The Plancherel Formula for complex quantum groups

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For $f \in L^1(\mathbb{R})$ the Fourier transform of f is defined by

$$\mathcal{F}(f)(p) = \int_{\mathbb{R}} e^{-ixp} f(x) dx,$$

where dx denotes (suitably normalised) Lebesgue measure.

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Theorem (Plancherel) Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then

$$\|\mathcal{F}(f)\|_{2}^{2} = \int_{\mathbb{R}} |\mathcal{F}(f)(p)|^{2} dp = \int_{\mathbb{R}} |f(x)|^{2} dx = \|f\|_{2}^{2}.$$

Hence \mathcal{F} induces a unitary isomorphism $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$.

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Since \mathbb{R} is a locally compact abelian group, it has a Pontrjagin dual group $\hat{\mathbb{R}}$, consisting of all unitary characters of \mathbb{R} .

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The unitary characters of $\ensuremath{\mathbb{R}}$ are of the form

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for $p \in \mathbb{R}$.

In this way one obtains $\hat{\mathbb{R}} \cong \mathbb{R}$.

The classical Plancherel Theorem

The group C^* -algebra $C^*(\mathbb{R})$ is a completion of $C^{\infty}_c(\mathbb{R})$, equipped with the convolution product

$$(f * g)(t) = \int_{\mathbb{R}} f(-s)g(s+t)ds$$

and *-structure

$$f^*(t)=\overline{f(-s)}.$$

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In particular, for the one-dimensional representations corresponding to the characters χ_p we obtain *-homomorphisms $\chi_p : C^*(\mathbb{R}) \to \mathbb{C}$ given by

$$\chi_p(f) = \int_{\mathbb{R}} f(x)\chi_p(x) = \int_{\mathbb{R}} f(x)e^{-ipx}dx = \mathcal{F}(f)(p)$$

for $f \in C_c^{\infty}(\mathbb{R})$.

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The classical Plancherel Theorem

For $f\in \mathit{C}^\infty_c(\mathbb{R})$ we have

$$(f^* * f)(0) = \int_{\mathbb{R}} \overline{f(s)} f(s) ds$$

= $\|f\|_2^2 = \|\mathcal{F}(f)\|_2^2 = \int_{\widehat{\mathbb{R}}} \overline{\mathcal{F}(f)(p)} \mathcal{F}(f)(p) dp$
= $\int_{\widehat{\mathbb{R}}} \chi_p(f)^* \chi_p(f) dp = \int_{\widehat{\mathbb{R}}} \chi_p(f^* * f) dp,$

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or equivalently,

Theorem (Plancherel formula) For any $h \in C_c^{\infty}(\mathbb{R})$ we have

$$h(0)=\int_{\hat{\mathbb{R}}}\chi_p(h)dp.$$

Plancherel versus Peter-Weyl

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Write Irr(G) for the set of equivalence classes of irreducible representations of G, and $\pi_{\lambda} : G \to U(\mathcal{H}_{\lambda})$ for $\lambda \in Irr(G)$.

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Theorem (Peter-Weyl) For $f \in L^1(G) \cap L^2(G)$ we have

$$\|f\|_2^2 = \sum_{\lambda \in \mathsf{Irr}(G)} \mathsf{tr}(\pi_\lambda(f)^* \pi_\lambda(f)) \dim(\mathcal{H}_\lambda)^{-1}$$

Hence the formula

$$\mathcal{F}(f) = igoplus_{\lambda \in \mathsf{Irr}(G)} \pi_{\lambda}(f)$$

for $f \in L^1(G) \cap L^2(G)$ extends to an isometric isomorphism

$$\mathcal{F}: L^2(G) \to \bigoplus_{\lambda \in \mathsf{Irr}(G)} \mathit{HS}(\mathcal{H}_{\lambda}),$$

if on Irr(G) we consider the (Plancherel) measure

$$dm = \sum_{\lambda \in \mathsf{Irr}(\mathcal{G})} \dim(\mathcal{H}_{\lambda})^{-1} \delta_{\lambda}.$$

Assume that G is a type I locally compact unimodular group.

