Emergent topology of insulators

Noncommutative index theory
Warsaw

Terry A. Loring

October, 2016

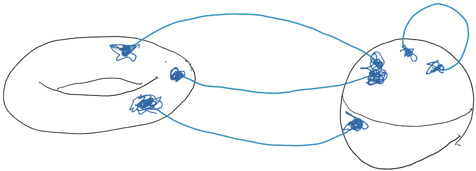
Strings and D-branes

Strings and D-branes

Strings interact on surfaces called D-branes.

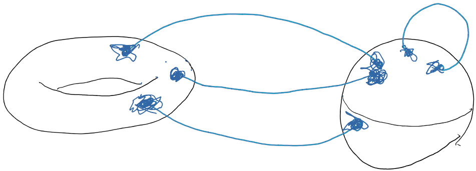
Strings and D-branes

Strings interact on surfaces called D-branes.



Strings and D-branes

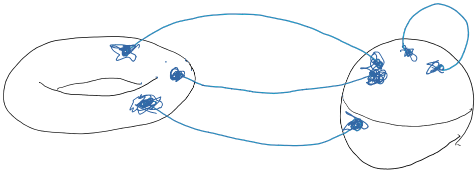
Strings interact on surfaces called D-branes.



Where string hits D-brane is not precisely determined.

Strings and D-branes

Strings interact on surfaces called D-branes.



Where string hits D-brane is not precisely determined.

Shown: fuzzy sphere, fuzzy torus.

Emergent geometry in string theory

Berenstein and Dzienkowski [1]: “emergent geometry”

Emergent geometry in string theory

Berenstein and Dzienkowski [1]: “emergent geometry”

D-branes might have three observables of position: noncommuting hermitian matrices X, Y, Z.

Emergent geometry in string theory

Berenstein and Dzienkowski [1]: “emergent geometry”

D-branes might have three observables of position: noncommuting hermitian matrices X, Y, Z.

A fermionic probe at position (x, y, z) leads to the effective Hamiltonian

$$\not{D}_{(x,y,z)} = \begin{bmatrix} (X-x) & (Y-y) - i(Z-z) \\ (Y-y) + i(Z-z) & -(X-x) \end{bmatrix}.$$

Emergent geometry in string theory

Berenstein and Dzienkowski [1]: "emergent geometry"

D-branes might have three observables of position: noncommuting hermitian matrices X, Y, Z.

A fermionic probe at position (x, y, z) leads to the effective Hamiltonian

$$\not{D}_{(x,y,z)} = \begin{bmatrix} (X - x) & (Y - y) - i(Z - z) \\ (Y - y) + i(Z - z) & -(X - x) \end{bmatrix}.$$

They call this the (shifted) Dirac operator.

Emergent geometry in string theory

Berenstein and Dzienkowski [1]: “emergent geometry”

D-branes might have three observables of position: noncommuting hermitian matrices X, Y, Z.

A fermionic probe at position (x, y, z) leads to the effective Hamiltonian

$$\not{D}_{(x,y,z)} = \begin{bmatrix} (X-x) & (Y-y)-i(Z-z) \\ (Y-y)+i(Z-z) & -(X-x) \end{bmatrix}.$$

They call this the (shifted) Dirac operator.

It’s square is almost the Laplace operator:

$$\not{D}_{\mathbf{0}}^2 = \begin{bmatrix} X^2+Y^2+Z^2 & 0 \\ 0 & X^2+Y^2+Z^2 \end{bmatrix} + \begin{bmatrix} i\,[Y,Z] & [X,Y]-i\,[X,Z] \\ -\,[X,Y]-i\,[X,Z] & -i\,[Y,Z] \end{bmatrix}$$

Clifford spectrum – zero modes

Karczmarek and Yeh [2]:

Clifford spectrum – zero modes

Karczmarek and Yeh [2]:

> The emergent surface is defined as the locus of possible positions for the probe brane where a fermionic string stretched from the stack to the probe brane has a massless mode.

Clifford spectrum – zero modes

Karczmarek and Yeh [2]:

> The emergent surface is defined as the locus of possible positions for the probe brane where a fermionic string stretched from the stack to the probe brane has a massless mode.

So the emergent surface is what mathematicians call the Joint Clifford Spectrum.

Clifford spectrum – zero modes

Karczmarek and Yeh [2]:

> The emergent surface is defined as the locus of possible positions for the probe brane where a fermionic string stretched from the stack to the probe brane has a massless mode.

