# The Plancherel Formula for complex quantum groups 

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## The classical Plancherel Theorem

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For $f \in L^{1}(\mathbb{R})$ the Fourier transform of $f$ is defined by

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\mathcal{F}(f)(p)=\int_{\mathbb{R}} e^{-i x p} f(x) d x
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where $d x$ denotes (suitably normalised) Lebesgue measure.

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where $d x$ denotes (suitably normalised) Lebesgue measure.

Theorem (Plancherel)
Let $f \in L^{1}(\mathbb{R}) \cap L^{2}(\mathbb{R})$. Then

$$
\|\mathcal{F}(f)\|_{2}^{2}=\int_{\mathbb{R}}|\mathcal{F}(f)(p)|^{2} d p=\int_{\mathbb{R}}|f(x)|^{2} d x=\|f\|_{2}^{2}
$$

Hence $\mathcal{F}$ induces a unitary isomorphism $L^{2}\left(\mathbb{R}^{2}\right) \rightarrow L^{2}\left(\mathbb{R}^{2}\right)$.

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for $p \in \mathbb{R}$.
In this way one obtains $\hat{\mathbb{R}} \cong \mathbb{R}$.

## The classical Plancherel Theorem

The group $C^{*}$-algebra $C^{*}(\mathbb{R})$ is a completion of $C_{c}^{\infty}(\mathbb{R})$, equipped with the convolution product

$$
(f * g)(t)=\int_{\mathbb{R}} f(-s) g(s+t) d s
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In particular, for the one-dimensional representations corresponding to the characters $\chi_{p}$ we obtain $*$-homomorphisms $\chi_{p}: C^{*}(\mathbb{R}) \rightarrow \mathbb{C}$ given by

$$
\chi_{p}(f)=\int_{\mathbb{R}} f(x) \chi_{p}(x)=\int_{\mathbb{R}} f(x) e^{-i p x} d x=\mathcal{F}(f)(p)
$$

for $f \in C_{c}^{\infty}(\mathbb{R})$.

## The classical Plancherel Theorem

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For $f \in C_{c}^{\infty}(\mathbb{R})$ we have

$$
\begin{aligned}
\left(f^{*} * f\right)(0) & =\int_{\mathbb{R}} \overline{f(s)} f(s) d s \\
& =\|f\|_{2}^{2}=\|\mathcal{F}(f)\|_{2}^{2}=\int_{\hat{\mathbb{R}}} \overline{\mathcal{F}(f)(p)} \mathcal{F}(f)(p) d p \\
& =\int_{\hat{\mathbb{R}}} \chi_{p}(f)^{*} \chi_{p}(f) d p=\int_{\hat{\mathbb{R}}} \chi_{p}\left(f^{*} * f\right) d p,
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$$

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Theorem (Plancherel formula)
For any $h \in C_{c}^{\infty}(\mathbb{R})$ we have

$$
h(0)=\int_{\hat{\mathbb{R}}} \chi_{p}(h) d p .
$$

## Plancherel versus Peter-Weyl

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Now let $G$ be a compact group.

Write $\operatorname{lrr}(G)$ for the set of equivalence classes of irreducible representations of $G$, and $\pi_{\lambda}: G \rightarrow U\left(\mathcal{H}_{\lambda}\right)$ for $\lambda \in \operatorname{Irr}(G)$.

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Theorem (Peter-Weyl)
For $f \in L^{1}(G) \cap L^{2}(G)$ we have

$$
\|f\|_{2}^{2}=\sum_{\lambda \in \operatorname{lrr}(G)} \operatorname{tr}\left(\pi_{\lambda}(f)^{*} \pi_{\lambda}(f)\right) \operatorname{dim}\left(\mathcal{H}_{\lambda}\right)^{-1}
$$

## Plancherel versus Peter-Weyl

Hence the formula

$$
\mathcal{F}(f)=\bigoplus_{\lambda \in \operatorname{lrr}(G)} \pi_{\lambda}(f)
$$

for $f \in L^{1}(G) \cap L^{2}(G)$ extends to an isometric isomorphism

$$
\mathcal{F}: L^{2}(G) \rightarrow \bigoplus_{\lambda \in \operatorname{lrr}(G)} H S\left(\mathcal{H}_{\lambda}\right)
$$

if on $\operatorname{lrr}(G)$ we consider the (Plancherel) measure

$$
d m=\sum_{\lambda \in \operatorname{lrr}(G)} \operatorname{dim}\left(\mathcal{H}_{\lambda}\right)^{-1} \delta_{\lambda} .
$$

