The Plancherel Formula for complex quantum groups

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The classical Plancherel Theorem

For $f \in L^1(\mathbb{R})$ the Fourier transform of $f$ is defined by

$$F(f)(p) = \int_{\mathbb{R}} e^{-ixp} f(x) \, dx,$$

where $dx$ denotes (suitably normalised) Lebesgue measure.

**Theorem (Plancherel)**

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then

$$\|F(f)\|_2^2 = \int_{\mathbb{R}} |F(f)(p)|^2 \, dp = \int_{\mathbb{R}} |f(x)|^2 \, dx = \|f\|_2^2.$$

Hence $F$ induces a unitary isomorphism $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$.
The classical Plancherel Theorem

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Hence $\mathcal{F}$ induces a unitary isomorphism $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. 

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Let us reinterpret the Plancherel Theorem from a slightly more general perspective.
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Since \( \mathbb{R} \) is a locally compact abelian group, it has a Pontrjagin dual group \( \hat{\mathbb{R}} \), consisting of all unitary characters of \( \mathbb{R} \).
Let us reinterpret the Plancherel Theorem from a slightly more general perspective.

Since $\mathbb{R}$ is a locally compact abelian group, it has a Pontrjagin dual group $\hat{\mathbb{R}}$, consisting of all unitary characters of $\mathbb{R}$.

The unitary characters of $\mathbb{R}$ are of the form

$$\chi_p(x) = e^{-ixp}$$

for $p \in \mathbb{R}$. 

The classical Plancherel Theorem

Let us reinterpret the Plancherel Theorem from a slightly more general perspective.

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The unitary characters of $\mathbb{R}$ are of the form

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for $p \in \mathbb{R}$.

In this way one obtains $\hat{\mathbb{R}} \cong \mathbb{R}$. 

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The classical Plancherel Theorem

The group $C^*$-algebra $C^*(\mathbb{R})$ is a completion of $C_c^\infty(\mathbb{R})$, equipped with the convolution product

$$(f \ast g)(t) = \int_{\mathbb{R}} f(-s)g(s + t)ds$$

and $\ast$-structure

$$f^*(t) = \overline{f(-s)}.$$
The classical Plancherel Theorem

The group $C^*$-algebra $C^*(\mathbb{R})$ is a completion of $C^\infty_c(\mathbb{R})$, equipped with the convolution product

$$(f * g)(t) = \int_{\mathbb{R}} f(-s)g(s + t) \, ds$$

and $*$-structure

$$f^*(t) = \overline{f(-s)}.$$

In particular, for the one-dimensional representations corresponding to the characters $\chi_p$ we obtain $*$-homomorphisms $\chi_p : C^*(\mathbb{R}) \to \mathbb{C}$ given by

$$\chi_p(f) = \int_{\mathbb{R}} f(x)\chi_p(x) = \int_{\mathbb{R}} f(x)e^{-ipx} \, dx = \mathcal{F}(f)(p)$$

for $f \in C^\infty_c(\mathbb{R})$. 

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The classical Plancherel Theorem

For $f \in C^\infty_c(\mathbb{R})$ we have

$$
(f \ast f)(0) = \int_{\mathbb{R}} f(s)f(s) \, ds = \|f\|_2^2 = \|\mathcal{F}(f)\|_2^2 = \int_{\widehat{\mathbb{R}}} \mathcal{F}(f)(p)\mathcal{F}(f)(p) \, dp = \int_{\widehat{\mathbb{R}}} \chi_p(f)^\ast \chi_p(f) \, dp = \int_{\widehat{\mathbb{R}}} \chi_p(f \ast f) \, dp,
$$

or equivalently,

**Theorem (Plancherel formula)**

For any $h \in C^\infty_c(\mathbb{R})$ we have

$$
h(0) = \int_{\widehat{\mathbb{R}}} \chi_p(h) \, dp.
$$

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The classical Plancherel Theorem

