converges in operator norm to* b *for all* $b \in B$.

Example (Localization of spectral triples)

Let $(\mathcal{B}, H, D)$ be a spectral triple ($\mathcal{B}$ is an operator $*$-algebra and $[D, \cdot] : \mathcal{B} \to \mathscr{L}(H)$ is completely bounded). Let $x \in \mathcal{B}$.

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Then the triple $(\overline{\mathrm{Im}(x)}, D_x, xx^*)$ is a modular spectral triple over $\mathcal{B}_x$.

Unbounded K-homology

Definition

The **unbounded K-homology** *over an operator $*$-algebra $\mathcal{B}$ consists of modular spectral triples modulo bounded perturbations and unitary equivalences. The unbounded K-homology over $\mathcal{B}$ is an abelian monoid denoted by $UK_*(\mathcal{B}, \mathbb{C})$.*

The bounded transform

Theorem (K.)

1. *Suppose that* $\mathscr{D} := (G, D, \Gamma)$ *is a modular spectral triple over an operator* $*$*-algebra* $\mathcal{B}$*. Then the pair* $F_{\mathscr{D}} := (G, D(1 + D^2)^{-1/2})$ *is a Kasparov module over the* C^**-completion* B*.*

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2. *The assignment* $\mathscr{D} \mapsto F_{\mathscr{D}}$ *induces a well-defined homomorphism*

$$F : UK_*(\mathcal{B}, \mathbb{C}) \to K^*(B)$$

with values in analytic K*-homology.*

The unbounded Kasparov product

Theorem (K.)

1. *The* **unbounded Kasparov product** *yields a well-defined and explicit associative and bilinear pairing:*

$$\widehat{\otimes}_{\mathcal{B}} : M(\mathcal{A},\mathcal{B}) \times UK_*(\mathcal{B},\mathbb{C}) \to UK_*(\mathcal{A},\mathbb{C})$$

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2. *This pairing is* **compatible** *with the* **bounded Kasparov product**

$$\widehat{\otimes}_B : KK_0(A, B) \times K^*(B) \to K^*(A)$$

after taking bounded transforms.