## Abstract Plancherel Theorem

Assume that $G$ is a type I locally compact unimodular group.
Theorem (Segal-Mautner)
Then there exists a standard measure $m$ on $\operatorname{lrr}(G)$, a measurable field of Hilbert spaces $\left(\mathcal{H}_{\lambda}\right)_{\lambda \in \operatorname{lrr}(G)}$, and an isometric G-equivariant isomorphism

$$
\mathcal{F}: L^{2}(G) \rightarrow \int_{\operatorname{lrr}(G)}^{\oplus} H S\left(\mathcal{H}_{\lambda}\right) d m(\lambda)
$$

given by

$$
\mathcal{F}(f)=\int_{\operatorname{lrr}(G)}^{\oplus} \pi_{\lambda}(f) d m(\lambda)
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on a dense subspace of $L^{1}(G) \cap L^{2}(G)$.

## Abstract Plancherel Theorem

Assume that $G$ is a type I locally compact possibly non-unimodular group.

## Theorem (Segal-Mautner, Duflo-Moore)

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Assume that $G$ is a type I locally compact possibly non-unimodular quantum group.

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## Remark on Duflo-Moore operators

The appearance of Duflo-Moore operators is not really due to non-unimodularity, but rather related to the question of whether the (left) Haar weight of the group algebra is a trace or not. In the group case, this is equivalent to (non-) unimodularity.

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For instance, for a compact quantum group, there are Duflo-Moore operators in the Plancherel formula. These are trivial iff the quantum group is of Kac type - note that compact quantum groups are always unimodular.

If $G$ is a compact quantum group the Plancherel formula becomes

$$
\epsilon(f)=\sum_{\lambda \in \operatorname{lrr}(G)} \operatorname{dim}_{q}\left(\mathcal{H}_{\lambda}\right) \operatorname{tr}\left(\pi_{\lambda}(f) D_{\lambda}^{-2}\right)
$$

for $f \in \mathcal{O}(G)$.

## Complex semisimple quantum groups

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- Podleś-Woronowicz (1990) construct complex semisimple quantum groups on the $C^{*}$-algebra level.
- Pusz $(1993)$, Pusz-Woronowicz $(1994,2000)$ completely classify the irreducible unitary representations of $S L_{q}(2, \mathbb{C})$.
- Buffenoir-Roche (1999) determine the Plancherel formula for $S L_{q}(2, \mathbb{C})$.
- Arano $(2014,2016)$ completely classifies the irreducible unitary representations of $S L_{q}(n, \mathbb{C})$, and most of the full dual in general.


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- Start from the Iwasawa decomposition $G=K A N$.
- For the compact part $K$ there exists a deformation $K_{q}$ obtained using quantized enveloping algebras.
- According to Drinfeld duality, a quantization of the Poisson dual $A N$ of $K$ is given by the Pontrjagin dual $\hat{K}_{q}$ of $K_{q}$.


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- For the compact part $K$ there exists a deformation $K_{q}$ obtained using quantized enveloping algebras.
- According to Drinfeld duality, a quantization of the Poisson dual $A N$ of $K$ is given by the Pontrjagin dual $\hat{K}_{q}$ of $K_{q}$.
- The complex quantum group $G_{q}$ is the quantum double

$$
G_{q}=K_{q} \bowtie \hat{K}_{q} .
$$

We shall now explain the ingredients in these constructions in more detail.

