On function spaces and extension results for nonlocal Dirichlet problems

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Let us consider the question, for which functions $g : \mathbf{R}^d \setminus \Omega \to \mathbf{R}$ there is a function $u : \mathbf{R}^d \to \mathbf{R}$ satisfying

$$Lu(x) := \text{p.v.} \int_{\mathbf{R}^d} \frac{u(y) - u(x)}{|y-x|^{1+2s}} \mathrm{d}y = 0 \qquad \text{for } x \in \Omega \,, \tag{1}$$

$$u(x) = g(x) \qquad \text{for } x \in \mathbf{R}^d \setminus \Omega \,. \tag{2}$$

For open $\Omega, G \subset \mathbf{R}^d$ define two vector spaces by

$$V(\Omega, G) = \Big\{ v \in L^2_{\text{loc}}(\mathbf{R}^d) \cap L^2(\Omega) \Big| \int_\Omega \int_G \frac{\big(v(y) - v(x)\big)^2}{|x-y|^{d+2s}} \, dx \, dy < \infty \Big\},$$

$$H_\Omega(\mathbf{R}^d) = \{ v \in V(\Omega, \mathbf{R}^d) | v = 0 \text{ on } \Omega^c \} \,.$$

Let us define the notion of a variational solution.

Definition 1 (cf. Definition 2.5 in [1])**.** Let $\Omega \subset \mathbf{R}^d$ be open and bounded. Let $g \in V(\Omega, \mathbf{R}^d)$. Then $u \in V(\Omega, \mathbf{R}^d)$ is called a variational solution to (1)–(2), if $u - g \in H_\Omega(\mathbf{R}^d)$ and for every $\varphi \in H_\Omega(\mathbf{R}^d)$

$$\int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \frac{\big(u(y) - u(x)\big)\big(\varphi(y) - \varphi(x)\big)}{|x-y|^{d+2s}} \, dy \, dx = 0 \,. \tag{3}$$

In [1] it is proved that such a variational solution u exists. However, in order to apply Definition 1 one needs to prescribe the data function g in the vector space $V(\Omega, \mathbf{R}^d)$, i.e. in particular one needs to prescribe all values of g in $\mathbf{R}^d$. This leads to two obvious questions:

Questions. For which space of functions $g : \Omega^c \to \mathbf{R}$ is there an extension operator $g \mapsto \text{ext}(g) \in V(\Omega, \mathbf{R}^d)$? Do elements of $V(\Omega, \mathbf{R}^d)$ have a trace in this space?

In the talk we will answer these questions.
Joint work with Moritz Kassmann (Universität Bielefeld).

REFERENCES

[1] M. Felsinger, M. Kassmann, and P. Voigt. The Dirichlet problem for nonlocal operators. *Math. Z.*, 279(3-4):779–809, 2015.