For \( f \in C_c^\infty(\mathbb{R}) \) we have

\[
(f^* \ast f)(0) = \int_{\mathbb{R}} \overline{f(s)} f(s) ds = \|f\|_2^2 = \|\mathcal{F}(f)\|_2^2 = \int_{\hat{\mathbb{R}}} \overline{\mathcal{F}(f)(p)} \mathcal{F}(f)(p) dp
\]

\[
= \int_{\hat{\mathbb{R}}} \chi_p(f)^* \chi_p(f) dp = \int_{\hat{\mathbb{R}}} \chi_p(f^* \ast f) dp,
\]

or equivalently,
The classical Plancherel Theorem

For $f \in C_c^\infty(\mathbb{R})$ we have

$$(f^* \ast f)(0) = \int_\mathbb{R} \overline{f(s)}f(s)ds$$

$$= \|f\|_2^2 = \|\mathcal{F}(f)\|_2^2 = \int_{\hat{\mathbb{R}}} \overline{\mathcal{F}(f)(p)}\mathcal{F}(f)(p)dp$$

$$= \int_{\hat{\mathbb{R}}} \chi_p(f)^*\chi_p(f)dp = \int_{\hat{\mathbb{R}}} \chi_p(f^* \ast f)dp,$$

or equivalently,

**Theorem (Plancherel formula)**

For any $h \in C_c^\infty(\mathbb{R})$ we have

$$h(0) = \int_{\hat{\mathbb{R}}} \chi_p(h)dp.$$
Now let $G$ be a compact group. Write $\text{Irr}(G)$ for the set of equivalence classes of irreducible representations of $G$, and $\pi_\lambda: G \to U(H_\lambda)$ for $\lambda \in \text{Irr}(G)$.

**Theorem (Peter-Weyl)**

For $f \in L^1(G) \cap L^2(G)$ we have

$$\|f\|_2^2 = \sum_{\lambda \in \text{Irr}(G)} \text{tr}(\pi_\lambda(f)^* \pi_\lambda(f)) \dim(H_\lambda) - 1.$$
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**Theorem (Peter-Weyl)**

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$$\|f\|_2^2 = \sum_{\lambda \in \text{Irr}(G)} \text{tr}(\pi_\lambda(f)^* \pi_\lambda(f)) \dim(\mathcal{H}_\lambda)^{-1}$$
Hence the formula

\[ \mathcal{F}(f) = \bigoplus_{\lambda \in \text{Irr}(G)} \pi_\lambda(f) \]

for \( f \in L^1(G) \cap L^2(G) \) extends to an isometric isomorphism

\[ \mathcal{F} : L^2(G) \rightarrow \bigoplus_{\lambda \in \text{Irr}(G)} HS(\mathcal{H}_\lambda), \]

if on \( \text{Irr}(G) \) we consider the (Plancherel) measure

\[ dm = \sum_{\lambda \in \text{Irr}(G)} \frac{\dim(\mathcal{H}_\lambda)^{-1}}{\delta_\lambda}. \]
Abstract Plancherel Theorem

Assume that $G$ is a type I locally compact unimodular group.

**Theorem (Segal-Mautner)**

*Then there exists a standard measure $m$ on $\text{Irr}(G)$, a measurable field of Hilbert spaces $(\mathcal{H}_\lambda)_{\lambda \in \text{Irr}(G)}$, and an isometric $G$-equivariant isomorphism*

$$
\mathcal{F} : L^2(G) \to \int_{\text{Irr}(G)}^{\oplus} HS(\mathcal{H}_\lambda)dm(\lambda),
$$

given by

$$
\mathcal{F}(f) = \int_{\text{Irr}(G)}^{\oplus} \pi_\lambda(f)dm(\lambda)
$$

*on a dense subspace of $L^1(G) \cap L^2(G)$.*
Assume that $G$ is a type I locally compact possibly non-unimodular group.

**Theorem (Segal-Mautner, Duflo-Moore)**

Then there exists a standard measure $m$ on $\text{Irr}(G)$, a measurable field of Hilbert spaces $(\mathcal{H}_\lambda)_{\lambda \in \text{Irr}(G)}$, a measurable field $(D_\lambda)_{\lambda \in \text{Irr}(G)}$ of self-adjoint strictly positive operators for $(\mathcal{H}_\lambda)_{\lambda \in \text{Irr}(G)}$, and an isometric $G$-equivariant isomorphism

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\mathcal{F} : L^2(G) \rightarrow \int_{\text{Irr}(G)} \bigoplus \text{HS}(\mathcal{H}_\lambda) \, dm(\lambda),
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given by

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on a dense subspace of $L^1(G) \cap L^2(G)$. 

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Assume that $G$ is a type I locally compact possibly non-unimodular quantum group.