## Notation

## Christian Voigt (joint with R. Yuncken)

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- Fix $q=e^{h} \in(0,1)$.
- Let $\mathfrak{g}$ be a semisimple complex Lie algebra of rank $N$ with Cartan matrix ( $a_{i j}$ ).
- $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra.
- $\Delta=\Delta^{+} \cup \Delta^{-}$the root system with simple roots $\alpha_{1}, \ldots, \alpha_{N} \subset \mathfrak{h}^{*}$.
- (,) the bilinear form on $\mathfrak{h}^{*}$ obtained by rescaling the Killing form such that all short roots $\alpha$ satisfy $(\alpha, \alpha)=2$.
- Set $d_{i}=\left(\alpha_{i}, \alpha_{i}\right) / 2$ and $q_{i}=q^{d_{i}}$.
- $\varpi_{1}, \ldots, \varpi_{N} \in \mathfrak{h}^{*}$ are the fundamental weights.
- $\mathbf{P}=\bigoplus_{j=1}^{N} \mathbb{Z} \varpi_{j}$ and $\mathbf{Q}=\bigoplus_{j=1}^{N} \mathbb{Z} \alpha_{j}$ are the weight and root lattices, respectively.
- $\mathbf{P}^{+}=\bigoplus_{j=1}^{N} \mathbb{N}_{0} \varpi_{j}$ are the dominant integral weights.
- $W$ is the Weyl group of $\mathfrak{g}$.


## The Drinfeld-Jimbo algebra associated to $\mathfrak{g}$

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The quantized universal enveloping algebra $U_{q}(\mathfrak{g})$ is the algebra with generators $E_{j}, F_{j}$ for $1 \leq j \leq N$ and $K_{\lambda}$ for $\lambda \in \mathbf{P}$ satisfying

$$
\begin{aligned}
& K_{0}=1, K_{\lambda} K_{\mu}=K_{\lambda+\mu}, \\
& K_{\lambda} E_{j} K_{\lambda}^{-1}=q^{\left(\lambda, \alpha_{j}\right)} E_{j}, \quad K_{\lambda} F_{j} K_{\lambda}^{-1}=q^{-\left(\lambda, \alpha_{j}\right)} F_{j}, \\
& {\left[E_{i}, F_{j}\right]=\delta_{i j} \frac{K_{i}-K_{i}^{-1}}{q_{i}-q_{i}^{-1}}, \quad \text { where } K_{i}=K_{\alpha_{i}}, } \\
& \sum_{k=0}^{1-a_{i j}}(-1)^{k}\left[\begin{array}{c}
1-a_{i j} \\
k
\end{array}\right]_{q_{i}} E_{i}^{k} E_{j} E_{i}^{1-a_{i j}-k}=0 \quad i \neq j, \\
& \sum_{k=0}^{1-a_{i j}}(-1)^{k}\left[\begin{array}{c}
1-a_{i j} \\
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\end{array}\right]_{q_{i}} F_{i}^{k} F_{j} F_{i}^{1-a_{i j}-k}=0 \quad i \neq j .
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The algebra $U_{q}(\mathfrak{g})$ is a Hopf algebra.
For instance, the coproduct $\hat{\Delta}: U_{q}(\mathfrak{g}) \rightarrow U_{q}(\mathfrak{g}) \otimes U_{q}(\mathfrak{g})$ is given by

$$
\begin{aligned}
\hat{\Delta}\left(K_{\lambda}\right) & =K_{\lambda} \otimes K_{\lambda} \\
\hat{\Delta}\left(E_{i}\right) & =E_{i} \otimes K_{i}+1 \otimes E_{i} \\
\hat{\Delta}\left(F_{i}\right) & =F_{i} \otimes 1+K_{i}^{-1} \otimes F_{i}
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\end{aligned}
$$

Moreover $U_{q}(\mathfrak{g})$ is a $*$-algebra with the $*$-structure

$$
E_{i}^{*}=K_{i} F_{i}, \quad F_{i}^{*}=E_{i} K_{i}^{-1}, \quad K_{\lambda}^{*}=K_{\lambda}
$$

As a Hopf $*$-algebra, $U_{q}(\mathfrak{g})$ should be viewed as quantization of the (complex) universal enveloping algebra of the (real) Lie algebra $\mathfrak{k}$.

## Representation theory and representative functions

The finite dimensional representation theory of $U_{q}(\mathfrak{g})$ is similar to the one for $U(\mathfrak{g})$. In particular, for every $\mu \in \mathbf{P}^{+}$there exists a unique irreducible representation $V(\mu)$ with a highest weight vector $v_{\mu}$, satisfying

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Using the representations $V(\mu)$ one defines a compact quantum group $K_{q}$ as follows.

## Definition

The algebra $\mathcal{O}\left(K_{q}\right) \subset U_{q}(\mathfrak{g})^{*}$ of representative functions on $K_{q}$ is the Hopf $*$-algebra of matrix coefficients of all $V(\mu)$ for $\mu \in \mathbf{P}^{+}$. We let $C\left(K_{q}\right)$ be its universal $C^{*}$-completion.
$\mathcal{O}\left(K_{q}\right)$ is a deformation of the algebra $\mathcal{O}(K)$ of representative functions on $K$, and $C\left(K_{q}\right)$ is a deformation of $C(K)$.

