Well-posedness and numerical approximation of distributional solutions of non-local equations of porous medium type

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Recently J. L. Vazquez and his group and others have obtained many results concerning so-called fractional porous medium equations. These nonlinear diffusion equations are degenerate, nonlocal, and can be written in the form:

$$u_t - L[\varphi(u)] = 0.$$

Typically the diffusion operator L is a fractional Laplacian, the nonlinearity φ is of power type, and the class of solutions have finite energy. Moreover, there seems to be only a few works on numerics and no results for so-called fast diffusions.

In this talk we present two sets of results:

(i) Much more general well-posedness results: We allow any nonlocal symmetric non-negative diffusion operator, any continuous nondecreasing nonlinearity, and solutions that a priori need not have finite energy [1].

(ii) A class of numerical schemes that is monotone, stable, consistent and *convergent*, both for local and general nonlocal problems (including fast diffusions) [2].

In the generality we consider, the equations may be very degenerate, both in L and φ, and include models of Stefan type and the full range of porous medium and fast diffusion nonlinearities. We obtain uniqueness, stability, existence, compactness, a priori estimates. Our generality is so that great that natural discrete in space approximations already belong to our class of equations! The convergence of such schemes then follows from our general compactness, stability and uniqueness results. In the second part of the talk we introduce a familiy of fully discrete scheme and show that similar arguments also work there. We note that our framework also contains natural finite difference approximations for local (generalized) porous medium equations, and that convergence holds in the full fast diffusion range.

A key result in this theory is the very general uniqueness result. This result is hard to prove because of our very general assumptions combined with a very weak solution concept, a combination that means that many classical techniques do not work. We will adapt a sort of "resolvent energy method" of Brezis-Crandall to our nonlocal setting - but because of the generality of our diffusion operators - we cannot rely on explicit fundamental solutions for our proofs as in the local case. We have to develop part of the theory from scratch, using instead the equation and the regularity that comes with our solutions concept. A key tool is to approximate the possibly unbounded nonlocal operator by a bounded operator and then carefully pass to the limit. This proceedure, and hence also the proof, is truly nonlocal. The proof necessarily becomes much more involved than in the local case and includes a number of approximations, a priori estimates, L1-contraction estimates, comparison principles, compactness and regularity arguments, new Stroock-Varoupolous inequalities and Liouville type of results. Both our approach and intermediate results should be of independent interest.

Joint work with Jørgen Endal and Felix del Teso.

References

[1] J. Endal, E. R. Jakobsen, and F. del Teso. Uniqueness and properties of distributional solutions of nonlocal equations of porous medium type. Submitted for publication. Preprint: `http://arxiv.org/abs/1507.04659`

[2] J. Endal, E. R. Jakobsen, and F. del Teso. On numerical approximations of equations of porous medium type. In preparation.