Theorem (Segal-Mautner)

Then there exists a standard measure m on Irr(G), a measurable field of Hilbert spaces $(\mathcal{H}_{\lambda})_{\lambda \in Irr(G)}$, and an isometric G-equivariant isomorphism

$$\mathcal{F}: L^2(G) \to \int_{\mathrm{Irr}(G)}^{\oplus} HS(\mathcal{H}_{\lambda}) dm(\lambda),$$

given by

$$\mathcal{F}(f) = \int_{\operatorname{Irr}(G)}^{\oplus} \pi_{\lambda}(f) dm(\lambda)$$

on a dense subspace of $L^1(G) \cap L^2(G)$.

Assume that G is a type I locally compact possibly non-unimodular group.

Theorem (Segal-Mautner, Duflo-Moore)

Then there exists a standard measure m on Irr(G), a measurable field of Hilbert spaces $(\mathcal{H}_{\lambda})_{\lambda \in Irr(G)}$, a measurable field $(D_{\lambda})_{\lambda \in Irr(G)}$ of self-adjoint strictly positive operators for $(\mathcal{H}_{\lambda})_{\lambda \in Irr(G)}$, and an isometric G-equivariant isomorphism

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Abstract Plancherel Theorem

Assume that G is a type I locally compact possibly non-unimodular quantum group.

Theorem (Segal-Mautner, Duflo-Moore, Desmedt)

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on a dense subspace of $L^1(G) \cap L^2(G)$.

The appearance of Duflo-Moore operators is not really due to non-unimodularity, but rather related to the question of whether the (left) Haar weight of the group algebra is a trace or not. In the group case, this is equivalent to (non-) unimodularity. The appearance of Duflo-Moore operators is not really due to non-unimodularity, but rather related to the question of whether the (left) Haar weight of the group algebra is a trace or not. In the group case, this is equivalent to (non-) unimodularity.

For instance, for a compact quantum group, there are Duflo-Moore operators in the Plancherel formula. These are trivial iff the quantum group is of Kac type - note that compact quantum groups are always unimodular.

If G is a compact quantum group the Plancherel formula becomes

$$\epsilon(f) = \sum_{\lambda \in \mathsf{Irr}(G)} \dim_q(\mathcal{H}_{\lambda}) \operatorname{tr}(\pi_{\lambda}(f) D_{\lambda}^{-2})$$

for $f \in \mathcal{O}(G)$.

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A little bit of history:

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A little bit of history:

- Podleś-Woronowicz (1990) construct complex semisimple quantum groups on the C*-algebra level.
- ► Pusz (1993), Pusz-Woronowicz (1994, 2000) completely classify the irreducible unitary representations of SL_q(2, C).
- ▶ Buffenoir-Roche (1999) determine the Plancherel formula for SL_q(2, ℂ).
- ► Arano (2014, 2016) completely classifies the irreducible unitary representations of SL_q(n, C), and most of the full dual in general.

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- According to Drinfeld duality, a quantization of the Poisson dual AN of K is given by the Pontrjagin dual K̂_q of K_q.

Here is a quick outline of the construction of the quantization G_q of a (simply connected) complex semisimple group G:

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- For the compact part K there exists a deformation K_q obtained using quantized enveloping algebras.
- According to Drinfeld duality, a quantization of the Poisson dual AN of K is given by the Pontrjagin dual K̂_q of K_q.
- The complex quantum group G_q is the quantum double

$$G_q = K_q \bowtie \hat{K}_q.$$

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We shall now explain the ingredients in these constructions in more detail.

Notation

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Notation

- Fix $q = e^h \in (0, 1)$.
- ▶ Let g be a semisimple complex Lie algebra of rank N with Cartan matrix (a_{ij}).
- $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra.
- $\Delta = \Delta^+ \cup \Delta^-$ the root system with simple roots $\alpha_1, \ldots, \alpha_N \subset \mathfrak{h}^*$.
- (,) the bilinear form on h^{*} obtained by rescaling the Killing form such that all short roots α satisfy (α, α) = 2.