## Example: the quantum group $S U_{q}(2)$

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The algebra $\mathcal{O}\left(S U_{q}(2)\right)$ can be identified with the $*$-algebra generated by elements $\alpha$ and $\gamma$ satisfying the relations

$$
\begin{gathered}
\alpha \gamma=q \gamma \alpha, \quad \alpha \gamma^{*}=q \gamma^{*} \alpha, \quad \gamma \gamma^{*}=\gamma^{*} \gamma, \\
\alpha^{*} \alpha+\gamma^{*} \gamma=1, \quad \alpha \alpha^{*}+q^{2} \gamma \gamma^{*}=1
\end{gathered}
$$

These relations are equivalent to saying that the fundamental matrix

$$
\left(\begin{array}{cc}
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is unitary.
The maximal torus survives the deformation untouched: There exists a $*$-homomorphism $\pi: \mathcal{O}\left(S U_{q}(2)\right) \rightarrow \mathcal{O}(T)=\mathbb{C}\left[z, z^{-1}\right]$ given by $\pi(\alpha)=z, \pi(\gamma)=0$.

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In the case of $K_{q}$, the dual $\hat{K}_{q}$ is encoded by the $*$-algebra

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C_{c}\left(\hat{K}_{q}\right)=\mathcal{D}\left(K_{q}\right)=\bigoplus_{\mu \in \mathbf{P}^{+}} \operatorname{End}(V(\mu))
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equipped with a suitable coproduct.

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equipped with a suitable coproduct.
To the classical group $A$ corresponds the quotient $\hat{T}$ of $\hat{K}_{q}$ obtained from the projection $\mathcal{O}\left(K_{q}\right) \rightarrow \mathcal{O}(T)$. Here $T \subset K_{q}$ is the classical maximal torus.

## Complex semisimple quantum groups

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Consider the vector space

$$
\mathcal{D}\left(G_{q}\right)=\mathcal{D}\left(K_{q}\right) \bowtie \mathcal{O}\left(K_{q}\right)
$$

equipped with the multiplication

$$
(x \bowtie f)(y \bowtie g)=x\left(f_{(1)}, y_{(1)}\right) y_{(2)} \bowtie f_{(2)}\left(f_{(3)}, \hat{S}\left(y_{(3)}\right)\right) g
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and the $*$-structure

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(x \bowtie f)^{*}=\left(1 \bowtie f^{*}\right)\left(x^{*} \bowtie 1\right) .
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## Definition

The group $C^{*}$-algebra $C^{*}\left(G_{q}\right)$ of the complex quantum group $G_{q}$ is the universal $C^{*}$-completion of $\mathcal{D}\left(G_{q}\right)$.

## The representation theory of $G_{q}$

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- Describe all irreducible representations of $G_{q}$ up to isomorphism.
- Describe the (reduced) unitary dual of $G_{q}$.
- Describe the Plancherel formula.
- Describe the Fell topology of the (reduced) dual.


## The representation theory of $G_{q}$

By construction, a nondegenerate representation of $C^{*}\left(G_{q}\right)$ on a Hilbert space $\mathcal{H}$ corresponds to a nondegenerate $*$-homomorphism $\mathcal{D}\left(G_{q}\right) \rightarrow \mathcal{L}(\mathcal{H})$.

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This is the same thing as a unitary Yetter-Drinfeld module, that is, a pair of a unital $*$-homomorphism $\mathcal{O}\left(K_{q}\right) \rightarrow \mathcal{L}(\mathcal{H})$ and a unitary corepresentation $V \in M\left(C\left(K_{q}\right) \otimes \mathcal{H}\right)$ satisfying the Yetter-Drinfeld compatibility condition, given by

$$
f_{(1)} \xi_{(-1)} S\left(f_{(3)}\right) \otimes f_{(2)} \cdot \xi_{(0)}=(f \cdot \xi)_{(-1)} \otimes(f \cdot \xi)_{(0)}
$$

for $f \in \mathcal{O}\left(K_{q}\right)$ and $\xi$ in (a certain dense subspace of) $\mathcal{H}$.