**Theorem (Segal-Mautner, Duflo-Moore, Desmedt)**

Then there exists a standard measure $m$ on $\text{Irr}(G)$, a measurable field of Hilbert spaces $(\mathcal{H}_\lambda)_{\lambda \in \text{Irr}(G)}$, a measurable field $(D_\lambda)_{\lambda \in \text{Irr}(G)}$ of self-adjoint strictly positive operators for $(\mathcal{H}_\lambda)_{\lambda \in \text{Irr}(G)}$, and an isometric $G$-equivariant isomorphism

$$\mathcal{F} : L^2(G) \rightarrow \int_{\text{Irr}(G)} \oplus \text{HS}(\mathcal{H}_\lambda) dm(\lambda),$$

given by

$$\mathcal{F}(f) = \int_{\text{Irr}(G)} \oplus \pi_\lambda(f) D_\lambda^{-1} dm(\lambda)$$

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Remark on Duflo-Moore operators

The appearance of Duflo-Moore operators is not really due to non-unimodularity, but rather related to the question of whether the (left) Haar weight of the group algebra is a trace or not. In the group case, this is equivalent to (non-) unimodularity.
The appearance of Duflo-Moore operators is not really due to non-unimodularity, but rather related to the question of whether the (left) Haar weight of the group algebra is a trace or not. In the group case, this is equivalent to (non-) unimodularity.

For instance, for a compact quantum group, there are Duflo-Moore operators in the Plancherel formula. These are trivial iff the quantum group is of Kac type - note that compact quantum groups are always unimodular.

If $G$ is a compact quantum group the Plancherel formula becomes

$$\epsilon(f) = \sum_{\lambda \in \text{Irr}(G)} \dim_q(H_\lambda) \text{tr}(\pi_\lambda(f) D^{-2}_\lambda)$$

for $f \in \mathcal{O}(G)$. 

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A little bit of history:

- Podleś-Woronowicz (1990) construct complex semisimple quantum groups on the $C^*$-algebra level.
- Buffenoir-Roche (1999) determine the Plancherel formula for $SL_q(2, \mathbb{C})$.
- Arano (2014, 2016) completely classifies the irreducible unitary representations of $SL_q(n, \mathbb{C})$, and most of the full dual in general.

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Here is a quick outline of the construction of the quantization $G^q$ of a (simply connected) complex semisimple group $G$:

▶ Start from the Iwasawa decomposition $G = KAN$.

▶ For the compact part $K$, there exists a deformation $K^q$ obtained using quantized enveloping algebras.

▶ According to Drinfeld duality, a quantization of the Poisson dual $\hat{AN}$ of $K$ is given by the Pontrjagin dual $\hat{K}^q$ of $K^q$.

▶ The complex quantum group $G^q$ is the quantum double $G^q = K^q ⊠ \hat{K}^q$.

We shall now explain the ingredients in these constructions in more detail.
Here is a quick outline of the construction of the quantization $G_q$ of a (simply connected) complex semisimple group $G$:

1. Start from the Iwasawa decomposition $G = K A N$.
2. For the compact part $K$, there exists a deformation $K_q$ obtained using quantized enveloping algebras.
3. According to Drinfeld duality, a quantization of the Poisson dual $A_N$ of $K$ is given by the Pontrjagin dual $\hat{K}_q$ of $K_q$.
4. The complex quantum group $G_q$ is the quantum double $G_q = K_q \bowtie \hat{K}_q$.

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Complex semisimple quantum groups

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- The complex quantum group $G_q$ is the quantum double

$$G_q = K_q \boxtimes \hat{K}_q.$$ 

We shall now explain the ingredients in these constructions in more detail.
Notation

- Fix $q = e^h \in (0, 1)$.
- Let $g$ be a semisimple complex Lie algebra of rank $N$ with Cartan matrix $(a_{ij})$.
- $h \subset g$ a Cartan subalgebra.
- $\Delta = \Delta^+ \cup \Delta^-$ the root system with simple roots $\alpha_1, \ldots, \alpha_N \subset h^*$.
- $(\cdot, \cdot)$ the bilinear form on $h^*$ obtained by rescaling the Killing form such that all short roots $\alpha_s$ satisfy $(\alpha, \alpha) = 2$.
- Set $d_i = (\alpha_i, \alpha_i)/2$ and $q_i = q^{d_i}$.
- $\varpi_1, \ldots, \varpi_N \in h^*$ are the fundamental weights.
- $P = \bigoplus_{j=1}^N \mathbb{Z} \varpi_j$ and $Q = \bigoplus_{j=1}^N \mathbb{Z} \alpha_j$ are the weight and root lattices, respectively.
- $P^+ = \bigoplus_{j=1}^N N_0 \varpi_j$ are the dominant integral weights.
- $W$ is the Weyl group of $g$. 