• Set
$$d_i = (\alpha_i, \alpha_i)/2$$
 and $q_i = q^{d_i}$

- $\varpi_1, \ldots, \varpi_N \in \mathfrak{h}^*$ are the fundamental weights.
- ▶ $\mathbf{P} = \bigoplus_{j=1}^{N} \mathbb{Z} \varpi_j$ and $\mathbf{Q} = \bigoplus_{j=1}^{N} \mathbb{Z} \alpha_j$ are the weight and root lattices, respectively.

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- $\mathbf{P}^+ = \bigoplus_{j=1}^N \mathbb{N}_0 \varpi_j$ are the dominant integral weights.
- W is the Weyl group of g.

The Drinfeld-Jimbo algebra associated to \mathfrak{g}

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The Drinfeld-Jimbo algebra associated to \mathfrak{g}

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is the algebra with generators E_j , F_j for $1 \leq j \leq N$ and K_λ for $\lambda \in \mathbf{P}$ satisfying

$$\begin{split} & \mathcal{K}_{0} = 1, \qquad \mathcal{K}_{\lambda}\mathcal{K}_{\mu} = \mathcal{K}_{\lambda+\mu}, \\ & \mathcal{K}_{\lambda}\mathcal{E}_{j}\mathcal{K}_{\lambda}^{-1} = q^{(\lambda,\alpha_{j})}\mathcal{E}_{j}, \qquad \mathcal{K}_{\lambda}\mathcal{F}_{j}\mathcal{K}_{\lambda}^{-1} = q^{-(\lambda,\alpha_{j})}\mathcal{F}_{j}, \\ & [\mathcal{E}_{i},\mathcal{F}_{j}] = \delta_{ij}\frac{\mathcal{K}_{i} - \mathcal{K}_{i}^{-1}}{q_{i} - q_{i}^{-1}}, \quad \text{where } \mathcal{K}_{i} = \mathcal{K}_{\alpha_{i}}, \\ & \sum_{k=0}^{1-a_{ij}} (-1)^{k} \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_{i}} \mathcal{E}_{i}^{k}\mathcal{E}_{j}\mathcal{E}_{i}^{1-a_{ij}-k} = 0 \qquad i \neq j, \\ & \sum_{k=0}^{1-a_{ij}} (-1)^{k} \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_{i}} \mathcal{F}_{i}^{k}\mathcal{F}_{j}\mathcal{F}_{i}^{1-a_{ij}-k} = 0 \qquad i \neq j. \end{split}$$

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The Drinfeld-Jimbo algebra associated to \mathfrak{g}

The algebra $U_q(\mathfrak{g})$ is a Hopf algebra. For instance, the coproduct $\hat{\Delta} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is given by

$$egin{aligned} \hat{\Delta}(\mathcal{K}_{\lambda}) &= \mathcal{K}_{\lambda} \otimes \mathcal{K}_{\lambda}, \ \hat{\Delta}(\mathcal{E}_{i}) &= \mathcal{E}_{i} \otimes \mathcal{K}_{i} + 1 \otimes \mathcal{E}_{i} \ \hat{\Delta}(\mathcal{F}_{i}) &= \mathcal{F}_{i} \otimes 1 + \mathcal{K}_{i}^{-1} \otimes \mathcal{F}_{i}. \end{aligned}$$

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Moreover $U_q(\mathfrak{g})$ is a *-algebra with the *-structure

$$E_i^* = K_i F_i, \qquad F_i^* = E_i K_i^{-1}, \qquad K_\lambda^* = K_\lambda.$$

As a Hopf *-algebra, $U_q(\mathfrak{g})$ should be viewed as quantization of the (complex) universal enveloping algebra of the (real) Lie algebra \mathfrak{k} .

The finite dimensional representation theory of $U_q(\mathfrak{g})$ is similar to the one for $U(\mathfrak{g})$. In particular, for every $\mu \in \mathbf{P}^+$ there exists a unique irreducible representation $V(\mu)$ with a highest weight vector v_{μ} , satisfying

$${\it K}_{\lambda}{\it v}_{\mu}={\it q}^{(\lambda,\mu)}{\it v}_{\mu}$$

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$$K_{\lambda}v_{\mu}=q^{(\lambda,\mu)}v_{\mu}$$

Using the representations $V(\mu)$ one defines a compact quantum group K_q as follows.