So the emergent surface is what mathematicians call the Joint Clifford Spectrum.

Definition

For Hermitian operators X, Y, Z the *Clifford spectrum* is the set

$$\Lambda(X, Y, Z) = \left\{ \lambda \in \mathbb{R}^d \,|\, \not{D}_\lambda(X, Y, Z) \text{ is singular} \right\}.$$

The unfuzzy case

The unfuzzy case

We have also the Laplacian:

$$\Delta_\lambda(X, Y, Z) = (X - \lambda_1)^2 + (Y - \lambda_2)^2 + (Z - \lambda_3)^2$$

The unfuzzy case

We have also the Laplacian:

$$\Delta_\lambda(X,Y,Z) = (X-\lambda_1)^2 + (Y-\lambda_2)^2 + (Z-\lambda_3)^2$$

In the case of *commuting* hermitian X, Y and Z,

$$\begin{aligned} & \not{D}_\lambda(X,Y,Z) \text{ is singular} \\ \iff & \Delta_\lambda(X,Y,Z) \text{ is singular} \\ \iff & \exists \mathbf{v} \text{ a unit vector with } X\mathbf{v} = \lambda_1\mathbf{v},\ Y\mathbf{v} = \lambda_2\mathbf{v},\ Z\mathbf{v} = \lambda_3\mathbf{v} \end{aligned}$$

The unfuzzy case

We have also the Laplacian:

$$\Delta_\lambda(X,Y,Z) = (X-\lambda_1)^2 + (Y-\lambda_2)^2 + (Z-\lambda_3)^2$$

In the case of *commuting* hermitian X, Y and Z,

$$\begin{aligned} & \not{D}_\lambda(X,Y,Z) \text{ is singular} \\ \iff & \Delta_\lambda(X,Y,Z) \text{ is singular} \\ \iff & \exists \mathbf{v} \text{ a unit vector with } X\mathbf{v} = \lambda_1\mathbf{v},\ Y\mathbf{v} = \lambda_2\mathbf{v},\ Z\mathbf{v} = \lambda_3\mathbf{v} \end{aligned}$$

This means $\Lambda(X,Y,Z)$ is a finite set.

Two fuzzy spheres

Two fuzzy spheres

At each positive half integer n, use a rescaled representation of $SU(2)$,

Two fuzzy spheres

At each positive half integer n, use a rescaled representation of $SU(2)$, so

$$X_n = \frac{1}{n}\begin{bmatrix} n & & & & \\ & n-1 & & & \\ & & \ddots & & \\ & & & 1-n & \\ & & & & -n \end{bmatrix},$$

$$Y_n = \frac{1}{2n}\left(T_n^* + T_n\right), \quad Z_n = \frac{i}{2n}\left(T_n^* - T_n\right)$$

Two fuzzy spheres

At each positive half integer n, use a rescaled representation of $SU(2)$, so

$$X_n = \frac{1}{n} \begin{bmatrix} n & & & & \\ & n-1 & & & \\ & & \ddots & & \\ & & & 1-n & \\ & & & & -n \end{bmatrix},$$

$$Y_n = \frac{1}{2n}\left(T_n^* + T_n\right), \quad Z_n = \frac{i}{2n}\left(T_n^* - T_n\right)$$

where we set $N_n = n(n+1)$ and

$$T_n = \begin{bmatrix} 0 & \sqrt{N_n - (n-1)n} & & & \\ & 0 & \sqrt{N_n - (n-2)(n-1)} & & \\ & & \ddots & \ddots & \\ & & & 0 & \sqrt{N_n - (-n)(1-n)} \\ & & & & 0 \end{bmatrix}.$$

Two fuzzy spheres

Two fuzzy spheres

The second example is $X_n, \tilde{Y}_n, \tilde{Z}_n$ where

$$Y_n = \frac{1}{2n}\left(S_n^* + S_n\right), \quad Z_n = \frac{i}{2n}\left(S_n^* - S_n\right)$$

Two fuzzy spheres

The second example is $X_n, \tilde{Y}_n, \tilde{Z}_n$ where

$$Y_n = \frac{1}{2n}\left(S_n^* + S_n\right), \quad Z_n = \frac{i}{2n}\left(S_n^* - S_n\right)$$

and

$$S_n = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}.$$

We'll see $\Lambda(X, Y, Z)$ and $\Lambda(X, \tilde{Y}, \tilde{Z})$ are uncountable sets.

