Strongly nonlinear multiplicative inequalities involving nonlocal operators

Agnieszka Kałamajska
University of Warsaw, Poland

I will discuss recently obtained inequality:

$$\int_{(a,b)} |f^{'}(x)|^q h(f(x))dx \leq C \int_{(a,b)} \left(\sqrt[p]{|f''(x)\mathcal{T}_{h,p}(f(x))|} \right)^q h(f(x))dx, \tag{1}$$

and its Orlicz variants, where $\mathcal{T}_{h,p}(\cdot)$ is certain transformation of function f with the property $\mathcal{T}_{h\equiv 1,2}(f) = f$, possibly nonlocal.
The example representative of such equation reads as

$$\int_{\{x:f(x)\neq 0\}} \left(\frac{|f^{'}|}{|f|^{\theta}} \right)^p dx \leq \left(\frac{p-1}{|1-\theta p|} \right)^{\frac{p}{2}} \int_{\{x:f(x)\neq 0\}} \left(\frac{\sqrt{|ff''|}}{|f|^{\theta}} \right)^p dx, \tag{2}$$

while inequality (1) with $h \equiv 1, p = 2$, implies the classical Gagliardo-Nirenberg multiplicative inequality

$$\left(\int_{(a,b)} |f^{'}(x)|^q dx \right)^{\frac{2}{q}} \leq C \left(\int_{(a,b)} |f(x)|^r \, dx \right)^{1/r} \left(\int_{(a,b)} |f^{''}(x)|^p \, dx \right)^{1/p},$$

where $\frac{2}{q} = \frac{1}{r} + \frac{1}{p}$.
Inequalities in the form (1) were obtained in the chain of my joint works with Katarzyna Pietruska-Pałuba, Jan Peszek, Katarzyna Mazowiecka, Tomasz Choczewski, Ignacy Lipka in local version, and with Alberto Fiorenza and Claudia Capogne in the nonlocal version. In some restricted variants the inequality (2) was earlier obtained by Oppial and Mazya in the 60ties and 70ties of the last century.

I will overview the variants of inequality (1), sketch its possible applications to regularity for singular elliptic pde's, and discuss its recent development toward inequalities involving nonlocal operators. These are achieved by exploiting invariances of inequality (1) in the local version.