Definition

The algebra $\mathcal{O}(K_q) \subset U_q(\mathfrak{g})^*$ of representative functions on K_q is the Hopf *-algebra of matrix coefficients of all $V(\mu)$ for $\mu \in \mathbf{P}^+$. We let $C(K_q)$ be its universal C^* -completion.

 $\mathcal{O}(K_q)$ is a deformation of the algebra $\mathcal{O}(K)$ of representative functions on K, and $C(K_q)$ is a deformation of C(K).

Example: the quantum group $SU_q(2)$

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Example: the quantum group $SU_q(2)$

The algebra $\mathcal{O}(SU_q(2))$ can be identified with the *-algebra generated by elements α and γ satisfying the relations

$$\begin{split} &\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma, \\ &\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1. \end{split}$$

These relations are equivalent to saying that the fundamental matrix

$$\begin{pmatrix} \alpha & -\boldsymbol{q}\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

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These relations are equivalent to saying that the fundamental matrix

$$\begin{pmatrix} \alpha & -\boldsymbol{q}\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is unitary.

The maximal torus survives the deformation untouched: There exists a *-homomorphism $\pi : \mathcal{O}(SU_q(2)) \to \mathcal{O}(T) = \mathbb{C}[z, z^{-1}]$ given by $\pi(\alpha) = z, \pi(\gamma) = 0$.

Every (locally compact) quantum group admits a Pontrjagin dual (locally compact) quantum group.

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In the case of K_q , the dual \hat{K}_q is encoded by the *-algebra

$$C_c(\hat{K}_q) = \mathcal{D}(K_q) = \bigoplus_{\mu \in \mathbf{P}^+} \operatorname{End}(V(\mu)),$$

equipped with a suitable coproduct.

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equipped with a suitable coproduct.

To the classical group A corresponds the quotient \hat{T} of \hat{K}_q obtained from the projection $\mathcal{O}(K_q) \to \mathcal{O}(T)$. Here $T \subset K_q$ is the classical maximal torus.

Complex semisimple quantum groups

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Complex semisimple quantum groups

Consider the vector space

$$\mathcal{D}(G_q) = \mathcal{D}(K_q) \bowtie \mathcal{O}(K_q),$$

equipped with the multiplication

$$(x \bowtie f)(y \bowtie g) = x(f_{(1)}, y_{(1)})y_{(2)} \bowtie f_{(2)}(f_{(3)}, \hat{S}(y_{(3)}))g$$

and the *-structure

$$(x \bowtie f)^* = (1 \bowtie f^*)(x^* \bowtie 1).$$

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Definition

The group C^* -algebra $C^*(G_q)$ of the complex quantum group G_q is the universal C^* -completion of $\mathcal{D}(G_q)$.

This leads to some natural tasks/questions.

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- ► Describe all irreducible representations of *G_q* up to isomorphism.
- Describe the (reduced) unitary dual of G_q .
- Describe the Plancherel formula.
- Describe the Fell topology of the (reduced) dual.

By construction, a nondegenerate representation of $C^*(G_q)$ on a Hilbert space \mathcal{H} corresponds to a nondegenerate *-homomorphism $\mathcal{D}(G_q) \to \mathcal{L}(\mathcal{H})$.

By construction, a nondegenerate representation of $C^*(G_q)$ on a Hilbert space \mathcal{H} corresponds to a nondegenerate *-homomorphism $\mathcal{D}(G_q) \to \mathcal{L}(\mathcal{H}).$

This is the same thing as a unitary Yetter-Drinfeld module, that is, a pair of a unital *-homomorphism $\mathcal{O}(K_q) \to \mathcal{L}(\mathcal{H})$ and a unitary corepresentation $V \in M(C(K_q) \otimes \mathcal{H})$ satisfying the Yetter-Drinfeld compatibility condition, given by

$$f_{(1)}\xi_{(-1)}S(f_{(3)}) \otimes f_{(2)} \cdot \xi_{(0)} = (f \cdot \xi)_{(-1)} \otimes (f \cdot \xi)_{(0)}$$

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for $f \in \mathcal{O}(K_q)$ and ξ in (a certain dense subspace of) \mathcal{H} .