We'll see $\Lambda(X, Y, Z)$ and $\Lambda(X, \tilde{Y}, \tilde{Z})$ are uncountable sets. Can we use a computer to calculate these? We can calculate

$$\left\| \not{D}_\lambda(X, Y, Z)^{-1} \right\|^{-1}$$

at a grid of points.

We'll see $\Lambda(X,Y,Z)$ and $\Lambda(X,\tilde{Y},\tilde{Z})$ are uncountable sets. Can we use a computer to calculate these? We can calculate

$$\left\| \not{D}_\lambda(X,Y,Z)^{-1} \right\|^{-1}$$

at a grid of points. Keep points where this is less than some cutoff, like 0.012.

We'll see $\Lambda(X, Y, Z)$ and $\Lambda(X, \tilde{Y}, \tilde{Z})$ are uncountable sets. Can we use a computer to calculate these? We can calculate

$$\left\| \not{D}_\lambda(X, Y, Z)^{-1} \right\|^{-1}$$

at a grid of points. Keep points where this is less than some cutoff, like 0.012.

Definition

For Hermitian operators X, Y, Z and $\epsilon \geq 0$, define the *Clifford ϵ-pseudospectrum* to be the set

$$\Lambda_\epsilon(X, Y, Z) = \left\{ \lambda \in \mathbb{R}^d \,\middle|\, \left\| \not{D}_\lambda(X, Y, Z)^{-1} \right\|^{-1} \leq \epsilon \right\}$$

with the convention that R singular means $\left\| R^{-1} \right\|^{-1} = 0$.

We'll see $\Lambda(X, Y, Z)$ and $\Lambda(X, \tilde{Y}, \tilde{Z})$ are uncountable sets. Can we use a computer to calculate these? We can calculate

$$\left\| \not{D}_{\lambda}(X, Y, Z)^{-1} \right\|^{-1}$$

at a grid of points. Keep points where this is less than some cutoff, like 0.012.

Definition

For Hermitian operators X, Y, Z and $\epsilon \geq 0$, define the *Clifford ϵ-pseudospectrum* to be the set

$$\Lambda_{\epsilon}(X, Y, Z) = \left\{ \lambda \in \mathbb{R}^d \,\middle|\, \left\| \not{D}_{\lambda}(X, Y, Z)^{-1} \right\|^{-1} \leq \epsilon \right\}$$

with the convention that R singular means $\left\| R^{-1} \right\|^{-1} = 0$.

Notice

$$\left\| \not{D}_{\lambda}(X, Y, Z)^{-1} \right\|^{-1} = \left| \mathrm{eig}_{\min} \left(D_{\lambda}(X, Y, Z) \right) \right|$$

called the gap at λ.

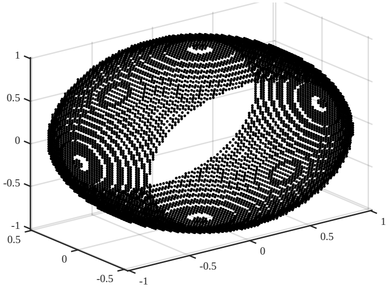


Figure : Left: approximation of part of the Clifford spectrum for the usual fuzzy sphere, spin=2. Right: Same spin but with modified Y and Z.

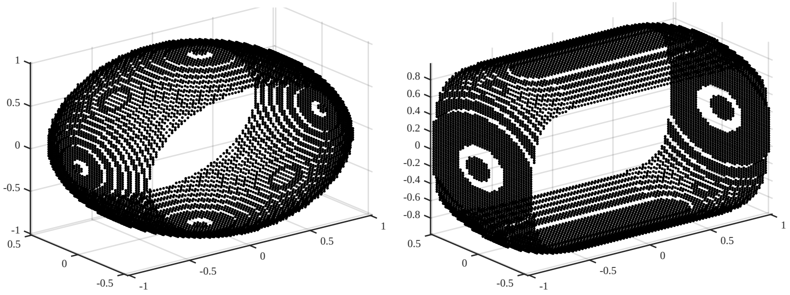


Figure : Now spin=20.