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Fix \( q = e^h \in (0, 1) \).

Let \( \mathfrak{g} \) be a semisimple complex Lie algebra of rank \( N \) with Cartan matrix \( (a_{ij}) \).

\( \mathfrak{h} \subset \mathfrak{g} \) a Cartan subalgebra.

\( \Delta = \Delta^+ \cup \Delta^- \) the root system with simple roots \( \alpha_1, \ldots, \alpha_N \subset \mathfrak{h}^* \).

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\( W \) is the Weyl group of \( \mathfrak{g} \).
The Drinfeld-Jimbo algebra associated to $\mathfrak{g}$

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is the algebra with generators $E_j$, $F_j$ for $1 \leq j \leq N$ and $K_\lambda$ for $\lambda \in \mathfrak{p}$ satisfying $K_0 = 1$, $K_\lambda K_\mu = K_{\lambda + \mu}$, $K_\lambda E_j K_{-1}^{\lambda} = q^{(\lambda,\alpha_j)} E_j$, $K_\lambda F_j K_{-1}^{\lambda} = q^{- (\lambda,\alpha_j)} F_j$, $[E_i, F_j] = \delta_{ij} K_i - K_{-1}^{i} q_i - q_{-1}^{i}$, where $K_i = K_{\alpha_i}$.
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K_0 = 1, \quad K_\lambda K_\mu = K_{\lambda + \mu},
\]

\[
K_\lambda E_j K^-1_\lambda = q^{(\lambda, \alpha_j)} E_j, \quad K_\lambda F_j K^-1_\lambda = q^{-(\lambda, \alpha_j)} F_j,
\]

\[
[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad \text{where } K_i = K_{\alpha_i},
\]

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} q_i E_i^{-a_{ij}-k} = 0 \quad i \neq j,
\]

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The Drinfeld-Jimbo algebra associated to $\mathfrak{g}$

The algebra $U_q(\mathfrak{g})$ is a Hopf algebra. For instance, the coproduct $\hat{\Delta}: U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is given by

\[
\hat{\Delta}(K_\lambda) = K_\lambda \otimes K_\lambda,
\]

\[
\hat{\Delta}(E_i) = E_i \otimes K_i + 1 \otimes E_i
\]

\[
\hat{\Delta}(F_i) = F_i \otimes 1 + K_{i-1} \otimes F_i
\]

Moreover $U_q(\mathfrak{g})$ is a $\ast$-algebra with the $\ast$-structure $E_i = K_i F_i$, $F_i = E_i K_{i-1}$, $K_\lambda \ast = K_\lambda$. As a Hopf $\ast$-algebra, $U_q(\mathfrak{g})$ should be viewed as quantization of the (complex) universal enveloping algebra of the (real) Lie algebra $\mathfrak{k}$.

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$$\hat{\Delta}(E_i) = E_i \otimes K_i + 1 \otimes E_i,$$

$$\hat{\Delta}(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i.$$

Moreover $U_q(\mathfrak{g})$ is a $*$-algebra with the $*$-structure

$$E_i^* = K_i F_i, \quad F_i^* = E_i K_i^{-1}, \quad K_\lambda^* = K_\lambda.$$

As a Hopf $*$-algebra, $U_q(\mathfrak{g})$ should be viewed as quantization of the (complex) universal enveloping algebra of the (real) Lie algebra $\mathfrak{k}$. 

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The finite dimensional representation theory of $U_q(g)$ is similar to the one for $U(g)$. In particular, for every $\mu \in P^+$ there exists a unique irreducible representation $V(\mu)$ with a highest weight vector $v_\mu$, satisfying

$$K_\lambda v_\mu = q^{(\lambda, \mu)} v_\mu$$
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Using the representations $V(\mu)$ one defines a compact quantum group $K_q$ as follows.

**Definition**

The algebra $\mathcal{O}(K_q) \subset U_q(g)^*$ of representative functions on $K_q$ is the Hopf $*$-algebra of matrix coefficients of all $V(\mu)$ for $\mu \in \mathbb{P}^+$. We let $C(K_q)$ be its universal $C^*$-completion.

$\mathcal{O}(K_q)$ is a deformation of the algebra $\mathcal{O}(K)$ of representative functions on $K$, and $C(K_q)$ is a deformation of $C(K)$. 