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For $\lambda \in \mathfrak{h}^*$ we define the twisted left adjoint representation of $\mathcal{O}(K_q)$ on $\mathcal{O}(\mathcal{E}_\mu)$ by

$$f \cdot \xi = f_{(1)} \xi S(f_{(3)})(K_{\lambda+2\rho}, f_{(2)}).$$

Together with the comultiplication of $\mathcal{O}(K_q)$ this turns $\mathcal{O}(\mathcal{E}_{\mu})$ into a Yetter-Drinfeld module, which we will denote by $\mathcal{O}(\mathcal{E}_{\mu,\lambda})$.

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This is called the *principal series Yetter-Drinfeld module* with parameter $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}^*$.

If $\lambda \in i\mathfrak{a}^* \subset \mathfrak{h}^*$ then this Yetter-Drinfeld module is unitary. It corresponds to a representation of $C^*(G_q)$ on the Hilbert space completion of $\mathcal{O}(\mathcal{E}_{\mu})$.

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In particular, $K_{\lambda} = K_{\lambda'}$ if $\lambda - \lambda' \in i\hbar^{-1}\mathbf{Q}^{\vee}$. Here \mathbf{Q}^{\vee} is the coroot lattice.

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Hence, by their very construction, the principal series modules $\mathcal{O}(\mathcal{E}_{\mu,\lambda})$ and $\mathcal{O}(\mathcal{E}_{\mu,\lambda'})$ are the same if $\lambda - \lambda' \in i\hbar^{-1}\mathbf{Q}^{\vee}$.

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Write

$$\mathfrak{h}_q^* = \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}, \qquad \mathfrak{a}_q^* = \mathfrak{a}^*/i\hbar^{-1}\mathbf{Q}.$$

This notation allows us to remove the "obvious" redundancies in the parametrisation of the principal series explained above.

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For $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta$ write $\lambda_{\alpha} = 2(\alpha, \lambda)/(\alpha, \alpha)$.

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Theorem

Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ such that $\lambda_{\alpha} \neq \pm (|\mu_{\alpha}| + j)$ modulo $i\hbar^{-1}\mathbb{Z}$ for all $j \in \mathbb{N}$ and all $\alpha \in \Delta^+$. Then the principal series module with parameter (μ, λ) is an irreducible Yetter-Drinfeld module.

For
$$\lambda \in \mathfrak{h}^*$$
 and $\alpha \in \Delta$ write $\lambda_{\alpha} = 2(\alpha, \lambda)/(\alpha, \alpha)$.

Theorem

Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ such that $\lambda_{\alpha} \neq \pm (|\mu_{\alpha}| + j)$ modulo $i\hbar^{-1}\mathbb{Z}$ for all $j \in \mathbb{N}$ and all $\alpha \in \Delta^+$. Then the principal series module with parameter (μ, λ) is an irreducible Yetter-Drinfeld module.

Theorem

Let $(\mu, \lambda) \in \mathbf{P} \times i\mathfrak{t}_q^*$. Then the principal series modules with parameters (μ, λ) and (μ', λ') are equivalent iff $(\mu', \lambda') = (w \cdot \mu, w \cdot \lambda)$ for some $w \in W$.

These results are (essentially) due to Joseph-Letzter and depend on deep facts about the structure of $U_q(\mathfrak{g})$.

The Plancherel formula

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Theorem

Let $q \in (0,1)$ and let G_q be a complex semisimple quantum group. Moreover let $\mathcal{H} = (\mathcal{H}_{\mu,i\nu})_{\mu,\nu}$ be the Hilbert space bundle of unitary principal series representations over $\mathbf{P} \times \mathfrak{a}_q^*$. Then there is a unitary isomorphism

$$Q: L^{2}(G_{q}) \cong \bigoplus_{\mu \in \mathbf{P}} \int_{\nu \in \mathfrak{a}_{q}^{*}}^{\oplus} HS(\mathcal{H}_{\mu,i\nu}) dm_{\mu}(\nu)$$

for the measures dm_{μ} on \mathfrak{a}_{q}^{*} given by

$$dm_{\mu}(\nu) = \prod_{\alpha \in \Delta^{+}} (q_{\alpha}^{1/2} - q_{\alpha}^{-1/2})^{2} [(\mu + i\nu)_{\alpha}]_{q_{\alpha}^{1/2}} [(\mu - i\nu)_{\alpha}]_{q_{\alpha}^{1/2}} d\nu,$$

where $d\nu$ denotes normalised Lebesgue measure on \mathfrak{a}_a^* .