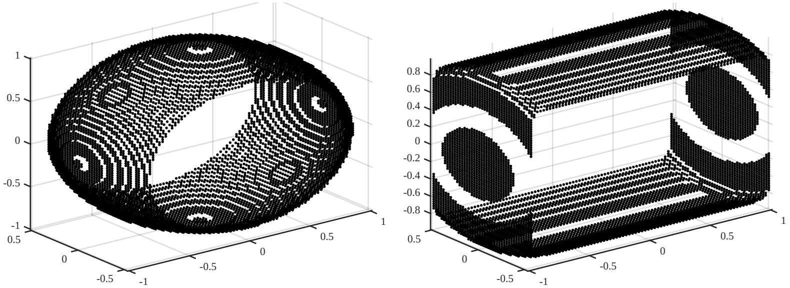


Figure : Now spin=200.

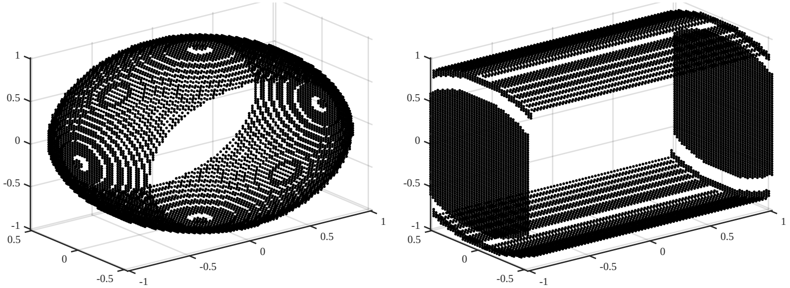


Figure : Now spin=2000.

Obviously $\Lambda(X, Y, Z)$ is a subset of $\Lambda_\epsilon(X, Y, Z)$. Could be a finite set?

Obviously $\Lambda(X, Y, Z)$ is a subset of $\Lambda_\epsilon(X, Y, Z)$. Could be a finite set?

Definition

The *local index* for the finite system (X, Y, Z) is then

$$\mathrm{Ind}_\lambda(X, Y, Z) = \frac{1}{2}\mathrm{Sig}(\not{D}_\lambda(X, Y, Z))$$

defined at every point where $\not{D}_\lambda$ is invertible.

Obviously $\Lambda(X, Y, Z)$ is a subset of $\Lambda_\epsilon(X, Y, Z)$. Could be a finite set?

Definition

The *local index* for the finite system (X, Y, Z) is then

$$\mathrm{Ind}_\lambda(X, Y, Z) = \frac{1}{2}\mathrm{Sig}(\not{D}_\lambda(X, Y, Z))$$

defined at every point where $\not{D}_\lambda$ is invertible.

Signature is the number of positive eigenvalues minus the number of negative eigenvalues.

Commuting case, again

Commuting case, again

The spectrum of

$$\not{D}_\lambda(r_1, r_2, r_3) = \begin{bmatrix} r_3 - \lambda_3 & r_1 - \lambda_1 - i(r_2 - \lambda_2) \\ r_1 - \lambda_1 + i(r_2 - \lambda_2) & -r_3 + \lambda_3 \end{bmatrix}$$

Commuting case, again

The spectrum of

$$\not{D}_\lambda(r_1, r_2, r_3) = \begin{bmatrix} r_3 - \lambda_3 & r_1 - \lambda_1 - i(r_2 - \lambda_2) \\ r_1 - \lambda_1 + i(r_2 - \lambda_2) & -r_3 + \lambda_3 \end{bmatrix}$$

is just

$$\left\{ \pm\sqrt{(r_1 - \lambda_1)^2 + (r_2 - \lambda_2)^2 + (r_3 - \lambda_3)^2} \right\}.$$

Commuting case, again

The spectrum of

$$\not{D}_\lambda(r_1, r_2, r_3) = \begin{bmatrix} r_3 - \lambda_3 & r_1 - \lambda_1 - i(r_2 - \lambda_2) \\ r_1 - \lambda_1 + i(r_2 - \lambda_2) & -r_3 + \lambda_3 \end{bmatrix}$$

is just

$$\left\{ \pm\sqrt{(r_1 - \lambda_1)^2 + (r_2 - \lambda_2)^2 + (r_3 - \lambda_3)^2} \right\}.$$

Thus

$$\lambda \in \Lambda(r_1, r_2, r_3) \quad \text{or} \quad \mathrm{Ind}_\lambda(\not{D}_\lambda(r_1, r_2, r_3)) = 0.$$

Commuting case, again

The spectrum of

$$\not{D}_\lambda(r_1, r_2, r_3) = \begin{bmatrix} r_3 - \lambda_3 & r_1 - \lambda_1 - i(r_2 - \lambda_2) \\ r_1 - \lambda_1 + i(r_2 - \lambda_2) & -r_3 + \lambda_3 \end{bmatrix}$$

is just

$$\left\{ \pm\sqrt{(r_1 - \lambda_1)^2 + (r_2 - \lambda_2)^2 + (r_3 - \lambda_3)^2} \right\}.$$

Thus

$$\lambda \in \Lambda(r_1, r_2, r_3) \quad \text{or} \quad \mathrm{Ind}_\lambda(\not{D}_\lambda(r_1, r_2, r_3)) = 0.$$

In the commuting case,

$$\mathrm{Ind}_\lambda(X, Y, Z) = 0$$

(or is undefined).