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Example: the quantum group $SU_q(2)$

The algebra $\mathcal{O}(SU_q(2))$ can be identified with the $\ast$-algebra generated by elements $\alpha$ and $\gamma$ satisfying the relations

\begin{align*}
\alpha \gamma &= q \gamma \alpha, \\
\alpha \gamma^\ast &= q \gamma^\ast \alpha, \\
\gamma \gamma^\ast &= \gamma^\ast \gamma,
\end{align*}

\begin{align*}
\alpha^\ast \alpha + \gamma^\ast \gamma &= 1, \\
\alpha \alpha^\ast + q^2 \gamma \gamma^\ast &= 1.
\end{align*}

These relations are equivalent to saying that the fundamental matrix $(\alpha - q \gamma^\ast \gamma \alpha^\ast)$ is unitary.

The maximal torus survives the deformation untouched: There exists a $\ast$-homomorphism $\pi: \mathcal{O}(SU_q(2)) \to \mathcal{O}(T) = \mathbb{C}[z, z^{-1}]$ given by $\pi(\alpha) = z$, $\pi(\gamma) = 0$. 

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These relations are equivalent to saying that the fundamental matrix

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\begin{pmatrix}
\alpha & -q \gamma^*
\
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The quantization of $AN$

Every (locally compact) quantum group admits a Pontrjagin dual (locally compact) quantum group.
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In the case of $K_q$, the dual $\hat{K}_q$ is encoded by the $*$-algebra

$$C_c(\hat{K}_q) = \mathcal{D}(K_q) = \bigoplus_{\mu \in P^+} \text{End}(V(\mu)),$$

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To the classical group $A$ corresponds the quotient $\hat{T}$ of $\hat{K}_q$ obtained from the projection $\mathcal{O}(K_q) \to \mathcal{O}(T)$. Here $T \subset K_q$ is the classical maximal torus.
Consider the vector space $D(G_q) = D(K_q) \triangleright\triangleleft O(K_q)$, equipped with the multiplication $(x \triangleright\triangleleft f)(y \triangleright\triangleleft g) = x(f(1), y(1))y(2) \triangleright\triangleleft f(2)(f(3), \hat{S}(y(3)))g$ and the $\ast$-structure $(x \triangleright\triangleleft f)^\ast = (1 \triangleright\triangleleft f^\ast)(x^\ast \triangleright\triangleleft 1)$.

Definition The group $C^\ast$-algebra $C^\ast(G_q)$ of the complex quantum group $G_q$ is the universal $C^\ast$-completion of $D(G_q)$. 

Christian Voigt  (joint with R. Yuncken)
Consider the vector space

$$\mathcal{D}(G_q) = \mathcal{D}(K_q) \otimes \mathcal{O}(K_q),$$

equipped with the multiplication

$$(x \otimes f)(y \otimes g) = x(f(1), y(1))y(2) \otimes f(2)(f(3), \hat{S}(y(3)))g$$

and the $*$-structure

$$(x \otimes f)^* = (1 \otimes f^*)(x^* \otimes 1).$$
Consider the vector space

\[ \mathcal{D}(G_q) = \mathcal{D}(K_q) \Join \mathcal{O}(K_q), \]

equipped with the multiplication

\[ (x \Join f)(y \Join g) = x(f_1, y_1)y_2 \Join f_2(f_3, \hat{S}(y_3))g \]

and the \( \ast \)-structure

\[ (x \Join f)^\ast = (1 \Join f^\ast)(x^\ast \Join 1). \]

**Definition**

The group \( \mathbb{C}^\ast \)-algebra \( \mathbb{C}^\ast(G_q) \) of the complex quantum group \( G_q \) is the universal \( \mathbb{C}^\ast \)-completion of \( \mathcal{D}(G_q) \).
The representation theory of $G_q$

This leads to some natural tasks/questions.

Describe all irreducible representations of $G_q$ up to isomorphism.

Describe the (reduced) unitary dual of $G_q$.

Describe the Plancherel formula.