## Principal series representations

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Let $\mathcal{O}\left(\mathcal{E}_{\mu}\right) \subset \mathcal{O}\left(K_{q}\right)$ be the spectral subspace of $\mathcal{O}\left(K_{q}\right)$ associated to $\mu \in \mathbf{P}$ with respect to the right action of $T$.

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For $\lambda \in \mathfrak{h}^{*}$ we define the twisted left adjoint representation of $\mathcal{O}\left(K_{q}\right)$ on $\mathcal{O}\left(\mathcal{E}_{\mu}\right)$ by

$$
f \cdot \xi=f_{(1)} \xi S\left(f_{(3)}\right)\left(K_{\lambda+2 \rho}, f_{(2)}\right)
$$

Together with the comultiplication of $\mathcal{O}\left(K_{q}\right)$ this turns $\mathcal{O}\left(\mathcal{E}_{\mu}\right)$ into a Yetter-Drinfeld module, which we will denote by $\mathcal{O}\left(\mathcal{E}_{\mu, \lambda}\right)$.

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This is called the principal series Yetter-Drinfeld module with parameter $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}^{*}$.
If $\lambda \in \mathfrak{i \mathfrak { a } ^ { * }} \subset \mathfrak{h}^{*}$ then this Yetter-Drinfeld module is unitary. It corresponds to a representation of $C^{*}\left(G_{q}\right)$ on the Hilbert space completion of $\mathcal{O}\left(\mathcal{E}_{\mu}\right)$.

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Recall that $q=e^{h}$, and let $\hbar=\frac{h}{2 \pi}$.

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For $\lambda \in \mathfrak{h}^{*}$, the operators $K_{\lambda}$ are defined by $K_{\lambda} v=q^{(\lambda, \nu)} v$.
Recall that $q=e^{h}$, and let $\hbar=\frac{h}{2 \pi}$.
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Hence, by their very construction, the principal series modules $\mathcal{O}\left(\mathcal{E}_{\mu, \lambda}\right)$ and $\mathcal{O}\left(\mathcal{E}_{\mu, \lambda^{\prime}}\right)$ are the same if $\lambda-\lambda^{\prime} \in i \hbar^{-1} \mathbf{Q}^{\vee}$.

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Write

$$
\mathfrak{h}_{q}^{*}=\mathfrak{h}^{*} / i \hbar^{-1} \mathbf{Q}, \quad \mathfrak{a}_{q}^{*}=\mathfrak{a}^{*} / i \hbar^{-1} \mathbf{Q} .
$$

This notation allows us to remove the "obvious" redundancies in the parametrisation of the principal series explained above.

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Theorem
Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_{q}^{*}$ such that $\lambda_{\alpha} \neq \pm\left(\left|\mu_{\alpha}\right|+j\right)$ modulo $i \hbar^{-1} \mathbb{Z}$ for all $j \in \mathbb{N}$ and all $\alpha \in \Delta^{+}$. Then the principal series module with parameter $(\mu, \lambda)$ is an irreducible Yetter-Drinfeld module.

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## Theorem

Let $(\mu, \lambda) \in \mathbf{P} \times i t_{q}^{*}$. Then the principal series modules with parameters $(\mu, \lambda)$ and ( $\mu^{\prime}, \lambda^{\prime}$ ) are equivalent iff
$\left(\mu^{\prime}, \lambda^{\prime}\right)=(w \cdot \mu, w \cdot \lambda)$ for some $w \in W$.
These results are (essentially) due to Joseph-Letzter and depend on deep facts about the structure of $U_{q}(\mathfrak{g})$.

## The Plancherel formula

## Christian Voigt (joint with R. Yuncken)

## The Plancherel formula

## Theorem

Let $q \in(0,1)$ and let $G_{q}$ be a complex semisimple quantum group. Moreover let $\mathcal{H}=\left(\mathcal{H}_{\mu, i \nu}\right)_{\mu, \nu}$ be the Hilbert space bundle of unitary principal series representations over $\mathbf{P} \times \mathfrak{a}_{q}^{*}$. Then there is a unitary isomorphism

$$
Q: L^{2}\left(G_{q}\right) \cong \bigoplus_{\mu \in \mathbf{P}} \int_{\nu \in \mathfrak{a}_{q}^{*}}^{\oplus} H S\left(\mathcal{H}_{\mu, i \nu}\right) d m_{\mu}(\nu)
$$