Commuting case, again

The spectrum of

$$\not{D}_\lambda(r_1, r_2, r_3) = \begin{bmatrix} r_3 - \lambda_3 & r_1 - \lambda_1 - i(r_2 - \lambda_2) \\ r_1 - \lambda_1 + i(r_2 - \lambda_2) & -r_3 + \lambda_3 \end{bmatrix}$$

is just

$$\left\{ \pm\sqrt{(r_1 - \lambda_1)^2 + (r_2 - \lambda_2)^2 + (r_3 - \lambda_3)^2} \right\}.$$

Thus

$$\lambda \in \Lambda(r_1, r_2, r_3) \quad \text{or} \quad \mathrm{Ind}_\lambda(\not{D}_\lambda(r_1, r_2, r_3)) = 0.$$

In the commuting case,

$$\mathrm{Ind}_\lambda(X, Y, Z) = 0$$

(or is undefined). From this one shows: for a given X, Y and Z there is some C so that

$$|\lambda| > C \implies \mathrm{Ind}_\lambda(X, Y, Z) = 0.$$

Theorem

If $\mathrm{Ind}_{\lambda}(X, Y, Z) \neq \mathrm{Ind}_{\mu}(X, Y, Z)$ *and* λ_t *is any curve between* λ *and* μ *then there is at least one value of* s *so that*

$$\lambda_s \notin \Lambda(X, Y, Z).$$

Theorem

If $\mathrm{Ind}_\lambda(X, Y, Z) \neq \mathrm{Ind}_\mu(X, Y, Z)$ *and* λ_t *is any curve between* λ *and* μ *then there is at least one value of s so that*

$$\lambda_s \notin \Lambda(X, Y, Z).$$

Corollary

If for any λ *we have* $\mathrm{Ind}_\lambda(X, Y, Z) \neq 0$ *then* $\Lambda(X, Y, Z)$ *separates* λ *from* ∞*.*

Fuzzy Tori

Fuzzy Tori

Consider

$$U = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \omega & & & & \\ & \omega^2 & & & \\ & & \ddots & & \\ & & & \omega^{n-1} & \\ & & & & 1 \end{bmatrix}, \tag{1}$$

$(\omega = e^{\frac{2\pi i}{n}})$

Fuzzy Tori

Consider

$$U = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \omega & & & & \\ & \omega^2 & & & \\ & & \ddots & & \\ & & & \omega^{n-1} & \\ & & & & 1 \end{bmatrix}, \tag{1}$$

($\omega = e^{\frac{2\pi i}{n}}$) which represent an embedding of a fuzzy torus into $\mathbb{R}^4$.

Fuzzy Tori

Consider

$$U = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \omega & & & & \\ & \omega^2 & & & \\ & & \ddots & & \\ & & & \omega^{n-1} & \\ & & & & 1 \end{bmatrix}, \tag{1}$$

($\omega = e^{\frac{2\pi i}{n}}$) which represent an embedding of a fuzzy torus into $\mathbb{R}^4$. We seek an embedding into $\mathbb{R}^3$.

Fuzzy Tori

Consider

$$U = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \omega & & & & \\ & \omega^2 & & & \\ & & \ddots & & \\ & & & \omega^{n-1} & \\ & & & & 1 \end{bmatrix}, \tag{1}$$

($\omega = e^{\frac{2\pi i}{n}}$) which represent an embedding of a fuzzy torus into $\mathbb{R}^4$. We seek an embedding into $\mathbb{R}^3$. Set

$$A_n = \frac{1}{2}\left(R + \frac{r}{2}U^* + \frac{r}{2}U\right)V^* + \frac{1}{2}V\left(R + \frac{r}{2}U^* + \frac{r}{2}U\right)$$

$$B_n = \frac{i}{2}\left(R + \frac{r}{2}U^* + \frac{r}{2}U\right)V^* - \frac{i}{2}V\left(R + \frac{r}{2}U^* + \frac{r}{2}U\right)$$

$$C_n = \frac{ri}{2}U^* - U\frac{ri}{2}.$$

Fuzzy Tori

Consider

$$U = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \omega & & & & \\ & \omega^2 & & & \\ & & \ddots & & \\ & & & \omega^{n-1} & \\ & & & & 1 \end{bmatrix}, \tag{1}$$