Describe the Fell topology of the (reduced) dual.
The representation theory of $G_q$

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- Describe all irreducible representations of $G_q$ up to isomorphism.
- Describe the (reduced) unitary dual of $G_q$.
- Describe the Plancherel formula.
- Describe the Fell topology of the (reduced) dual.
By construction, a nondegenerate representation of $C^*(G_q)$ on a Hilbert space $\mathcal{H}$ corresponds to a nondegenerate $\ast$-homomorphism $\mathcal{D}(G_q) \to \mathcal{L}(\mathcal{H})$. This is the same thing as a unitary Yetter-Drinfeld module, that is, a pair of a unital $\ast$-homomorphism $\mathcal{O}(K_q) \to \mathcal{L}(\mathcal{H})$ and a unitary corepresentation $V \in M(C(K_q) \otimes \mathcal{H})$ satisfying the Yetter-Drinfeld compatibility condition, given by

$$f(1)\xi(-1)S(f(3)) \otimes f(2) \cdot \xi(0) = (f \cdot \xi)(-1) \otimes (f \cdot \xi)(0)$$

for $f \in \mathcal{O}(K_q)$ and $\xi$ in (a certain dense subspace of) $\mathcal{H}$. 

Christian Voigt (joint with R. Yuncken)
The representation theory of $G_q$

By construction, a nondegenerate representation of $C^*(G_q)$ on a Hilbert space $\mathcal{H}$ corresponds to a nondegenerate $\ast$-homomorphism $D(G_q) \to L(\mathcal{H})$.

This is the same thing as a unitary Yetter-Drinfeld module, that is, a pair of a unital $\ast$-homomorphism $\mathcal{O}(K_q) \to L(\mathcal{H})$ and a unitary corepresentation $V \in M(C(K_q) \otimes \mathcal{H})$ satisfying the Yetter-Drinfeld compatibility condition, given by

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for $f \in \mathcal{O}(K_q)$ and $\xi$ in (a certain dense subspace of) $\mathcal{H}$. 
Let $O(E^\mu) \subset O(Kq)$ be the spectral subspace of $O(Kq)$ associated to $\mu \in P$ with respect to the right action of $T$.

For $\lambda \in h^*$ we define the twisted left adjoint representation of $O(Kq)$ on $O(E^\mu)$ by

$$f \cdot \xi = f(1) \xi S(f(3))(K\lambda + 2\rho, f(2)).$$

Together with the comultiplication of $O(Kq)$ this turns $O(E^\mu)$ into a Yetter-Drinfeld module, which we will denote by $O(E^\mu, \lambda)$. This is called the principal series Yetter-Drinfeld module with parameter $(\mu, \lambda) \in P \times h^*$. If $\lambda \in i_{a^*} \subset h^*$ then this Yetter-Drinfeld module is unitary. It corresponds to a representation of $C^*(G^q)$ on the Hilbert space completion of $O(E^\mu)$. 

Christian Voigt (joint with R. Yuncken)
Let $\mathcal{O}(\mathcal{E}_\mu) \subset \mathcal{O}(K_q)$ be the spectral subspace of $\mathcal{O}(K_q)$ associated to $\mu \in \mathbf{P}$ with respect to the right action of $T$. For $\lambda \in \mathfrak{h}^*$ we define the twisted left adjoint representation of $\mathcal{O}(K_q)$ on $\mathcal{O}(\mathcal{E}_\mu)$ by $f \cdot \xi = f(1)\xi S(f(3))(K_{\lambda}+2\rho, f(2))$. Together with the comultiplication of $\mathcal{O}(K_q)$ this turns $\mathcal{O}(\mathcal{E}_\mu)$ into a Yetter-Drinfeld module, which we will denote by $\mathcal{O}(\mathcal{E}_\mu,\lambda)$. This is called the principal series Yetter-Drinfeld module with parameter $(\mu,\lambda) \in \mathbf{P} \times \mathfrak{h}^*$. If $\lambda \in i\mathfrak{a}^* \subset \mathfrak{h}^*$ then this Yetter-Drinfeld module is unitary. It corresponds to a representation of $C^*(G_q)$ on the Hilbert space completion of $\mathcal{O}(\mathcal{E}_\mu)$. 

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f \cdot \xi = f_1(1) \xi S(f_3)(K_{\lambda+2\rho}, f_2).
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Together with the comultiplication of \( \mathcal{O}(K_q) \) this turns \( \mathcal{O}(\mathcal{E}_\mu) \) into a Yetter-Drinfeld module, which we will denote by \( \mathcal{O}(\mathcal{E}_\mu, \lambda) \).

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The structure of principal series representations

For $\lambda \in h^*$, the operators $K_\lambda$ are defined by $K_\lambda v = q(\lambda, \nu)v$. Recall that $q = e^{\hbar}$, and let $\hbar = \frac{h}{2\pi}$. In particular, $K_\lambda = K_{\lambda'}$ if $\lambda - \lambda' \in i\hbar - 1Q^\vee$. Here $Q^\vee$ is the coroot lattice. Hence, by their very construction, the principal series modules $O(E_{\mu, \lambda})$ and $O(E_{\mu, \lambda'})$ are the same if $\lambda - \lambda' \in i\hbar - 1Q^\vee$.