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Some remarks

The proof proceeds by verifying the Plancherel formula

$$\epsilon_{G_q}(f) = \sum_{\mu \in \mathbf{P}} \int_{\mathfrak{a}_q^*} \operatorname{tr}(\pi_{\mu,i
u}(f)D_{\mu,i
u}^{-2})dm_{\mu}(
u)$$

for elements of the form $f = u_{ij}^{\beta} \otimes \omega_{kl}^{\gamma} \in \mathcal{O}(K_q) \otimes \mathcal{D}(K_q).$

For this one starts by directly calculating the characters of principal series representations.

In this computation, the universal R-matrix of $U_q(\mathfrak{g})$ enters crucially.

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The lowest order contribution in h of the quantum Plancherel measure agrees with the classical Plancherel measure

$$\prod_{lpha\in {}^{\prime+}}|(\mu_{lpha}+i
u_{lpha})|^{2}d
u=(\mu+i
u)_{lpha}(\mu-i
u)_{lpha}d
u$$

on $\mathbf{P} \times \mathfrak{a}^*$.
The reduced dual of G_q

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The reduced group C^* -algebra of G_q is the norm closure of $\mathcal{D}(G_q)$ inside $\mathcal{L}(L^2(G_q))$ under the regular representation.

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Theorem

Let $q \in (0, 1)$ and let G_q be a complex semisimple quantum group. Moreover let $\mathcal{H} = (\mathcal{H}_{\mu,\lambda})_{\mu,\lambda}$ be the Hilbert space bundle of principal series representations of G_q over $\mathbf{P} \times \mathfrak{a}_q^*$. Then the canonical *-homomorphism

$$\pi: \mathit{C}^*_{\mathsf{r}}(\mathit{G}_q)
ightarrow \mathit{C}_0(\mathsf{P} imes \mathfrak{a}_q^*, \mathbb{K}(\mathcal{H}))^W$$

is an isomorphism.

Setting formally h = 0 here (corresponding to q = 1), and $\mathfrak{a}_1^* = \mathfrak{a}^*$ one obtains the corresponding statement for the classical reduced group C^* -algebra $C_r^*(G)$.

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The deformation picture of the Baum-Connes assembly map for the classical complex group G provides an isomorphism

$$\mathcal{K}_*(\mathcal{C}^*(\mathcal{K}\ltimes_{\mathsf{ad}}\mathfrak{k}^*)) = \mathcal{K}_*(\mathcal{K}\ltimes_{\mathsf{ad}}\mathcal{C}_0(\mathfrak{k})) \to \mathcal{K}_*(\mathcal{C}^*_\mathsf{r}(\mathcal{G})).$$

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Let us restrict attention to the case $G = SL(2, \mathbb{C})$.

Theorem

Fix $q \in (0,1)$. Then there is a commutative diagram

$$\begin{array}{cccc}
\mathcal{K}_{*}(\mathcal{K} \ltimes_{\mathsf{ad}} \mathcal{C}_{0}(\mathfrak{k})) & \stackrel{\mu}{\longrightarrow} \mathcal{K}_{*}(\mathcal{C}_{\mathsf{r}}^{*}(\mathcal{G})) \\
& & & \downarrow \\
\mathcal{K}_{*}(\mathcal{K} \ltimes_{\mathsf{ad}} \mathcal{C}(\mathcal{K})) & \stackrel{\mu_{q}}{\longrightarrow} \mathcal{K}_{*}(\mathcal{C}_{\mathsf{r}}^{*}(\mathcal{G}_{q}))
\end{array}$$

Both vertical maps are split injective, and the horizontal maps are isomorphisms.