($\omega = e^{\frac{2\pi i}{n}}$) which represent an embedding of a fuzzy torus into $\mathbb{R}^4$. We seek an embedding into $\mathbb{R}^3$. Set

$$A_n = \frac{1}{2}\left(R + \frac{r}{2}U^* + \frac{r}{2}U\right)V^* + \frac{1}{2}V\left(R + \frac{r}{2}U^* + \frac{r}{2}U\right)$$

$$B_n = \frac{i}{2}\left(R + \frac{r}{2}U^* + \frac{r}{2}U\right)V^* - \frac{i}{2}V\left(R + \frac{r}{2}U^* + \frac{r}{2}U\right)$$

$$C_n = \frac{ri}{2}U^* - U\frac{ri}{2}.$$

Good choices for the radii are $R = 0.8$ and $r = 0.4$.

Emergent tori, showing K-theory

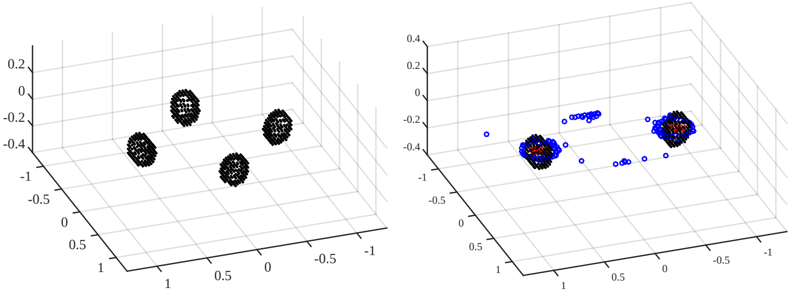


Figure : Left: approximation of part of a fuzzy torus, $n = 4$. Right: Same matrices, pseudospectrum cut open and showing some index values.

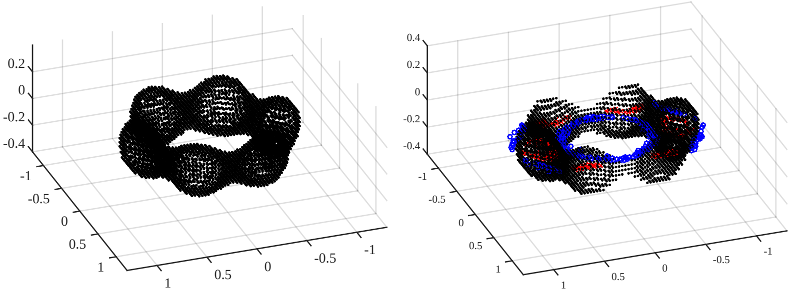


Figure : Now $n = 6$.

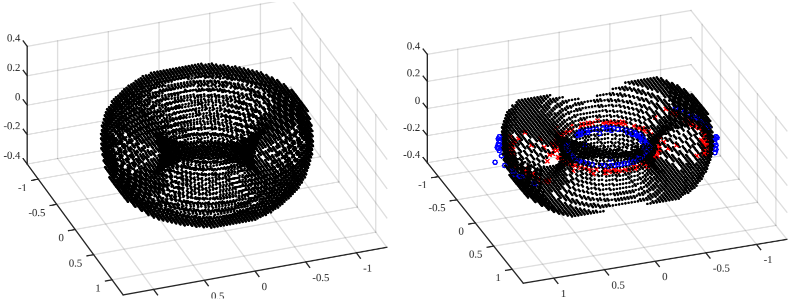


Figure : Now $n = 10$.