Write $h^*q = h^*/i\hbar - 1Q$, $a^*q = a^*/i\hbar - 1Q$. This notation allows us to remove the "obvious" redundancies in the parametrisation of the principal series explained above.

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Write

$$\mathfrak{h}_q^* = \mathfrak{h}^*/i\hbar^{-1}Q, \quad \mathfrak{a}_q^* = \mathfrak{a}^*/i\hbar^{-1}Q.$$  

This notation allows us to remove the “obvious” redundancies in the parametrisation of the principal series explained above.
The structure of principal series representations

For $\lambda \in h^*$ and $\alpha \in \Delta$, write $\lambda \alpha = 2(\alpha, \lambda) / (\alpha, \alpha)$.

Theorem

Let $(\mu, \lambda) \in P \times h^*$ such that $\lambda \alpha \neq \pm (|\mu\alpha| + j)$ modulo $i \hbar - 1 Z$ for all $j \in \mathbb{N}$ and all $\alpha \in \Delta^+$. Then the principal series module with parameter $(\mu, \lambda)$ is an irreducible Yetter-Drinfeld module.

Theorem

Let $(\mu, \lambda) \in P \times i^*$. Then the principal series modules with parameters $(\mu, \lambda)$ and $(\mu', \lambda')$ are equivalent iff $(\mu', \lambda') = (w \cdot \mu, w \cdot \lambda)$ for some $w \in W$.

These results are (essentially) due to Joseph-Letzter and depend on deep facts about the structure of $U_q(g)$.
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Let $(\mu, \lambda) \in P \times i\mathfrak{t}_q^*$. Then the principal series modules with parameters $(\mu, \lambda)$ and $(\mu', \lambda')$ are equivalent iff $(\mu', \lambda') = (w \cdot \mu, w \cdot \lambda)$ for some $w \in W$.

These results are (essentially) due to Joseph-Letzter and depend on deep facts about the structure of $U_q(\mathfrak{g})$. 
The Plancherel formula

Theorem

Let $q \in (0, 1)$ and let $G_q$ be a complex semisimple quantum group. Moreover let $H = (H_{\mu, \nu})_{\mu, \nu}$ be the Hilbert space bundle of unitary principal series representations over $P \times \mathfrak{a}^*$. Then there is a unitary isomorphism $Q : L^2(G_q) \cong \bigoplus_{\mu \in P} \int_{\mathfrak{a}^*} \otimes_{\nu \in \mathfrak{a}^*} HS(H_{\mu, \nu}) \, d\nu$ for the measures $d\mu \nu$ on $\mathfrak{a}^*$ given by

$$d\mu \nu = \prod_{\alpha \in \Delta^+} \left( q^{1/2} \alpha^{-q^{-1}/2} \right) \left( \mu + i \nu \right)^{\alpha} q^{1/2} \left( \mu - i \nu \right)^{\alpha} d\nu,$$

where $d\nu$ denotes normalised Lebesgue measure on $\mathfrak{a}^*$.

Christian Voigt (joint with R. Yuncken)
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$$Q : L^2(G_q) \cong \bigoplus_{\mu \in P} \int_{\nu \in a_q^*}^{\oplus} HS(\mathcal{H}_{\mu, i\nu}) dm_\mu(\nu)$$

for the measures $dm_\mu$ on $a_q^*$ given by

$$dm_\mu(\nu) = \prod_{\alpha \in \Delta^+} (q_\alpha^{1/2} - q_\alpha^{-1/2})^2 [(\mu + i\nu)_\alpha]_{q_\alpha^{1/2}} [(\mu - i\nu)_\alpha]_{q_\alpha^{1/2}} d\nu,$$

where $d\nu$ denotes normalised Lebesgue measure on $a_q^*$. 

Christian Voigt  (joint with R. Yuncken)
Some remarks

The proof proceeds by verifying the Plancherel formula
\[ \varepsilon_G q(f) = \sum_{\mu \in P} \int a^* q \, \text{tr}(\pi_{\mu}, i_{\nu}(f)) D^{-2}_{\mu, i_{\nu}} \, dm_{\mu}(\nu) \]
for elements of the form
\[ f = u_{\beta}^{ij} \otimes \omega_{\gamma}^{kl} \in \mathcal{O}(K q) \otimes D(K q). \]

For this one starts by directly calculating the characters of principal series representations.