for the measures $d m_{\mu}$ on $\mathfrak{a}_{q}^{*}$ given by

$$
d m_{\mu}(\nu)=\prod_{\alpha \in \Delta^{+}}\left(q_{\alpha}^{1 / 2}-q_{\alpha}^{-1 / 2}\right)^{2}\left[(\mu+i \nu)_{\alpha}\right]_{q_{\alpha}^{1 / 2}}\left[(\mu-i \nu)_{\alpha}\right]_{q_{\alpha}^{1 / 2}} d \nu
$$

where $d \nu$ denotes normalised Lebesgue measure on $\mathfrak{a}_{q}^{*}$.

## Some remarks

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The proof proceeds by verifying the Plancherel formula

$$
\epsilon_{G_{q}}(f)=\sum_{\mu \in \mathbf{P}} \int_{\mathfrak{a}_{q}^{*}} \operatorname{tr}\left(\pi_{\mu, i \nu}(f) D_{\mu, i \nu}^{-2}\right) d m_{\mu}(\nu)
$$

for elements of the form $f=u_{i j}^{\beta} \otimes \omega_{k l}^{\gamma} \in \mathcal{O}\left(K_{q}\right) \otimes \mathcal{D}\left(K_{q}\right)$.
For this one starts by directly calculating the characters of principal series representations.
In this computation, the universal $R$-matrix of $U_{q}(\mathfrak{g})$ enters
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For this one starts by directly calculating the characters of principal series representations.
In this computation, the universal $R$-matrix of $U_{q}(\mathfrak{g})$ enters
crucially.
The lowest order contribution in $h$ of the quantum Plancherel measure agrees with the classical Plancherel measure

$$
\prod_{\alpha \in^{+}+}\left|\left(\mu_{\alpha}+i \nu_{\alpha}\right)\right|^{2} d \nu=(\mu+i \nu)_{\alpha}(\mu-i \nu)_{\alpha} d \nu
$$

on $\mathbf{P} \times \mathfrak{a}^{*}$.

## The reduced dual of $G_{q}$

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## Theorem

Let $q \in(0,1)$ and let $G_{q}$ be a complex semisimple quantum group. Moreover let $\mathcal{H}=\left(\mathcal{H}_{\mu, \lambda}\right)_{\mu, \lambda}$ be the Hilbert space bundle of principal series representations of $G_{q}$ over $\mathbf{P} \times \mathfrak{a}_{q}^{*}$. Then the canonical *-homomorphism

$$
\pi: C_{r}^{*}\left(G_{q}\right) \rightarrow C_{0}\left(\mathbf{P} \times \mathfrak{a}_{q}^{*}, \mathbb{K}(\mathcal{H})\right)^{W}
$$

is an isomorphism.
Setting formally $h=0$ here (corresponding to $q=1$ ), and $\mathfrak{a}_{1}^{*}=\mathfrak{a}^{*}$ one obtains the corresponding statement for the classical reduced group $C^{*}$-algebra $C_{r}^{*}(G)$.

## Baum-Connes

## Christian Voigt (joint with R. Yuncken)

## Baum-Connes

The deformation picture of the Baum-Connes assembly map for the classical complex group $G$ provides an isomorphism

$$
K_{*}\left(C^{*}\left(K \ltimes_{\mathrm{ad}} \mathfrak{k}^{*}\right)\right)=K_{*}\left(K \ltimes_{\mathrm{ad}} C_{0}(\mathfrak{k})\right) \rightarrow K_{*}\left(C_{\mathrm{r}}^{*}(G)\right) .
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Let us restrict attention to the case $G=S L(2, \mathbb{C})$.
Theorem
Fix $q \in(0,1)$. Then there is a commutative diagram

$$
\begin{gathered}
K_{*}\left(K \ltimes_{\mathrm{ad}} C_{0}(\mathfrak{k})\right) \xrightarrow{\mu} K_{*}\left(C_{\mathrm{r}}^{*}(G)\right) \\
\downarrow \\
K_{*}\left(K \ltimes_{\mathrm{ad}} C(K)\right) \xrightarrow{\mu_{q}} K_{*}\left(C_{\mathrm{r}}^{*}\left(G_{q}\right)\right)
\end{gathered}
$$

Both vertical maps are split injective, and the horizontal maps are isomorphisms.