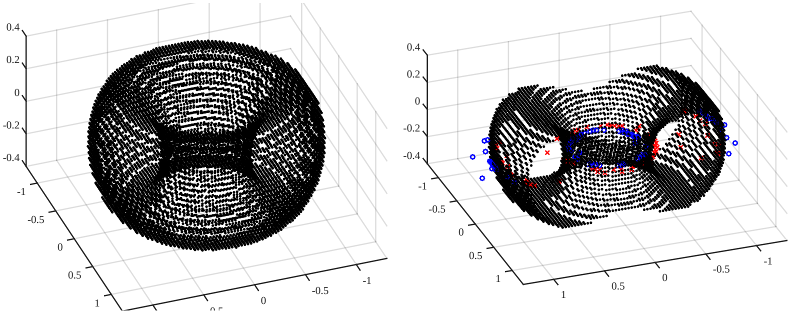


Figure : Now $n = 20$.

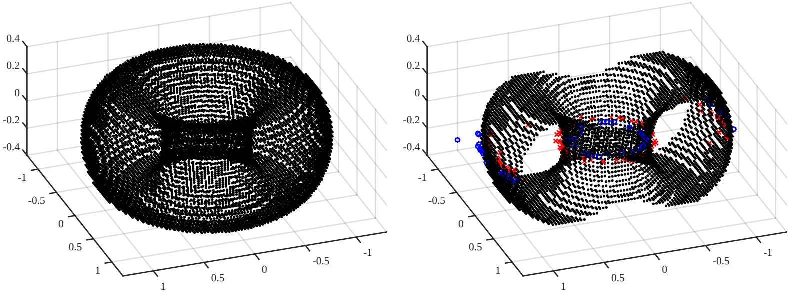


Figure : Now $n = 50$.

Insulators

Insulators

In finite models of condensed matter, many use periodic boundary conditions.

Insulators

In finite models of condensed matter, many use periodic boundary conditions.

Edge modes are important; I prefer open boundaries.

Insulators

In finite models of condensed matter, many use periodic boundary conditions.

Edge modes are important; I prefer open boundaries.

For a $(d-1)$-dimensional physical system, the matrix model is then

$$(X_1, \ldots, X_{d-1}, H)$$

where the X_j are commuting matrices for position, and H is the Hamiltonian, corresponding to energy.

Topological insulators

Topological insulators

In the case $d = 3$, there is a simple model of what is called a Chern insulator.

Topological insulators

In the case $d = 3$, there is a simple model of what is called a Chern insulator. A homologically non-trivial surface emerged, as $\Lambda(X, Y, H)$.

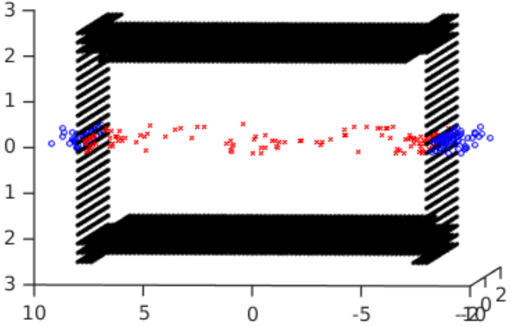


Figure : Part of the pseudospectrum of a Chern insulator on an 18-by-18 lattice with no disorder. The vertical axis is energy, the others are position. The crosses at the Fermi level (energy zero) indicate nontrivial index, while the circles indicate trivial index. Reproduced from [3].

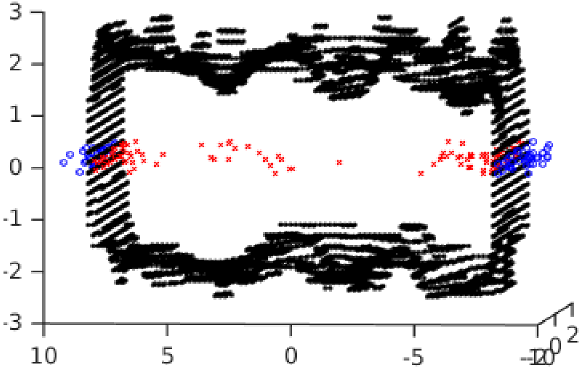


Figure : Part of the pseudospectrum of a Chern insulator on an 18-by-18 lattice with disorder sufficient to half close the gap in the bulk spectrum. Reproduced from [3].

A finite volume index theorem

A finite volume index theorem

Theorem (L. & Schulz-Baldes)

Suppose $H = H^*$ is bounded on $\mathbb{H} = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^r$ is invertible and local. Specifically assume

$$\delta = \max\left(\|[H,X]\|, \|[H,Y]\|\right)$$

is finite and $\sigma(H) \cap (-\Delta, \Delta) = \emptyset$. There exists L_0 so that for $L \geq L_0$ we have

$$\mathrm{Ind}_\lambda(X_L, Y_L, H_L) = \mathrm{Ind}_{P\mathbb{H}}\left(P\frac{X+iY}{|X+iY|}P\right)$$

where X_L, Y_L and H_L are formed by restriction to $\mathbb{H} = \ell^2([-L,L] \times [-L,L]) \otimes \mathbb{C}^r$, with H_L given Dirichlet boundary conditions, and where P is the spectral projection for H corresponding to $[0,\infty)$.