In this computation, the universal \( R \)-matrix of \( U_q(g) \) enters crucially.

The lowest order contribution in \( h \) of the quantum Plancherel measure agrees with the classical Plancherel measure
\[ \prod_{\alpha \in \dot{\alpha}} |(\mu + i\nu_{\alpha})|^{2} \, d\nu_{\alpha} = (\mu + i\nu_{\alpha})^{\alpha}(\mu - i\nu_{\alpha})^{\alpha} \]
on \( P \times a^* \).

Christian Voigt (joint with R. Yuncken)
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$$\epsilon_{G_q}(f) = \sum_{\mu \in P} \int_{a_q^*} \text{tr}(\pi_{\mu,i\nu}(f)D_{\mu,i\nu}^{-2})dm_\mu(\nu)$$

for elements of the form $f = u_{ij}^\beta \otimes \omega_{kl}^{\gamma} \in \mathcal{O}(K_q) \otimes \mathcal{D}(K_q)$.

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$$\prod_{\alpha \in \check{\gamma}^+} |(\mu_\alpha + i\nu_\alpha)|^2 d\nu = (\mu + i\nu)_\alpha (\mu - i\nu)_\alpha d\nu$$

on $P \times a^*$.
The reduced dual of $G_q$ is the norm closure of $D(G_q)$ inside $L(L^2(G_q))$ under the regular representation.

Theorem

Let $q \in (0,1)$ and let $G_q$ be a complex semisimple quantum group. Moreover let $H = (H_{\mu,\lambda})_{\mu,\lambda}$ be the Hilbert space bundle of principal series representations of $G_q$ over $P \times a^*$. Then the canonical $\ast$-homomorphism $\pi : C^*_r(G_q) \to C^0(P \times a^*, K(H))$ is an isomorphism.

Setting formally $h = 0$ here (corresponding to $q = 1$), and $a^*1 = a^*$ one obtains the corresponding statement for the classical reduced $\ast$-algebra $C^*_r(G)$. 

Christian Voigt (joint with R. Yuncken)
The reduced dual of $G_q$

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**Theorem**

Let $q \in (0, 1)$ and let $G_q$ be a complex semisimple quantum group. Moreover let $\mathcal{H} = (\mathcal{H}_{\mu, \lambda})_{\mu, \lambda}$ be the Hilbert space bundle of principal series representations of $G_q$ over $P \times a^*_q$. Then the canonical $*$-homomorphism

\[
\pi : C^*_r(G_q) \to C_0(P \times a^*_q, \mathbb{K}(\mathcal{H}))^W
\]

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Christian Voigt (joint with R. Yuncken)
The deformation picture of the Baum-Connes assembly map for the classical complex group $G$ provides an isomorphism $K^*(C^*(K \rtimes \text{ad}k)) \to K^*(C^r(G))$. Let us restrict attention to the case $G = \text{SL}(2, \mathbb{C})$.

Theorem

Fix $q \in (0,1)$. Then there is a commutative diagram

$$
\begin{array}{ccc}
K^*(C^*(k)) & \xrightarrow{\mu} & K^*(C^r(G)) \\
\downarrow & & \downarrow \\
K^*(C^*(K \rtimes \text{ad}C)) & \xrightarrow{\mu_q} & K^*(C^r(G_q))
\end{array}
$$

Both vertical maps are split injective, and the horizontal maps are isomorphisms.
The deformation picture of the Baum-Connes assembly map for the classical complex group $G$ provides an isomorphism

$$ K_\ast(C^*(K \rtimes_{ad} \mathfrak{k}^*)) = K_\ast(K \rtimes_{ad} C_0(\mathfrak{k})) \to K_\ast(C_\ast_r(G)). $$

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**Theorem**

*Fix $q \in (0, 1)$. Then there is a commutative diagram*

$$
\begin{array}{ccc}
K_\ast(K \rtimes_{ad} C_0(\mathfrak{k})) & \xrightarrow{\mu} & K_\ast(C_r^\ast(G)) \\
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K_\ast(K \rtimes_{ad} C(K)) & \xrightarrow{\mu_q} & K_\ast(C_r^\ast(G_q))
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$$

*Both vertical maps are split injective, and the horizontal maps are isomorphisms.*