A finite volume index theorem

Theorem (L. & Schulz-Baldes)

Suppose $H = H^*$ is bounded on $\mathbb{H} = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^r$ is invertible and local. Specifically assume

$$\delta = \max\left(\|[H, X]\|, \|[H, Y]\|\right)$$

is finite and $\sigma(H) \cap (-\Delta, \Delta) = \emptyset$. There exists L_0 so that for $L \geq L_0$ we have

$$\mathrm{Ind}_\lambda(X_L, Y_L, H_L) = \mathrm{Ind}_{P\mathbb{H}}\left(P \frac{X + iY}{|X + iY|} P\right)$$

where X_L, Y_L and H_L are formed by restriction to $\mathbb{H} = \ell^2([-L, L] \times [-L, L]) \otimes \mathbb{C}^r$, with H_L given Dirichlet boundary conditions, and where P is the spectral projection for H corresponding to $[0, \infty)$.

Also have results for H invertible and local on $\mathbb{H} = \ell^2(\mathbb{Z}^{2d}) \otimes \mathbb{C}^r$, etc.

Used the following to understand the boundary map.

$$0 \longrightarrow J \longrightarrow C_\delta(\mathbb{D}_d) \longrightarrow C_\delta(S^{d-1}) \longrightarrow 0$$

where

$$C_\delta(\mathbb{D}) = C_1^* \left\langle x_1, \ldots, x_d \,\middle|\, \begin{array}{c} x_j^* = x_j \\ \|x_j x_k - x_k x_j\| \le \delta \\ \sum x_j^2 \le 1 \end{array} \right\rangle$$

is a “soft d-ball” which is projective (L. & Shulman).

Used the following to understand the boundary map.

$$0 \longrightarrow J \longrightarrow C_\delta(\mathbb{D}_d) \longrightarrow C_\delta(S^{d-1}) \longrightarrow 0$$

where

$$C_\delta(\mathbb{D}) = C^*_1 \left\langle x_1, \ldots, x_d \;\middle|\; \begin{array}{c} x_j^* = x_j \\ \|x_j x_k - x_k x_j\| \leq \delta \\ \sum x_j^2 \leq 1 \end{array} \right\rangle$$

is a "soft d-ball" which is projective (L. & Shulman). Also

$$C_\delta(S^{d-1}) = C^*_1 \left\langle x_1, \ldots, x_d \;\middle|\; \begin{array}{c} x_j^* = x_j \\ \|x_j x_k - x_k x_j\| \leq \delta \\ 1 - \delta \leq \sum x_j^2 \leq 1 \end{array} \right\rangle$$

about which we know less.

The usual formulas for mapping the *d*-ball onto the *d*-sphere lead to a coboundary map for

$$0 \to I \to A \to B \to 0$$

in term of ”fuzzy” or “soft” spheres:

$$C_{\delta'}(S^{d-1}) \to B \quad \looparrowright \quad C_{\delta}(S^{d}) \to \tilde{J}$$

Following the ideas in cohomotopy:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] & A \arrow[r] & B \arrow[r] & 0 \\
0 \arrow[r] & J \arrow[r] \arrow[u, dashed, two heads] & C_\delta(\mathbb{D}_d) \arrow[r] \arrow[u, dashed, two heads] & C_\delta(S^{d-1}) \arrow[r] \arrow[u] & 0 \\
 & C_{\delta'}(\mathbb{R}^d) \arrow[u] & & &
\end{tikzcd}$$

References

David Berenstein and Eric Dzienkowski.
Matrix embeddings on flat $\mathbb{R}^3$ and the geometry of membranes.
Physical Review D, 86(8):086001, 2012.

Joanna L. Karczmarek and Ken Huai-Che Yeh.
Noncommutative spaces and matrix embeddings on flat $\mathbb{R}^{2n+1}$.
Journal of High Energy Physics, 2015(11):1–15, 2015.

Terry A. Loring.
K-theory and pseudospectra for topological insulators.
Ann. Physics, 356:383–416, 2015.